Appendix A
Multilinear Algebra

In this appendix, we recall some basic definitions and properties related to multilinear algebra, including (graded) algebra and coalgebra structures, derivations and coderivations. We need them in two particular cases, namely for vector spaces over a field $\mathbb{F}$, and for $\mathscr{A}$-modules, where $\mathscr{A}$ is a commutative associative algebra over a field $\mathbb{F}$. Notice that the latter modules can also be thought of as vector spaces over $\mathbb{F}$, a fact which we often use, since many operations which we consider are $\mathbb{F}$-linear, rather than $\mathscr{A}$-linear. In order to cover both cases, we consider in this appendix the structures which we need on modules over an arbitrary commutative ring R with unit; the reader may find it useful to keep in mind the example of the $C^\infty(M)$-module of vector fields or differential forms over a manifold M.

Throughout the appendix, $\mathbb{F}$ denotes a field of characteristic zero and R denotes an arbitrary commutative ring with unit, denoted by 1.

A.1 Tensor Algebra

For R-modules V and W, the set of linear[1] maps $V \to W$ is denoted by $\mathrm{Hom}_R(V,W)$, or by $\mathrm{Hom}(V,W)$ when it is clear that V and W are considered as R-modules. $\mathrm{Hom}(V,W)$ is itself an R-module in a natural way. When V is a free R-module (for example when $R = \mathbb{F}$, so that V is an $\mathbb{F}$-vector space) and $\mathscr{B}$ is a basis of V, every map $\mathscr{B} \to W$ extends to a unique element of $\mathrm{Hom}(V,W)$ by linearity; when both V and W are finite-dimensional $\mathbb{F}$-vector spaces, and bases for V and W have been fixed, we often think of elements of $\mathrm{Hom}(V,W)$ as matrices (with $\dim W$ rows and $\dim V$ columns).

The *dual* of an R-module V is the R-module $V^* := \mathrm{Hom}(V,R)$. For a finite-dimensional $\mathbb{F}$-vector space V, the dual V^* is an $\mathbb{F}$-vector space, isomorphic to V; the isomorphism is however not canonical, since it depends on the choice of a basis

[1] We rarely use the word R-linear, to avoid confusion with the terminology for multilinear maps, i.e., k-linear maps, with $k \in \mathbb{N}^*$.

C. Laurent-Gengoux et al., *Poisson Structures*,
Grundlehren der mathematischen Wissenschaften 347,
DOI 10.1007/978-3-642-31090-4,

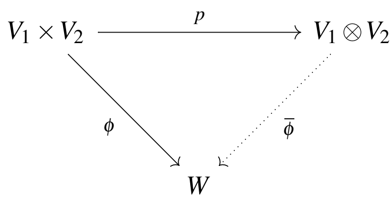
for V. The natural pairing (evaluation map)

$$\begin{aligned} \langle \cdot , \cdot \rangle : V^* \times V &\to R \\ (\xi , v) &\mapsto \langle \xi , v \rangle := \xi (v) \, , \end{aligned} \tag{A.1}$$

leads for fixed $v \in V$ to a linear map $\langle \cdot , v \rangle : V^* \to R$, i.e., to an element of $(V^*)^* = \mathrm{Hom}(V^*, R)$. The R-module $(V^*)^*$ is called the *bidual* of V. The resulting linear map $V \to (V^*)^*$ is, in general, neither injective nor surjective, but there are two important particular cases in which it is an isomorphism:

- When V is a finite-dimensional vector space over $R = \mathbb{F}$;
- When $R = C^\infty(M)$, the algebra of smooth functions on a smooth manifold M, with V the space of smooth differential k-forms on M; in this case, V^* can be identified with the R-module of smooth vector fields on M.

We will often deal with bilinear maps, sometimes with more general multilinear maps between R-modules, where multilinear means (R-)linear in each of its arguments, keeping the other arguments fixed. The language of tensor products, which we introduce now, is very useful for this. Let V_1, V_2, V and W be R-modules and let $p : V_1 \times V_2 \to V$ be a bilinear map. If we compose p with a linear map $V \to W$, then we obtain a bilinear map $V_1 \times V_2 \to W$; one may wonder if, given V_1 and V_2, there exists an R-module V, such that, for every R-module W, every bilinear map $V_1 \times V_2 \to W$ can be obtained in this way from a linear map on $V \to W$. In fact, there is a unique (up to isomorphism) such R-module, called the *tensor product* of V_1 and V_2, denoted by $V_1 \otimes_R V_2$, or $V_1 \otimes V_2$, and it comes with a natural bilinear map $p : V_1 \times V_2 \to V_1 \otimes V_2$. Formally, the stated property means that every bilinear map $\phi : V_1 \times V_2 \to W$ factors uniquely via p, meaning that there exists a unique linear map $\overline{\phi}$, such that $\overline{\phi} \circ p = \phi$. This property is displayed in the following diagram:



In practice we do not make a distinction between the bilinear map ϕ and the linear map $\overline{\phi}$, so we simply write $\phi \in \mathrm{Hom}(V_1 \otimes V_2, W)$. A natural construction of the tensor product $V_1 \otimes V_2$ is as the quotient of the free R-module which is generated by all formal expressions $v_1 \otimes v_2$, with $v_1 \in V_1$ and $v_2 \in V_2$, divided by the equivalence relation defined by

$$\begin{aligned} &(v_1 + v_1') \otimes v_2 = v_1 \otimes v_2 + v_1' \otimes v_2 , \\ &v_1 \otimes (v_2 + v_2') = v_1 \otimes v_2 + v_1 \otimes v_2' , \\ &a(v_1 \otimes v_2) = (a v_1) \otimes v_2 = v_1 \otimes (a v_2) , \end{aligned}$$

where $v_1, v_1' \in V_1$ and $v_2, v_2' \in V_2$ and $a \in R$. One says that $v_1 \otimes v_2$ is the *tensor product* of v_1 and v_2. The maps p and $\overline{\phi}$ are in this notation simply given by

$p(v_1, v_2) = v_1 \otimes v_2$ and $\overline{\phi}(v_1 \otimes v_2) = \phi(v_1, v_2)$. In view of the above three properties, p is a bilinear map.

As an application, consider the bilinear map $V_1 \times V_2 \to V_2 \otimes V_1$, which is given by $(v_1, v_2) \mapsto v_2 \otimes v_1$. It factors via p to yield an R-module isomorphism $S : V_1 \otimes V_2 \to V_2 \otimes V_1$, given by $S(v_1 \otimes v_2) = v_2 \otimes v_1$, and called the *twist map*.

One easily shows that for arbitrary R-modules V_1, V_2 and V_3, one has a natural isomorphism $(V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes (V_2 \otimes V_3)$. The latter is also denoted as $V_1 \otimes V_2 \otimes V_3$ as it can also be described, like in the case of bilinear maps, as a universal object for trilinear maps, defined on $V_1 \times V_2 \times V_3$. Under the natural isomorphisms, $(V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes (V_2 \otimes V_3) \simeq V_1 \otimes V_2 \otimes V_3$, one has that

$$(v_1 \otimes v_2) \otimes v_3 \leftrightarrow v_1 \otimes (v_2 \otimes v_3) \leftrightarrow v_1 \otimes v_2 \otimes v_3 \, , \tag{A.2}$$

so that, in the sequel, we will not make a notational distinction between the elements in (A.2). The extension to several R-modules is clear. For a given R-module V, we obtain a natural sequence of R-modules $T^k V := V^{\otimes k}$, where $k = 0, 1, 2, \dots$, which is defined, as the notation suggests, by $V^{\otimes k} := V \otimes \cdots \otimes V$ (k factors), when $k \geqslant 1$ and $V^{\otimes 0} := R$. Equipped with the product $(X, Y) \mapsto X \otimes Y$, where $X \in V^{\otimes k}$ and $Y \in V^{\otimes \ell}$, the R-module

$$T^\bullet V := \bigoplus_{k=0}^{\infty} T^k V = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

becomes a graded R-algebra, called the *tensor algebra* of V. See Section A.3 below for the basic definitions on graded algebras.

Fixing one R-module Z, it is useful to think of taking the tensor product with Z as a functor, which means on the one hand that a linear map $\phi \in \mathrm{Hom}(V, W)$ yields, in a natural way, a linear map

$$\tilde{\phi} \in \mathrm{Hom}(V \otimes Z, W \otimes Z) \, , \tag{A.3}$$

simply by putting $\tilde{\phi}(v \otimes z) := \phi(v) \otimes z$, for $v \in V$ and $z \in Z$, which is well-defined; the map $\tilde{\phi}$ is usually denoted by $\phi \otimes \mathbb{1}_Z$, where $\mathbb{1}_Z$ stands for the identity map on Z. On the other hand, it means that taking the tensor product with Z has the usual functorial properties, which make it into a covariant functor. In formulas, this is written as follows:

$$\mathbb{1}_V \otimes \mathbb{1}_Z = \mathbb{1}_{V \otimes Z}, \qquad (\phi \circ \psi) \otimes \mathbb{1}_Z = (\phi \otimes \mathbb{1}_Z) \circ (\psi \otimes \mathbb{1}_Z) \, ,$$

where $\phi \in \mathrm{Hom}(V, W)$ and $\psi \in \mathrm{Hom}(U, V)$. Above, we tensored with Z on the right, but we could also have tensored with Z on the left.

For R-modules V_i and W_i, with $i = 1, 2$, there is also a natural injective morphism (which is not surjective, in general)

$$\mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2) \to \mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2) \, ,$$

where $\phi_1 \otimes \phi_2 \in \mathrm{Hom}(V_1,W_1) \otimes \mathrm{Hom}(V_2,W_2)$ is, as a linear map from $V_1 \otimes V_2$ to $W_1 \otimes W_2$, given by

$$(\phi_1 \otimes \phi_2)(v_1 \otimes v_2) := \phi_1(v_1) \otimes \phi_2(v_2) ,$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. This justifies the notation $\phi \otimes \mathbb{1}_Z$ introduced above.

For R-modules V_1, V_2 and W, bilinear maps $V_1 \times V_2 \to W$ are also in natural correspondence with linear maps $V_1 \to \mathrm{Hom}(V_2,W)$, or with linear maps $V_2 \to \mathrm{Hom}(V_1,W)$. We use this usually in the form of the two natural isomorphisms

$$\mathrm{Hom}(V_1 \otimes V_2, W) \simeq \mathrm{Hom}(V_1, \mathrm{Hom}(V_2,W)) \simeq \mathrm{Hom}(V_2, \mathrm{Hom}(V_1,W)) ,$$

which allows one to use a single bilinear map $V_1 \times V_2 \to W$ to associate to each element of V_1 (respectively V_2) a linear map $V_2 \to W$ (respectively a linear map $V_1 \to W$). For example, for arbitrary R-modules V and W,

$$(V \otimes W)^* \simeq \mathrm{Hom}(V,W^*) \simeq \mathrm{Hom}(W,V^*) . \tag{A.4}$$

For given R-modules V and W, there is also a natural linear map $\Psi : V^* \otimes W \to \mathrm{Hom}(V,W)$, which associates to an element $\xi \otimes w$, with $\xi \in V^*$ and $w \in W$, the linear map

$$\begin{aligned} \Psi(\xi \otimes w) : V &\to W \\ v &\mapsto \xi(v)\, w = \langle \xi, v \rangle\, w . \end{aligned}$$

The map Ψ is always injective, but is in general not surjective. When $R = \mathbb{F}$, then Ψ is an isomorphism if and only if V or W is finite-dimensional. Combining the injection $V^* \otimes W^* \to \mathrm{Hom}(V,W^*)$ with the first isomorphism in (A.4), we obtain a natural inclusion

$$V^* \otimes W^* \to (V \otimes W)^* ,$$

which is an isomorphism when $R = \mathbb{F}$ and V or W is finite-dimensional.

When $\phi_1 \in \mathrm{Hom}(V_1, \mathscr{A})$ and $\phi_2 \in \mathrm{Hom}(V_2, \mathscr{A})$, where $\mathscr{A}$ is an R-algebra, then $\phi_1 \otimes \phi_2$ is often implicitly combined with the multiplication map $\mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$, yielding $\phi_1 \otimes \phi_2 \in \mathrm{Hom}(V_1 \otimes V_2, \mathscr{A})$, given for $v_1 \in V_1$ and $v_2 \in V_2$, by

$$(\phi_1 \otimes \phi_2)(v_1 \otimes v_2) := \phi_1(v_1)\phi_2(v_2) , \tag{A.5}$$

where the latter product is the multiplication in $\mathscr{A}$. When this multiplication has extra properties, they yield similar properties for the product of maps: for example, if $\mathscr{A}$ is commutative, then

$$\phi_1 \otimes \phi_2 = (\phi_2 \otimes \phi_1) \circ S ,$$

as elements of $\mathrm{Hom}(V_1 \otimes V_2, \mathscr{A})$. Associativity of $\mathscr{A}$ implies that

$$(\phi_1 \otimes \phi_2) \otimes \phi_3 = \phi_1 \otimes (\phi_2 \otimes \phi_3) ,$$

as elements of $\mathrm{Hom}((V_1 \otimes V_2) \otimes V_3, \mathscr{A}) \simeq \mathrm{Hom}(V_1 \otimes (V_2 \otimes V_3), \mathscr{A})$.

A.2 Exterior and Symmetric Algebra

Our multilinear maps are usually skew-symmetric k-linear maps $V^k \to V$, where V is an R-module, so we recall here the corresponding tensorial notions. In $T^\bullet V$, consider the submodule N which is generated by all ℓ-tensors ($\ell \in \mathbb{N}^*$) of the form $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$, where $v_{i_s} = v_{i_t}$ for some $1 \leqslant s < t \leqslant \ell$, and let $N_k := N \cap T^k V$ for $k \in \mathbb{N}^*$ and $N_0 := \{0\}$. Then we may consider

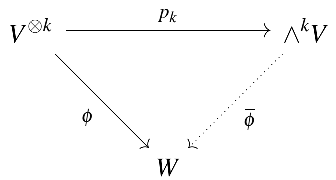
$$\wedge^\bullet V := T^\bullet V/N = \bigoplus_{k=0}^{\infty} T^k V/N_k = \bigoplus_{k=0}^{\infty} \wedge^k V ,$$

where $\wedge^k V := T^k V/N_k$, for $k \in \mathbb{N}$. Notice that $\wedge^k V = \{0\}$ as soon as k is bigger than the (minimal) number of generators of V. We denote the quotient maps by $p : T^\bullet V \to \wedge^\bullet V$ and $p_k : T^k V \to \wedge^k V$. Since N is a two-sided ideal of $T^\bullet V$, we have that the associative product $\otimes$ on $T^\bullet V$ induces an associative product on $\wedge^\bullet V$, which is denoted by $\wedge$. Thus $p(v_1 \otimes \cdots \otimes v_k) = p(v_1) \wedge \cdots \wedge p(v_k)$, which we also write as $v_1 \wedge \cdots \wedge v_k$, because p_1 (i.e., the restriction of p to V) is injective. One easily verifies that, if $X \in \wedge^i V$ and $Y \in \wedge^j V$, then $X \wedge Y \in \wedge^{i+j} V$ and

$$X \wedge Y = (-1)^{ij} Y \wedge X . \tag{A.6}$$

In the language of the next section, this property is called graded commutativity. The associative, graded commutative R-algebra $(\wedge^\bullet V, \wedge)$ is called the *exterior algebra* of V and elements of $\wedge^\bullet V$ are called *multivectors*. One similarly constructs the *symmetric algebra* $(S^\bullet V, \cdot)$ which is the associative commutative graded R-algebra obtained as $T^\bullet V/N'$, where N' is the two-sided ideal of $T^\bullet V$, generated by all $v \otimes w - w \otimes v$, where $v, w \in V$.

It is clear that every skew-symmetric k-linear map $\phi \in \operatorname{Hom}(V^{\otimes k}, W)$ vanishes on N_k. Therefore, for R-modules V and W, we have that every skew-symmetric k-linear map $\phi \in \operatorname{Hom}(V^{\otimes k}, W)$ corresponds in a canonical way to a linear map $\overline{\phi} : \wedge^k V \to W$, as in the following commutative diagram:



In formulas, $\overline{\phi}(v_1 \wedge \cdots \wedge v_k) = \phi(v_1, \ldots, v_k)$. From now on, we do not distinguish notationally between the maps $\overline{\phi}$ and ϕ: we write $\phi \in \operatorname{Hom}(\wedge^k V, W)$ and we simply say that ϕ is a skew-symmetric k-linear map.

One usually thinks of elements of $\wedge^k V$ as skew-symmetric tensors: the permutation group S_k defines a natural linear action on $V^{\otimes k}$ which is defined for $\sigma \in S_k$ and $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ by

$$\sigma(v_1 \otimes \cdots \otimes v_k) := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} .$$

An element $X \in V^{\otimes k}$ is called a *symmetric tensor* when $\sigma(X) = X$ for all $\sigma \in S_k$, while it is called a *skew-symmetric tensor* when $\sigma(X) = \operatorname{sgn}(\sigma)X$ for all $\sigma \in S_k$, where $\operatorname{sgn}(\sigma)$ denotes the signature of σ. In order to identify the skew-symmetric k-tensors with $\wedge^k V$, one defines a linear map ρ_k^-, the *skew-symmetrization map*, by

$$\begin{aligned} \rho_k^- : \quad \wedge^k V \quad &\to T^k V \\ v_1 \wedge \cdots \wedge v_k &\mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} . \end{aligned} \tag{A.7}$$

It is easy to see that this map is well-defined and injective, and that its image consists precisely of all skew-symmetric k-tensors. One similarly identifies the symmetric k-tensors with $S^k V$ by using the *symmetrization map*

$$\rho_k^+ : S^k V \to T^k V ,$$

whose definition is formally the same as the above definition (A.7) of the skew-symmetrization map, except that one leaves out the factor $\operatorname{sgn}(\sigma)$.

There are two types of *internal products* related to the exterior algebra. Let V and W be arbitrary R-modules. For $X \in \wedge^j V$, the *internal product* ι_X yields, for every $i \in \mathbb{N}$, a linear map

$$\iota_X : \operatorname{Hom}(\wedge^i V, W) \to \operatorname{Hom}(\wedge^{i-j} V, W) ,$$

which is given by

$$\iota_X \phi\, (Z) := \phi(X \wedge Z)$$

where $\phi : \wedge^i V \to W$ and $Z \in \wedge^{i-j} V$, assuming $i \geqslant j$; otherwise, i.e., when $i < j$, then $\iota_X \phi := 0$. It is easily verified that

$$\iota_{X \wedge Y} = \iota_Y \circ \iota_X ,$$

for $X \in \wedge^j V$ and $Y \in \wedge^k V$. For $\phi \in \operatorname{Hom}(\wedge^i V, R)$ the *internal product* ι_ϕ, is the family of linear maps, indexed by $j \in \mathbb{N}$,

$$\iota_\phi : \wedge^j V \to \wedge^{j-i} V ,$$

defined for all $v_1 \wedge \cdots \wedge v_j \in \wedge^j V$ by

$$\iota_\phi(v_1 \wedge \cdots \wedge v_j) := \sum_{\sigma \in S_{i,j-i}} \operatorname{sgn}(\sigma)\, \phi(v_{\sigma(1)}, \ldots, v_{\sigma(i)})\, v_{\sigma(i+1)} \wedge \cdots \wedge v_{\sigma(j)}$$

when $i \leqslant j$, and ι_ϕ is the zero map otherwise. In this formula, $S_{i,k}$ denotes the set of all (i,k)-*shuffles*, i.e., all permutations $\sigma \in S_{i+k}$ for which $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(i+k)$; $\operatorname{sgn}(\sigma)$ is the signature of σ as a permutation.

To finish this section, we define a product on skew-symmetric multilinear maps which take values in a commutative associative R-algebra $\mathscr{A}$. We write FG for the product of elements F and G in $\mathscr{A}$. As above, V denotes an arbitrary R-module; often, in the body of the book, this product will be considered for $V = \mathscr{A}$, viewed as an $\mathbb{F}$-vector space ($R = \mathbb{F}$). For $\phi \in \operatorname{Hom}(\wedge^i V, \mathscr{A})$ and $\psi \in \operatorname{Hom}(\wedge^j V, \mathscr{A})$, where $i, j > 0$, we define $\phi \wedge \psi \in \operatorname{Hom}(\wedge^{i+j} V, \mathscr{A})$ by

$$(\phi \wedge \psi)(v_1, \ldots, v_{i+j}) := \sum_{\sigma \in S_{i,j}} \operatorname{sgn}(\sigma)\, \phi(v_{\sigma(1)}, \ldots, v_{\sigma(i)})\, \psi(v_{\sigma(i+1)}, \ldots, v_{\sigma(i+j)}) ,$$

for all $v_1, \ldots, v_{i+j} \in V$. We also write

$$\langle \phi \wedge \psi, v_1 \wedge \cdots \wedge v_{i+j} \rangle := (\phi \wedge \psi)(v_1, \ldots, v_{i+j}) ,$$

thereby extending the notation of (A.1).

Notice that, if $\mathscr{A} = R$, so that $\phi \in (\wedge^i V)^*$ and $\psi \in (\wedge^j V)^*$, then we can alternatively define $\phi \wedge \psi \in (\wedge^{i+j} V)^*$ as being the linear map $\psi \circ \iota_\phi$. In this particular case, the definition implies the following relation between ι and $\wedge$:

$$\iota_{\phi \wedge \psi} = \iota_\psi \circ \iota_\phi ,$$

for $\phi, \psi \in \oplus_{k \in \mathbb{N}} (\wedge^k V)^*$. Indeed, let $\phi \in \operatorname{Hom}(\wedge^i V, \mathscr{A})$ and $\psi \in \operatorname{Hom}(\wedge^j V, \mathscr{A})$, where $i, j > 0$, and let $v_1, \ldots, v_k \in V$, with $k \geqslant i + j$. Then,

$$\begin{aligned}
&\iota_{\phi \wedge \psi}(v_1, \ldots, v_k) \\
&= \sum_{\sigma \in S_{i+j,k-i-j}} \operatorname{sgn}(\sigma) (\phi \wedge \psi)(v_{\sigma(1)}, \ldots, v_{\sigma(i+j)})\, v_{\sigma(i+j+1)} \wedge \cdots \wedge v_{\sigma(k)} \\
&= \sum_{\tau \in S_{i,j,k-i-j}} \operatorname{sgn}(\tau)\, \phi(v_{\tau(1)}, \ldots, v_{\tau(i)})\, \psi(v_{\tau(i+1)}, \ldots, v_{\tau(i+j)}) \\
&\qquad\qquad v_{\tau(i+j+1)} \wedge \cdots \wedge v_{\tau(k)} \\
&= \iota_\psi \left(\sum_{\rho \in S_{i,k-i}} \operatorname{sgn}(\rho) \phi(v_{\rho(1)}, \ldots, v_{\rho(i)})\, v_{\rho(i+1)} \wedge \cdots \wedge v_{\rho(k)} \right) \\
&= (\iota_\psi \circ \iota_\phi)(v_1, \ldots, v_k) ,
\end{aligned} \tag{A.8}$$

where we have used the notation $S_{i,j,k-i-j}$ for the set of all permutations of $\{1, \ldots, k\}$ that are increasing on the intervals $1, \ldots, i$ and $i+1, \ldots, i+j$ and $i+j+1, \ldots, k$.

We return to the general case for which $\mathscr{A}$ is not necessarily R, but an arbitrary commutative associative R-algebra. We point out that, if $\phi \in \operatorname{Hom}(\wedge^i V, \mathscr{A})$, $\psi \in \operatorname{Hom}(\wedge^j V, \mathscr{A})$ and $\chi \in \operatorname{Hom}(\wedge^k V, \mathscr{A})$, where $i, j, k \geqslant 0$, then

$$(\phi \wedge \psi) \wedge \chi = \phi \wedge (\psi \wedge \chi) ,$$

i.e., $\wedge$ is an associative product on $\oplus_{\ell \in \mathbb{N}} \operatorname{Hom}(\wedge^\ell, \mathscr{A})$. This is obtained by exactly the same computation as in (A.8), with $k = \ell - i - j$ and with χ acting on the last part (i.e., on the last v_s that appear out of the arguments of ϕ and ψ). Moreover, for $\phi \in \mathrm{Hom}(\wedge^i V, \mathscr{A})$ and $\psi \in \mathrm{Hom}(\wedge^j V, \mathscr{A})$, we have

$$\phi \wedge \psi = (-1)^{ij} \psi \wedge \phi \ .$$

In the language of the next section, the product $\wedge$ makes $\oplus_{k\in\mathbb{N}} \mathrm{Hom}(\wedge^k V, \mathscr{A})$ into a graded R-algebra which is associative and graded commutative. Notice that, in the notation which we use, if $\phi_1, \ldots, \phi_k \in \mathrm{Hom}(V, \mathscr{A})$ and $v_1, \ldots, v_k \in V$, then

$$\langle \phi_1 \wedge \cdots \wedge \phi_k, v_1 \wedge \cdots \wedge v_k \rangle = \det\big(\langle \phi_i, v_j \rangle\big)_{1\leqslant i,j\leqslant k} \ . \tag{A.9}$$

A.3 Algebras and Graded Algebras

In this section we recall the basic definitions of algebras and graded algebras. These definitions will be dualized in the next section, to obtain the notions of a coalgebra and of a graded coalgebra.

Let V be an R-module. An algebra structure on V is a bilinear map $\mu : V \times V \to V$, called a *product*. We also view μ as an element of $\mathrm{Hom}(V \otimes V, V)$, and we say that (V, μ) is an R-algebra. Usually, μ is assumed to have additional properties; the typical extra properties that μ may be supposed to have are summarized in Table A.1.

Table A.1 A product μ on an R-module, which makes it into an R-algebra, is usually assumed to have one or two additional properties, taken from the list which appears in this table. We write the properties in their usual form (with $u, v, w \in V$) and in their functional form; the latter is useful for obtaining the "co"-version (see Section A.4). S is the twist map $u \otimes v \mapsto v \otimes u$ and $\mathscr{S}$ is the cycle map $u \otimes v \otimes w \mapsto v \otimes w \otimes u$.

Property	Usual / functional form
commutative	$\mu(v,u) = \mu(u,v)$ $\mu \circ S = \mu$
skew-symmetric	$\mu(v,u) = -\mu(u,v)$ $\mu \circ S = -\mu$
associative	$\mu(u,\mu(v,w)) = \mu(\mu(u,v),w)$ $\mu \circ (\mathbb{1}_V \otimes \mu) = \mu \circ (\mu \otimes \mathbb{1}_V)$
Jacobi identity	$\mu(u,\mu(v,w)) + \circlearrowleft (u,v,w) = 0$ $\sum_{\ell=0}^{2} \mu \circ (\mathbb{1}_V \otimes \mu) \circ \mathscr{S}^\ell = 0$

The usual combinations of adjectives are the following: (1) If μ is skew-symmetric and satisfies the Jacobi identity, then (V, μ) is called a *Lie algebra*, and μ is called a *Lie bracket* on V. We employ the standard custom of using brackets, such as $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$, for the product. (2) An equally important combination of properties that μ

may have are commutativity and associativity; some authors call (V,μ) in this case simply an *algebra* (or R-algebra), but we will not use this convention here since our modules will usually have two algebra structures, one of which is a Lie algebra structure, and the other one is associative and commutative.

Example A.1. A simple example of a Lie algebra structure is given by the vector space $\mathrm{Hom}(W,W)$, where W is an arbitrary vector space, equipped with the *commutator*

$$[\phi_1,\phi_2] := \phi_1\circ\phi_2 - \phi_2\circ\phi_1 \ ,$$

where ϕ_1, $\phi_2 \in \mathrm{Hom}(W,W)$. The Jacobi identity for $[\cdot\,,\cdot]$ is a direct consequence of the associativity of the composition of (linear) maps.

A linear map $\phi : V \to W$ between R-algebras (V,μ) and (W,μ') is called an *algebra homomorphism* if $\phi(\mu(v_1,v_2)) = \mu'(\phi(v_1),\phi(v_2))$, for all v_1, $v_2 \in V$, which is written in functional form as $\phi\circ\mu = \mu'\circ(\phi\otimes\phi)$, and which corresponds to the commutativity of the following diagram.

$$\begin{array}{ccc} V\otimes V & \xrightarrow{\ \mu\ } & V \\ {\scriptstyle \phi\otimes\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ W\otimes W & \xrightarrow{\ \mu'\ } & W \end{array} \tag{A.10}$$

In the case of Lie algebras $(V,[\cdot\,,\cdot])$ and $(W,[\cdot\,,\cdot]')$, such a linear map ϕ is called a *Lie algebra homomorphism*. The homomorphism property then takes the form $\phi([v_1,v_2]) = [\phi(v_1),\phi(v_2)]'$ for all $v_1, v_2 \in V$.

We now turn to the graded version of these definitions. For this, it is assumed that V is a *graded R-module*,

$$V = \bigoplus_{i\in\mathbb{Z}} V_i \ , \tag{A.11}$$

where each of the subspaces V_i is invariant under the action of R. The notation $V_\bullet$ (or $V^\bullet$, when the subspaces are indexed by superscripts) is also used for V. In many cases, one has $V = \oplus_{i\in\mathbb{N}} V_i$, or even $V = \oplus_{i=0}^{k} V_i$, the other V_i being undefined, but one easily arrives at the form (A.11) by defining $V_i := \{0\}$, for those values of i where V_i was undefined. An element of V_i is called a *homogeneous element* of V of *degree i*. A linear map $\phi : V \to W$ between graded R-modules is said to be *graded* of *degree r* if $\phi(V_i) \subset W_{i+r}$ for every $i \in \mathbb{Z}$. When V and W are written as $V_\bullet$ and $W_\bullet$, the suggestive notation $\phi : V_\bullet \to W_{\bullet+r}$ is also used. We denote the R-module of all graded linear maps from V to W of degree r by $\mathrm{Hom}_r(V,W)$. A *graded product* on V is a product μ on V such that

$$\mu(V_i\otimes V_j) \subset V_{i+j} \ , \text{ for all } i,j\in\mathbb{Z} \ .$$

Then V, equipped with μ, becomes a *graded R-algebra* and we have induced maps $\mu_{i,j} : V_i\otimes V_j \to V_{i+j}$ for all i, j. A graded linear map of degree zero $\phi : V \to W$ which is a (Lie) algebra homomorphism is called a *graded (Lie) algebra homomorphism*. The graded analog of Table A.1 is given by Table A.2.

Table A.2 The graded analog of Table A.1 is displayed. The only difference between the graded and ungraded notions lies in the signs; in fact, as there are no signs in the case of graded associativity, the notion of associativity and graded associativity coincide. For the graded Jacobi identity, it is understood that when one sums over the three cyclic permutations of (i,j,k), the exponent ℓ takes the consecutive values $0,1$ and 2. The elements u,v and w are assumed to be homogeneous of respective degrees i,j and k.

Property	Usual / functional form
graded commutative	$\mu(v,u) = (-1)^{ij}\mu(u,v)$ $\mu_{j,i}\circ S = (-1)^{ij}\mu_{i,j}$
graded skew-symmetric	$\mu(v,u) = -(-1)^{ij}\mu(u,v)$ $\mu_{j,i}\circ S = -(-1)^{ij}\mu_{i,j}$
(graded) associative	$\mu(u,\mu(v,w)) = \mu(\mu(u,v),w)$ $\mu_{i,j+k}\circ(\mathbb{1}_{V_i}\otimes\mu_{j,k}) = \mu_{i+j,k}\circ(\mu_{i,j}\otimes\mathbb{1}_{V_k})$
graded Jacobi identity	$(-1)^{ik}\mu(u,\mu(v,w)) + \circlearrowleft(u,v,w) = 0$ $(-1)^{ik}\mu_{i,j+k}\circ(\mathbb{1}_{V_i}\otimes\mu_{j,k})\circ\mathscr{S}^{\ell} + \circlearrowleft(i,j,k) = 0$

As in the ungraded case, the combination of graded skew-symmetric and the graded Jacobi identity leads to the notion of a *graded Lie bracket* and of a *graded Lie algebra*. The combination of graded commutativity and associativity leads to the notion of an *associative, graded commutative algebra*.

Example A.2. To give a simple example of a graded Lie algebra, we define for graded linear maps $\phi_i \in \mathrm{Hom}_{r_i}(V,V)$, where $i = 1,2$, their *graded commutator* $[\phi_1,\phi_2]$ as the graded linear map of degree r_1+r_2, given by

$$[\phi_1,\phi_2] := \phi_1\circ\phi_2 - (-1)^{r_1 r_2}\phi_2\circ\phi_1 \ . \tag{A.12}$$

The graded R-module $\oplus_{r\in\mathbb{Z}}\mathrm{Hom}_r(V,V)$, equipped with this bracket, is a graded Lie algebra. For graded linear maps ϕ_1,ϕ_2,ϕ_3 of degree r_1,r_2,r_3, the graded Jacobi identity takes the following form

$$(-1)^{r_1 r_3}[\phi_1,[\phi_2,\phi_3]] + (-1)^{r_2 r_1}[\phi_2,[\phi_3,\phi_1]] + (-1)^{r_3 r_2}[\phi_3,[\phi_1,\phi_2]] = 0\ , \tag{A.13}$$

which can also be written, in view of the graded skew-symmetry of $[\cdot,\cdot]$, as

$$[\phi_1,[\phi_2,\phi_3]] = [[\phi_1,\phi_2],\phi_3] + (-1)^{r_1 r_2}[\phi_2,[\phi_1,\phi_3]]\ .$$

In the language of Section A.5, this means that $[\phi_1,\cdot]$, which is a graded linear map of degree r_1, is a graded derivation of $[\cdot,\cdot]$.

We have in this appendix already met the following three examples of graded algebras, associated to an R-module V.

$(T^\bullet V, \otimes)$ is a graded R-algebra, which is associative;
$(\wedge^\bullet V, \wedge)$ is a graded R-algebra, which is associative and graded commutative;
$(S^\bullet V, \cdot)$ is a graded R-algebra, which is associative and commutative.

The grading on $T^\bullet V$ comes from the natural decomposition $T^\bullet V = \oplus_{i\in\mathbb{N}} V^{\otimes i}$; for $\wedge^\bullet V$ and $S^\bullet V$, the induced grading is used. The commutativity of the graded algebra $(S^\bullet V, \cdot)$ should not be confused with the *graded* commutativity of the graded algebra $(\wedge^\bullet V, \wedge)$: in the former the arguments commute, but in the latter they only commute up to a sign, see (A.6). We think of $T^\bullet$ as a functor: given a linear map $\phi \in \mathrm{Hom}(V,W)$, we obtain a homomorphism of graded algebras $T^\bullet\phi : T^\bullet V \to T^\bullet W$, whose restriction to $T^k V$ is the linear map $T^k V \to T^k W$, defined by $T^k\phi := \phi\otimes\phi\otimes\cdots\otimes\phi$, i.e.,

$$T^k\phi\,(v_1\otimes v_2\otimes\cdots\otimes v_k) := \phi(v_1)\otimes\phi(v_2)\otimes\cdots\otimes\phi(v_k)\ .$$

$T^\bullet$ is a covariant functor: for $\phi \in \mathrm{Hom}(V,W)$ and $\psi \in \mathrm{Hom}(W,Z)$ one obtains

$$T^\bullet(\mathbb{1}_V) = \mathbb{1}_{T^\bullet V}, \qquad T^\bullet(\psi\circ\phi) = T^\bullet(\psi)\circ T^\bullet(\phi)\ .$$

Similarly, $\wedge^\bullet$ and $S^\bullet$ are covariant functors.

Associated to a graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$ there is a graded exterior algebra which takes into account the grading on V. This algebra, denoted by $\wedge^\bullet V$ is the associative, graded commutative algebra, obtained by dividing the tensor algebra $T^\bullet V = \bigoplus_{k\in\mathbb{N}} T^k V$ of V by the ideal generated by the elements of the form $x\otimes y + (-1)^{ij} y\otimes x$, with $x\in V_i$ and $y\in V_j$. Denoting by $\wedge$ the product in $\wedge^\bullet V$, one then has

$$x\wedge y = -(-1)^{ij} y\wedge x\ ,$$

for all $x\in V_i$ and $y\in V_j$. For integers $i_1,\ldots,i_k$ and for $\sigma\in\mathscr{S}_k$ an arbitrary permutation of $\{1,\ldots,k\}$, let $\mathrm{sgn}(\sigma; i_1,\ldots,i_k)\in\{1,-1\}$ denote the sign, defined by the equality

$$x_1\wedge\cdots\wedge x_k = \mathrm{sgn}(\sigma; i_1,\ldots,i_k)\, x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k)}\ ,$$

where x_ℓ is an arbitrary homogeneous element of V of degree i_ℓ, for $\ell = 1,\ldots,k$. As in the ungraded case, if $\phi : V^{\otimes k}\to W$ is a linear map, satisfying

$$\phi(x_1\otimes\cdots\otimes x_k) = \mathrm{sgn}(\sigma; i_1,\ldots,i_k)\ \phi(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(k)})\ ,$$

for all homogeneous elements $x_1,\ldots,x_k$, with i_ℓ denoting the degree of x_ℓ for $\ell = 1,\ldots,k$, then ϕ descends to a linear map $\wedge^\bullet V\to W$, which we also denote by ϕ.

A.4 Coalgebras and Graded Coalgebras

We now consider the "co"-versions of the concepts which were introduced in the previous section. This is done in the customary way, namely we obtain the "co"-versions by dualizing the definitions of the previous section, written in their functional forms (see Tables A.1 and A.2), which is done by reversing all arrows and switching the order of their composition. Let us first dualize the definition of an algebra: a *coalgebra* structure on an R-module V is an element Δ of $\operatorname{Hom}(V, V \otimes V)$, called a *coproduct*. The most important additional properties which Δ may have, are summarized in Table A.3.

Table A.3 Dualizing the usual properties of the product of an algebra, written in functional form, as in Table A.1, we obtain the usual properties which the coproduct Δ, defining a coalgebra structure on an R-module, can have. As before, S is the twist map and $\mathscr{S}$ is the cycle map.

Property	Functional form
cocommutative	$S \circ \Delta = \Delta$
co-skew-symmetric	$S \circ \Delta = -\Delta$
coassociative	$(\mathbb{1}_V \otimes \Delta) \circ \Delta = (\Delta \otimes \mathbb{1}_V) \circ \Delta$
co-Jacobi identity	$\sum_{\ell=0}^{2} \mathscr{S}^{\ell} \circ (\mathbb{1}_V \otimes \Delta) \circ \Delta = 0$

In order to define a *homomorphism of coalgebras*, we just reverse the arrows in (A.10): for given coalgebras (V, Δ) and (W, Δ') we call a linear map $\phi : W \to V$ a homomorphism if the following diagram is commutative.

$$
\begin{array}{ccc}
V \otimes V & \stackrel{\Delta}{\longleftarrow} & V \\
\uparrow {\scriptstyle \phi \otimes \phi} & & \uparrow {\scriptstyle \phi} \\
W \otimes W & \underset{\Delta'}{\longleftarrow} & W
\end{array}
$$

We now consider the graded versions of the above "co"-concepts. Let V be a graded R-module, $V = \oplus_{i \in \mathbb{Z}} V_i$. A *graded coproduct* is a coproduct Δ on V such that for every $k \in \mathbb{Z}$,

$$\Delta(V_k) \subset \bigoplus_{i+j=k} V_i \otimes V_j \; ,$$

is finitely supported, i.e., $\Delta(V_k)$ has a non-trivial intersection with only a finite number of $V_i \otimes V_j$. Notice that $\Delta(V_k)$ is automatically finitely supported when $V_i = \{0\}$ for all $i < 0$. A graded R-module V, equipped with a graded coproduct Δ, is called a *graded R-coalgebra*. For fixed $i, j \in \mathbb{Z}$, we will need the linear map $\Delta_{i,j} : V_{i+j} \to V_i \otimes V_j$, which is obtained by composing Δ, restricted to V_{i+j}, with the natural projection $V \otimes V \to V_i \otimes V_j$. The usual properties of a graded coproduct are displayed in Table A.4.

Table A.4 This last table deals with the case of graded coalgebras. We recall that S is the twist map and $\mathscr{S}$ is the cycle map (see Table A.2).

Property	Functional form
graded cocommutative	$S \circ \Delta_{j,i} = (-1)^{ij} \Delta_{i,j}$
graded co-skew-symmetric	$S \circ \Delta_{j,i} = -(-1)^{ij} \Delta_{i,j}$
graded coassociative	$(\mathbb{1}_{V_i} \otimes \Delta_{j,k}) \circ \Delta_{i,j+k} = (\Delta_{i,j} \otimes \mathbb{1}_{V_k}) \circ \Delta_{i+j,k}$
graded co-Jacobi identity	$(-1)^{ik} \mathscr{S}^\ell \circ (\mathbb{1}_{V_i} \otimes \Delta_{j,k}) \circ \Delta_{i,j+k} + \circlearrowleft (i,j,k) = 0$

We have seen in the previous section that the graded R-modules $T^\bullet V$, $\wedge^\bullet V$ and $S^\bullet V$ have a natural (graded) algebra structure. We now show that they also have a graded coalgebra structure; the latter structure is important at a few places in this book. The coalgebra structure Δ on $T^\bullet V$ is called *de-concatenation* and is defined, for $k \in \mathbb{N}$ and for $v_1, \ldots, v_k \in V$, by

$$\Delta(v_1 \otimes \cdots \otimes v_k) := \sum_{i=0}^{k} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_k) , \tag{A.14}$$

which means that the linear maps $\Delta_{i,j}$ are given by

$$\begin{aligned} \Delta_{i,j} : \quad V^{\otimes(i+j)} &\to V^{\otimes i} \otimes V^{\otimes j} \\ v_1 \otimes \cdots \otimes v_{i+j} &\mapsto (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_{i+j}) . \end{aligned} \tag{A.15}$$

Graded coassociativity of Δ is an immediate consequence of the associativity of $\otimes$:

$$\begin{aligned} &((\Delta_{i,j} \otimes \mathbb{1}_{V_{k-i-j}}) \circ \Delta_{i+j,k-i-j})(v_1 \otimes \cdots \otimes v_k) \\ &\quad = ((v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_j)) \otimes (v_{j+1} \otimes \cdots \otimes v_k) \\ &\quad = (v_1 \otimes \cdots \otimes v_i) \otimes ((v_{i+1} \otimes \cdots \otimes v_j) \otimes (v_{j+1} \otimes \cdots \otimes v_k)) \\ &\quad = ((\mathbb{1}_{V_i} \otimes \Delta_{j,k-i-j}) \circ \Delta_{i,k-i})(v_1 \otimes \cdots \otimes v_k) . \end{aligned}$$

We now turn to the natural coalgebra structure of the exterior algebra $\wedge^\bullet V$. Let us denote by δ the diagonal map $V \to V \times V : v \mapsto (v,v)$. By functoriality of $\wedge^\bullet$, it induces a linear map

$$\wedge^\bullet \delta : \wedge^\bullet V \to \wedge^\bullet (V \times V) ,$$

which we view as a linear map

$$\Delta := \rho \circ \wedge^\bullet \delta : \wedge^\bullet V \to \wedge^\bullet V \otimes \wedge^\bullet V ,$$

where $\rho : \wedge^\bullet(V \times V) \to \wedge^\bullet V \otimes \wedge^\bullet V$ is the natural isomorphism, given by

$$\rho\big((v_1,0)\wedge\cdots\wedge(v_i,0)\wedge(0,v_{i+1})\wedge\cdots\wedge(0,v_k)\big) := (v_1\wedge\cdots\wedge v_i)\otimes(v_{i+1}\wedge\cdots\wedge v_k)\ ,$$

where $v_1,\dots,v_k \in V$. We denote the natural product on $\wedge^\bullet V\otimes\wedge^\bullet V$ which makes ρ into a homomorphism of graded algebras by $\triangle$ (not to be confused with Δ). It follows easily from the graded commutativity of $\wedge$ that, if $v_i \in \wedge^{r_i}V$ for $i = 1,\dots,4$, then

$$(v_1\otimes v_2)\triangle(v_3\otimes v_4) = (-1)^{r_2 r_3}(v_1\wedge v_3)\otimes(v_2\wedge v_4)\ . \tag{A.16}$$

The commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \delta\ } & V\times V \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta\times\mathbb{1}_V} \\ V\times V & \xrightarrow[\ 1_V\times\delta\ ]{} & V\times V\times V \end{array}$$

leads to a commutative diagram

$$\begin{array}{ccc} \wedge^\bullet V & \xrightarrow{\ \wedge^\bullet\delta\ } & \wedge^\bullet(V\times V) \\ {\scriptstyle\wedge^\bullet\delta}\downarrow & & \downarrow{\scriptstyle\wedge^\bullet(\delta\times\mathbb{1}_V)} \\ \wedge^\bullet(V\times V) & \xrightarrow[\ \wedge^\bullet(1_V\times\delta)\ ]{} & \wedge^\bullet(V\times V\times V) \end{array}$$

which, in terms of Δ, becomes the coassociativity property of the coproduct Δ.

$$\begin{array}{ccc} \wedge^\bullet V & \xrightarrow{\ \Delta\ } & \wedge^\bullet V\otimes\wedge^\bullet V \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathbb{1}_{\wedge^\bullet V}} \\ \wedge^\bullet V\otimes\wedge^\bullet V & \xrightarrow[\ 1_{\wedge^\bullet V}\otimes\Delta\ ]{} & \wedge^\bullet V\otimes\wedge^\bullet V\otimes\wedge^\bullet V \end{array}$$

The graded algebra structure on $\wedge^\bullet V\otimes\wedge^\bullet V\otimes\wedge^\bullet V$ is defined as in (A.16). An explicit formula for Δ is given by

$$\begin{aligned} \Delta(v_1\wedge v_2\wedge\cdots\wedge v_k) &= \rho(\wedge^\bullet\delta(v_1\wedge\cdots\wedge v_k)) = \rho(\delta(v_1)\wedge\cdots\wedge\delta(v_k)) \\ &= \rho\big(((v_1,0)+(0,v_1))\wedge\cdots\wedge((v_k,0)+(0,v_k))\big) \\ &= (v_1\otimes 1 + 1\otimes v_1)\triangle\cdots\triangle(v_k\otimes\mathbb{1}_V + \mathbb{1}_V\otimes v_k) \end{aligned}$$

$$= \sum_{i+j=k} \sum_{\sigma \in S_{i,j}} \operatorname{sgn}(\sigma)(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \wedge \cdots \wedge v_{\sigma(k)}) .$$

In particular, $\Delta_{i,j}$ is given by

$$\Delta_{i,j}(v_1 \wedge \cdots \wedge v_{i+j}) = \sum_{\sigma \in S_{i,j}} \operatorname{sgn}(\sigma)(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \wedge \cdots \wedge v_{\sigma(i+j)}) , \tag{A.17}$$

when $i, j \in \mathbb{N}$ and $\Delta_{i,j} = 0$ when $i < 0$ or $j < 0$. The above formula is similar to the de-concatenation formula (A.14) which we have introduced in the case of the tensor algebra.

A.5 Graded Derivations and Coderivations

Let (V, μ) be a (not necessarily associative) graded algebra, where $V = \oplus_{i \in \mathbb{Z}} V_i$ is a graded R-module. A graded linear map $\phi \in \operatorname{Hom}_r(V,V)$ of degree r is said to be a *graded derivation* of degree r if

$$\phi(vw) = \phi(v)w + (-1)^{rp} v\phi(w) ,$$

for all $v \in V_p$ and $w \in V_q$, where vw is a shorthand for $\mu(v,w)$. In functional notation this means that for every $p, q \in \mathbb{Z}$,

$$\phi \circ \mu_{p,q} = \mu_{p+r,q} \circ (\phi \otimes \mathbb{1}_{V_q}) + (-1)^{rp} \mu_{p,r+q} \circ (\mathbb{1}_{V_p} \otimes \phi) , \tag{A.18}$$

as linear maps $V_p \otimes V_q \to V_{p+r+q}$. We denote the R-module of all graded derivations of degree r of V by $\operatorname{Der}_r(V)$. It is easily verified by direct computation that the graded commutator of two graded derivations of degrees r_1 and r_2 is a graded derivation of degree $r_1 + r_2$. This implies that $\oplus_{r \in \mathbb{Z}} \operatorname{Der}_r(V)$ is a graded Lie algebra, with the graded commutator as Lie bracket. It is a Lie subalgebra of the graded Lie algebra $\oplus_{r \in \mathbb{Z}} \operatorname{Hom}_r(V,V)$, equipped with the graded commutator, which we considered in Section A.3. Clearly, a derivation is completely determined by its values on an arbitrary set of elements of V, which generates V as a (graded) algebra.

Example A.3. Let V be an R-module and consider the graded algebra $\wedge^\bullet V$. Equipped with the graded commutator, the R-module $\oplus_{r \in \mathbb{Z}} \operatorname{Der}_r(\wedge^\bullet V)$ is a graded Lie algebra. Particular elements of $\operatorname{Der}_r(\wedge^\bullet V)$ can be constructed from linear maps $V \to \wedge^{r+1} V$: every linear map

$$\phi : V \to \wedge^{r+1} V$$

extends to a derivation of degree r of the graded algebra $(\wedge^\bullet V, \wedge)$ by putting

$$\begin{aligned} \tilde{\phi} : \quad \wedge^\bullet V &\to \wedge^{\bullet + r} V \\ v_1 \wedge \cdots \wedge v_k &\mapsto \sum_{i=1}^{k} (-1)^{i-1} \phi(v_i) \wedge v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k . \end{aligned}$$

Indeed, for $X \in \wedge^p V$ and $Y \in \wedge^q V$,

$$\tilde{\phi}(X \wedge Y) = \tilde{\phi}(X) \wedge Y + (-1)^{pr} X \wedge \tilde{\phi}(Y),$$

which is an easy consequence of the fact that ϕ will either be applied to a factor which appears in X, leaving Y untouched, or vice versa.

We now formulate the notion of a derivation for the case of a graded coalgebra. As before, (V, Δ) is a graded coalgebra, where $V = \oplus_{i\in\mathbb{Z}} V_i$ is a graded R-module. Dualizing (A.18), a linear map $\phi : V \to V$ of degree $-r$ is called a *graded coderivation* of degree r if for every $p, q \in \mathbb{Z}$,

$$\Delta_{p,q} \circ \phi = (\phi \otimes \mathbb{1}_{V_q}) \circ \Delta_{p+r,q} + (-1)^{rp} (\mathbb{1}_{V_p} \otimes \phi) \circ \Delta_{p,r+q} ,$$

as maps from V_{p+r+q} to $V_p \otimes V_q$. We denote the R-module of all graded coderivations of degree r of V by $\mathrm{CoDer}_r(V)$. Again, it follows by direct computation that the graded commutator of two graded coderivations of degrees r_1 and r_2 is a graded coderivation of degree $r_1 + r_2$. This implies that $\oplus_{r\in\mathbb{Z}} \mathrm{CoDer}_r(V)$ is also a graded Lie algebra, with the graded commutator as Lie bracket. Like $\oplus_{r\in\mathbb{Z}} \mathrm{Der}_r(V)$, it is a Lie subalgebra of the graded Lie algebra $\oplus_{r\in\mathbb{Z}} \mathrm{Hom}_r(V,V)$, equipped with the graded commutator. The following example is the "co"-version of Example A.3.

Example A.4. Let V be an R-module and consider the graded coalgebra $\wedge^\bullet V$. Equipped with the graded commutator, the R-module $\oplus_{r\in\mathbb{Z}} \mathrm{CoDer}_r(\wedge^\bullet V)$ is a graded Lie algebra. Particular elements of $\mathrm{CoDer}_r(\wedge^\bullet V)$ can be constructed from linear maps $\wedge^{r+1} V \to V$: every linear map

$$\phi : \wedge^{r+1} V \to V$$

extends to a coderivation of degree r of the graded R-coalgebra $(\wedge^\bullet V, \Delta)$ by putting

$$\begin{array}{rccl} \tilde{\phi} : & \wedge^\bullet V & \to & \wedge^{\bullet - r} V \\ & v_1 \wedge \cdots \wedge v_k & \mapsto & \displaystyle\sum_{\tau \in S_{r+1,k-1-r}} \mathrm{sgn}(\tau)\, \phi(v_{\tau(1)}, \ldots, v_{\tau(r+1)}) \wedge v_{\tau(r+2)} \wedge \cdots \wedge v_{\tau(k)} \ . \end{array}$$

It is understood that the above definition means that $\tilde{\phi} = 0$ on $\wedge^k V$ for $k \leqslant r$. It follows that, for $p, q \in \mathbb{N}$ we have that

$$\Delta_{p,q} \circ \tilde{\phi} = (\tilde{\phi} \otimes \mathbb{1}_{\wedge^q V}) \circ \Delta_{p+r,q} + (-1)^{rp} (\mathbb{1}_{\wedge^p V} \otimes \tilde{\phi}) \circ \Delta_{p,r+q} \ . \tag{A.19}$$

All coderivations of $\wedge^\bullet V$ are obtained in this way: if $\Phi : \wedge^\bullet V \to \wedge^{\bullet - r} V$ is a coderivation of degree r of $(\wedge^\bullet V, \Delta)$, then $\Phi = \tilde{\phi}$, where $\phi : \wedge^{r+1} V \to V$ is the restriction of Φ to $\wedge^{r+1} V$. Indeed, since Φ and $\tilde{\phi}$ agree on $\wedge^k V$, for $k \leqslant r+1$ (they are both zero when $k \leqslant r$, for degree reasons), they also agree, in view of (A.19), on $\wedge^{r+2} V$ (take $p = q = 1$ in (A.19)), and similarly for the higher exterior powers of V.

Appendix B
Real and Complex Differential Geometry

In this appendix we recall the basic notions of differential geometry: the definition of a real manifold, of a complex manifold and of a vector field on such a manifold. We also recall briefly the main properties of vector fields on manifolds: the existence of integral curves of a vector field, the flow of a vector field, the bracket of vector fields and the straightening theorem, which says that a vector field takes, in well-chosen coordinates, a simple form. Our definition of vector fields on a manifold is based on the concept of a pointwise derivation. This approach easily generalizes to the introduction of the concept of a bivector field on a manifold, a crucial element in the (geometrical!) definition of the notion of a Poisson structure on a (real or complex) manifold (see Section 1.3).

B.1 Real and Complex Manifolds

We adopt the following geometric point of view: a differentiable manifold is a (second countable, Hausdorff) topological space which is covered by a family of *coordinate charts* (U,x), where U is an open subset of M, called the *domain* of the chart, and $x=(x_1,\dots,x_d)$ is a homeomorphism from U to an open subset of $\mathbb{R}^d$. The functions $x_1,\dots,x_d$ are called *local coordinates*; they are said to be *centered* at m if $x(m)=o$, where o stands for the origin of $\mathbb{R}^d$. The coordinate charts are demanded to be compatible in the sense that, if (U,x) and (V,y) are coordinate charts, with $U\cap V\neq\emptyset$, then the homeomorphism

$$y\circ x^{-1}\big|_{x(U\cap V)} : x(U\cap V)\to y(U\cap V)$$

is a smooth map, see Fig. B.1. This homeomorphism is called a *transition map*. A collection of compatible coordinate charts of M, whose domains cover M, is called an *atlas* of M. For connected differentiable manifolds, the integer d is independent of the coordinate chart; it is called the *dimension* of M, a terminology which we also use in the non-connected case, when d is independent of the coordinate chart; it is

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Grundlehren der mathematischen Wissenschaften 347,
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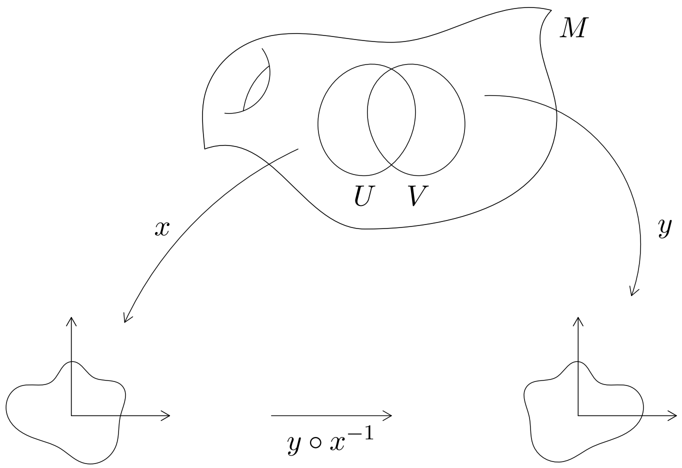

Fig. B.1 A manifold comes equipped with an atlas, a collection of coordinate charts (U,x), where the transition maps $y \circ x^{-1}$ between coordinate charts (U,x) and (V,y) (with $U \cap V \neq \emptyset$) are demanded to be smooth.

denoted by $\dim M$. When the integer d is even, we may interpret the homeomorphisms x as taking values in $\mathbb{C}^{d/2}$; in this case, if all transition maps are complex analytic (holomorphic), then M is called a *complex manifold* and $d/2$ is called the (complex) *dimension* of M. It is a trivial, but important, fact that every non-empty open subset of a real or complex manifold is itself, in a natural way, a real or complex manifold. In particular, every open subset of a (real or complex) vector space is a (real or complex) manifold.

The main virtue of manifolds is that we can do calculus on them, hence also analytic geometry. Roughly speaking, the coordinate charts allow us to identify objects on the manifold, locally, with standard objects on open subsets of $\mathbb{F}^d$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and the transition maps allow us to compare these standard objects. The first object one thinks of is that of a smooth function: a function $F : M \to \mathbb{F}$ on a real (respectively complex) manifold M is called a *smooth function* (respectively a *holomorphic function*) if for every coordinate chart (U,x) of M the function

$$\tilde{F} = F \circ x^{-1} : x(U) \to \mathbb{F}$$

is smooth (respectively holomorphic). The function $\tilde{F}$ is called the *coordinate expression* of F in the coordinate chart (U,x). See Fig. B.2. The (commutative associative) algebra of all such functions F on M will be denoted by $\mathscr{F}(M)$. For an open subset U, viewing U itself as a manifold, we have an algebra of functions $\mathscr{F}(U)$ and there are obvious restriction maps $\mathscr{F}(M) \to \mathscr{F}(U)$, which are in general neither injective nor surjective; the restriction of $F \in \mathscr{F}(M)$ to U will be denoted by $F_{|_U}$. Notice that $\mathscr{F}(M)$ may consist only of the constant functions, for example when M is a compact complex manifold.

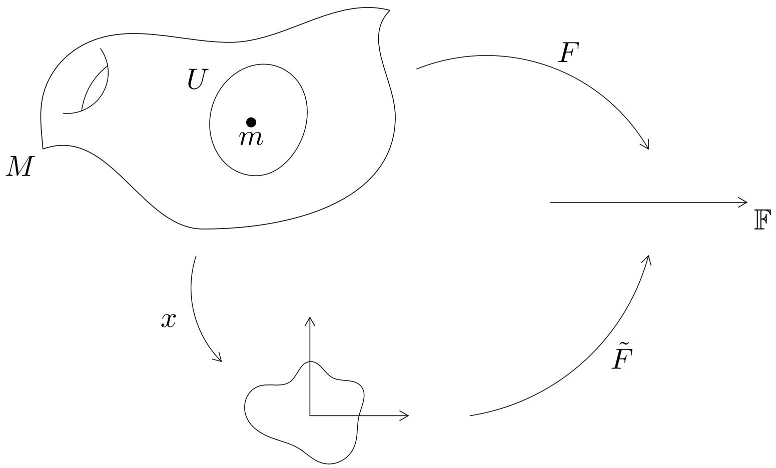

Fig. B.2 Smooth functions on a manifold M are functions on M which are smooth in terms of local coordinates.

One similarly defines the notion of a *smooth map* between real manifolds and a *holomorphic map* between complex manifolds. It leads to two categories: the category of real manifolds, whose objects are real manifolds with smooth maps as morphisms, and the category of complex manifolds, whose objects are complex manifolds and whose morphisms are holomorphic maps. Since complex manifolds are in a natural way also real manifolds, and since holomorphic maps are smooth, there is a natural forgetful functor from the latter category to the former.

B.2 The Tangent Space

Let M be a manifold and let $m \in M$ be an arbitrary point. We define the tangent space T_mM of M at m. To do this, we consider the set $\mathscr{F}_m(M)$ of all pairs (F,U), where U is an open subset of M which contains m, and F is an element of $\mathscr{F}(U)$. Two elements (F,U) and (G,V) of $\mathscr{F}_m(M)$ are defined to be equivalent, denoted $(F,U) \sim (G,V)$, if there exists a pair $(H,W) \in \mathscr{F}_m(M)$, such that $W \subset U \cap V$ and $H = F|_W = G|_W$. For $(F,U) \in \mathscr{F}_m(M)$, we denote its equivalence class by F_m and we call it the *germ* of F (or of (F,U)) at m. It is clear that the quotient set $\mathscr{F}_m(M)/\sim$ of all function germs at m inherits from $\mathscr{F}(M)$ the structure of an associative $\mathbb{F}$-algebra. For example, in the complex case, the algebra $\mathscr{F}_m(M)/\sim$ is isomorphic to the algebra of power series in d variables, whose radius of convergence is positive. Notice that a function germ F_m has a well-defined value at m, which is simply $F(m)$.

Definition B.1. Let M be a manifold and let $m \in M$. A *pointwise derivation* δ_m of $\mathscr{F}(M)$ at m is a linear function

$$\delta_m : \frac{\mathscr{F}_m(M)}{\sim} \to \mathbb{F} \,,$$

satisfying, for all functions F and G, defined on a neighborhood of m in M,

$$\delta_m(F_m G_m) = F(m)\,\delta_m G_m + G(m)\,\delta_m F_m \,. \tag{B.1}$$

The vector space of all pointwise derivations of $\mathscr{F}(M)$ at m is denoted by $T_m M$, and is called the *tangent space* of M at m, while the dual space $T_m^* M$ of linear forms $T_m M \to \mathbb{F}$ is called the *cotangent space* of M at m. The canonical pairing between the dual vector spaces $T_m M$ and $T_m^* M$, which amounts to evaluating elements of $T_m^* M$ on elements of $T_m M$, is denoted by $\langle \cdot , \cdot \rangle$.

One easily deduces from (B.1) that if δ_m is a pointwise derivation at m, and F is constant in a neighborhood of m, then $\delta_m F_m = 0$. For a given function F, defined on a neighborhood of m in M, consider the function $\mathrm{d}_m F$ on $T_m M$, defined by

$$\begin{aligned} \mathrm{d}_m F \,:\, T_m M &\to \mathbb{F} \\ \delta_m &\mapsto \delta_m F_m . \end{aligned} \tag{B.2}$$

Clearly, $\mathrm{d}_m F$ is a linear function, hence it is an element of the cotangent space $T_m^* M$. It is called the *differential* of F at m. The differential of a function admits a natural generalization to the case of maps between manifolds. Let M and N be two manifolds and let Ψ be a map, defined on a neighborhood of a point $m \in M$, with values in N. The linear map $T_m \Psi : T_m M \to T_{\Psi(m)} N$, called the *tangent map* of Ψ at m, associates to a pointwise derivation δ_m of $\mathscr{F}(M)$ at m, the pointwise derivation $T_m \Psi(\delta_m)$ of $\mathscr{F}(N)$ at $\Psi(m)$, defined for every germ $G_{\Psi(m)}$ at $\Psi(m)$ by

$$(T_m \Psi)(\delta_m)\, G_{\Psi(m)} := \delta_m (G \circ \Psi)_m \,,$$

see Fig. B.3. This is well-defined, because the germ $(G \circ \Psi)_m$ is independent of the function G which represents the germ $G_{\Psi(m)}$. When F is a function, defined on a neighborhood of m, the tangent map at m is a linear map $T_m F : T_m M \to T_{F(m)} \mathbb{F}$, and we can recover the differential $\mathrm{d}_m F$, upon composing $T_m F$ with the canonical isomorphism $T_{F(m)} \mathbb{F} \simeq \mathbb{F}$, which will be explained below.

The tangent map obeys the usual rules of calculus: for example, if M, N and P are manifolds, $\Psi : M \to N$ and $\Xi : N \to P$ are maps and $m \in M$, then

$$T_m(\Xi \circ \Psi) = (T_{\Psi(m)} \Xi) \circ T_m \Psi \,.$$

In particular, if Ψ is a diffeomorphism (at least in the neighborhood of m), then $T_m \Psi$ is invertible and

$$(T_m \Psi)^{-1} = T_{\Psi(m)} \Psi^{-1} \,.$$

Consider the vector space $\mathbb{F}^d$, viewed as a d-dimensional manifold, and let m be a point of $\mathbb{F}^d$. There is a natural isomorphism between the vector spaces $T_m \mathbb{F}^d$ and $\mathbb{F}^d$. Namely to a vector $v \in \mathbb{F}^d$ we can associate a pointwise derivation v_m at m by setting,

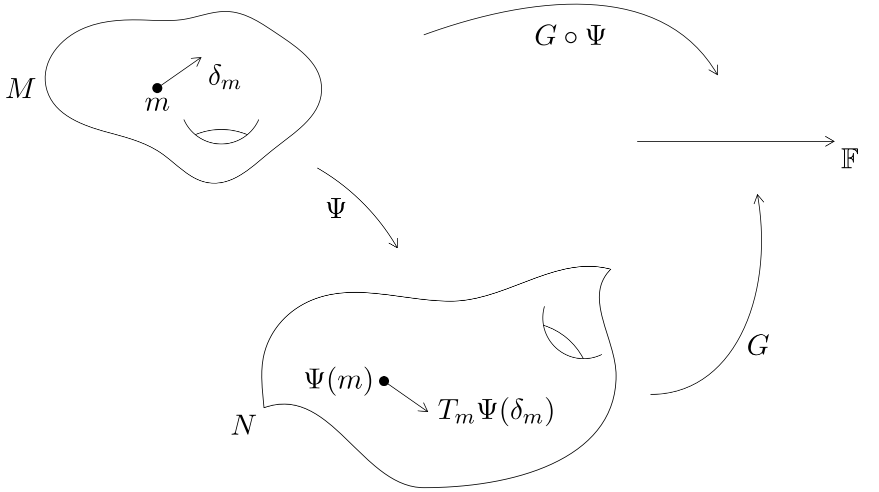

Fig. B.3 A map Ψ between two manifolds M and N leads for every point $m \in M$ to a tangent map $T_m\Psi$, which is a linear map between the tangent space T_mM and $T_{\Psi(m)}N$.

for every germ F_m at m,

$$v_m F_m := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F(m+tv) = \lim_{t\to 0} \frac{F(m+tv)-F(m)}{t} .$$

It is clear that $v \mapsto v_m$ defines an injective linear map from $\mathbb{F}^d$ to $T_m\mathbb{F}^d$. The surjectivity of this map follows from the Hadamard lemma.

Lemma B.2 (Hadamard's lemma). *Let $F \in \mathscr{F}(U)$, where U is an open subset of $\mathbb{F}^d$ and let $m \in U$. On a small neighborhood $V \subset U$ of m in $\mathbb{F}^d$,*

$$F = F(m) + \sum_{i=1}^{d} (x_i - x_i(m))\, F^{(i)} , \tag{B.3}$$

where each of the functions $F^{(i)}$ belongs to $\mathscr{F}(V)$. In particular,

$$F^{(i)}(m) = (e_i)_m F_m ,$$

where $(e_1, \ldots, e_d)$ denotes the natural basis of $\mathbb{F}^d$.

For a proof of this lemma, which is essentially a first-order form of Taylor's theorem, see [155, p. 17].

Let $F \in \mathscr{F}(U)$ and consider (B.3), germified at $m \in U$,

$$F_m = F(m) + \sum_{i=1}^{d} ((x_i)_m - x_i(m)) F_m^{(i)} . \tag{B.4}$$

Let δ_m be a pointwise derivation at m. Applying δ_m to (B.4) we find, using (B.1), that

$$\delta_m F_m = \sum_{i=1}^{d} \delta_m (x_i)_m F^{(i)}(m) = \sum_{i=1}^{d} \delta_m (x_i)_m (e_i)_m F_m ,$$

for all germs F_m at m, so that

$$\delta_m = \sum_{i=1}^{d} \delta_m (x_i)_m (e_i)_m . \tag{B.5}$$

This shows that $T_m\mathbb{F}^d$ is spanned by the pointwise derivations $(e_i)_m$, so that the map, defined by $v \mapsto v_m$, is an isomorphism between $\mathbb{F}^d$ and $T_m\mathbb{F}^d$.

If (U,x) is a coordinate chart of a manifold M, centered at m, then each of the d vectors of the natural basis $(e_1,\ldots,e_d)$ of $\mathbb{F}^d$ leads to a pointwise derivation of $\mathscr{F}(M)$ at m, defined by

$$\left(\frac{\partial}{\partial x_i}\right)_m := \left(T_o x^{-1}\right)(e_i)_o .$$

For a function F, defined on an open neighborhood of m in M, this means that

$$\left(\frac{\partial}{\partial x_i}\right)_m F_m = \partial_i \tilde{F}(x(m)) ,$$

where $\tilde{F} : x(U) \to \mathbb{F}$ is the coordinate expression of a representative $F \in \mathscr{F}(U)$ of the germ F_m (see Fig. B.2), and $1 \leqslant i \leqslant d$; also, $\partial_i \tilde{F}$ denotes the derivative of $\tilde{F}$ with respect to its i-th variable (real or complex). Since $T_o\mathbb{F}^d$ is spanned by the pointwise derivations $(e_1)_o,\ldots,(e_d)_o$ and since $T_o x^{-1}$ is an isomorphism, the pointwise derivations $\left(\frac{\partial}{\partial x_1}\right)_m,\ldots,\left(\frac{\partial}{\partial x_d}\right)_m$ span T_mM.

Remark B.3. When the tangent space of M at m is viewed as an equivalence class of curves, passing through m, as is done in a more analytic approach to elementary differential geometry, then the differential d_mF of a function $F : M \to \mathbb{F}$ at m, is

$$\mathrm{d}_m F(\tilde{\gamma}_m) := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} F(\gamma(t))$$

where $\gamma : I \to M$ is a curve, defined on a neighborhood I of 0 in $\mathbb{F}$, with $\gamma(0) = m$, whose equivalence class has been denoted by $\tilde{\gamma}_m$. As we have seen, in our approach to defining the tangent space, the definition of the differential takes the more algebraic form

$$\langle \mathrm{d}_m F, \delta_m \rangle := \delta_m F_m ,$$

for all $\delta_m \in T_mM$. Thus, in our setup, we view tangent vectors as objects which act on equivalence classes of functions, rather than viewing functions as objects which define linear forms on equivalence classes of curves, although both points of view are equivalent (see [198, Ch. 1]).

B.3 Vector Fields

Let us consider a manifold M and a map $\mathscr{V}$, which assigns to every $m \in M$ an element $\mathscr{V}_m$ of T_mM. To each function $F \in \mathscr{F}(U)$, where U is an open subset of M, we can associate a function $\mathscr{V}[F]$ on U by defining, for all $m \in U$,

$$\mathscr{V}[F](m) := \mathscr{V}_m F_m \in \mathbb{F} . \tag{B.6}$$

We also write $\mathscr{V}_m[F]$ for $\mathscr{V}_m F_m$, so that $\mathscr{V}_m[F] = \mathscr{V}[F](m)$. We say that $\mathscr{V}$ is a *smooth vector field* (respectively *holomorphic vector field*) on M if for every open subset $U \subset M$ and for every function $F \in \mathscr{F}(U)$, the function $\mathscr{V}[F]$, defined by (B.6), belongs to $\mathscr{F}(U)$ (i.e., it is a smooth, respectively holomorphic function on U). When the type of manifold which is considered is irrelevant or is clear from the context, we simply say *vector field* for smooth or holomorphic vector field. Notice that we use square brackets to denote the action of a vector field on a function.

With respect to pointwise multiplication, the vector fields on M form an $\mathscr{F}(M)$-module, which is denoted by $\mathfrak{X}^1(M)$. Viewed as a vector space, $\mathfrak{X}^1(M)$ is a Lie algebra, where the Lie bracket is the commutator of vector fields, defined as follows. Let $\mathscr{V}$ and $\mathscr{W}$ be vector fields on M, and let $m \in M$. For every function F, defined in a neighborhood of m, letting

$$[\mathscr{V}, \mathscr{W}]_m (F_m) := \mathscr{V}_m (\mathscr{W}[F])_m - \mathscr{W}_m (\mathscr{V}[F])_m ,$$

leads to a well-defined linear map $[\mathscr{V}, \mathscr{W}]_m : \mathscr{F}_m(M)/\sim \to \mathbb{F}$, which is easily shown to be a pointwise derivation at m. For given vector fields $\mathscr{V}$ and $\mathscr{W}$ on M, the map which assigns to every $m \in M$ the element $[\mathscr{V}, \mathscr{W}]_m$ of the tangent space T_mM is a (smooth or holomorphic) vector field, hence we have a map $[\cdot,\cdot] : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \to \mathfrak{X}^1(M)$. Clearly, $[\cdot,\cdot]$ is a skew-symmetric bilinear map, which satisfies the Jacobi identity, hence it defines a Lie algebra structure on $\mathfrak{X}^1(M)$; it is called the *Lie bracket* on vector fields.

We have seen that a map $\Psi : M \to N$ leads for every $m \in M$ to a linear map $T_m\Psi : T_mM \to T_{\Psi(m)}N$. However, since Ψ is in general neither injective nor surjective, this collection of linear maps cannot be used to associate to a vector field $\mathscr{V}$ on M, a vector field on N. Nevertheless, when Ψ is bijective, so that Ψ is a diffeomorphism (or biholomorphism), we get a vector field $\Psi_*\mathscr{V}$ on N by setting

$$(\Psi_*\mathscr{V})_{\Psi(m)} := (T_m\Psi)\mathscr{V}_m ,$$

for all $m \in M$. The vector field $\Psi_*\mathscr{V}$ is called the *pushforward* of $\mathscr{V}$ by Ψ.

It is clear that vector fields can be restricted to open subsets; we usually do not make a notational distinction between a vector field on M and its restriction to some open subset of M. It is also clear from (B.6) that $\mathscr{V}$ defines, for every open subset U of M, a derivation of $\mathscr{F}(U)$, i.e., we have

$$\mathscr{V}[FG] = F\,\mathscr{V}[G] + G\,\mathscr{V}[F]\,,$$

for all $F, G \in \mathscr{F}(U)$. In particular, a vector field on M defines a derivation of $\mathscr{F}(M)$.

Remark B.4. It is shown in standard books on differential geometry that for a *real manifold* M, the above natural correspondence between (smooth) vector fields on M and derivations of $\mathscr{F}(M)$ is bijective. For complex manifolds however, this is not true in general: think of a compact complex torus $\mathbb{C}^d/\mathbb{Z}^{2d}$, which has non-trivial holomorphic vector fields, but whose algebra of holomorphic functions consists of constant functions only, so that all its derivations are trivial. The same phenomenon occurs for skew-symmetric biderivations and bivector fields (e.g., Poisson structures), introduced in Chapter 1.

Remark B.5. The set of all tangent vectors at m, for m ranging through M, has a natural vector bundle structure over M, denoted $TM \to M$. The fiber over m is the vector space T_mM and the vector fields on M can be defined as the (smooth, holomorphic) sections of $TM \to M$. In abstract geometrical constructions, it is the latter point of view on vector fields which is often the most appropriate.

On a coordinate chart (U,x) of M, there are d distinguished vector fields $\partial/\partial x_i$, $i = 1,\dots,d$, which are defined by $m \mapsto \left(\frac{\partial}{\partial x_i}\right)_m$, for all $m \in U$, which amounts to defining

$$\frac{\partial}{\partial x_i}[F](m) := \partial_i \tilde{F}(x(m))\,, \tag{B.7}$$

for all $F \in \mathscr{F}(M)$, where $\tilde{F} : x(U) \to \mathbb{F}$ is the coordinate expression of F, as in Fig. B.2. By a slight abuse of notation, we usually write $\frac{\partial F}{\partial x_i}(m)$ instead of either expression in (B.7). If $\mathscr{V}$ is a map which assigns to every $m \in U$ an element $\mathscr{V}_m \in T_mM$, then $\mathscr{V}$ can be written in a unique way as $\mathscr{V} = \sum_{i=1}^d \mathscr{V}^{(i)}\,\partial/\partial x_i$, since the pointwise derivations $\left(\frac{\partial}{\partial x_i}\right)_m$ form a basis of T_mM, for every $m \in U$. Then $\mathscr{V}$ is a (smooth) vector field on U if and only if the coefficients $\mathscr{V}^{(i)}$ in this expression belong to $\mathscr{F}(U)$. It is clear that these coefficients $\mathscr{V}^{(i)}$ are given by $\mathscr{V}^{(i)} = \mathscr{V}[x_i]$. By a slight abuse of language, we often refer to the expression

$$\mathscr{V} = \sum_{i=1}^d \mathscr{V}[x_i]\frac{\partial}{\partial x_i} \tag{B.8}$$

as a *coordinate expression* of $\mathscr{V}$ in the coordinate chart (U,x). It is very useful for explicit computations, as it allows us to compute with vector fields on a manifold, locally, in the same way as on $\mathbb{F}^d$.

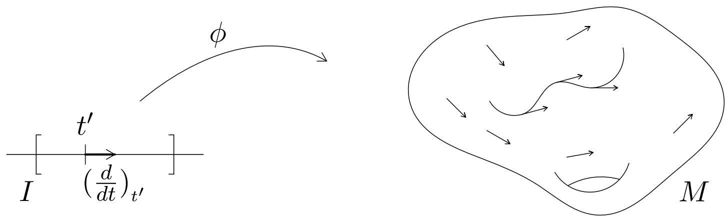

Fig. B.4 For a given vector field on a manifold, there passes through every point of the manifold a unique integral curve.

B.4 The Flow of a Vector Field

Let $\mathscr{V}$ be a vector field on a manifold M and let $m \in M$. The fundamental theorem on the existence and uniqueness of solutions of first order ordinary differential equations with initial conditions, tells us that there exists a connected open neighborhood I of 0 in $\mathbb{F}$, and there exists a map $\phi : I \to M$, such that $\phi(0) = m$ and such that

$$(T_{t'}\phi)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t'} = \mathscr{V}_{\phi(t')} \,, \tag{B.9}$$

for all $t' \in I$; the map is unique in the sense that if $\phi_1 : I_1 \to M$ and $\phi_2 : I_2 \to M$ are two such maps, then they coincide on $(I_1 \cap I_2)^0$, the connected component of their common intersection, which contains 0,

$$\phi_1|_{(I_1\cap I_2)^0} = \phi_2|_{(I_1\cap I_2)^0} \,.$$

The map ϕ or the pair (I, ϕ) is called an *integral curve* of $\mathscr{V}$, passing through m, see Fig. B.4. By a slight abuse of notation, the left-hand side in (B.9) is often denoted by $\frac{\mathrm{d}\phi}{\mathrm{d}t}(t')$; using this notation (B.9) takes the more familiar form

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}(t') = \mathscr{V}_{\phi(t')} \,.$$

The integral curves of a vector field depend smoothly on the initial data; this is stated in a precise way in the following theorem (see [187, Ch. 5] for a proof).

Theorem B.6. *Let $\mathscr{V}$ be a vector field on a manifold M and let $m \in M$. There exists a neighborhood U of $(0,m)$ in $\mathbb{F} \times M$ and there exists a map $\Phi : U \to M$ such that, for every $(0,m') \in U$, the restriction*

$$\Phi|_{I_{m'}} : I_{m'} \to M \,, \tag{B.10}$$

is an integral curve of $\mathscr{V}$, passing through m'; the subset $I_{m'}$ in (B.10) is the connected component of $U \cap (\mathbb{F} \times \{m'\})$ which contains $(0,m')$. Moreover, Φ has the following property: for every (t',m) in the connected component of $(0,m)$ in

$U \cap (\mathbb{F} \times \{m\})$, there exists a neighborhood $U_{t'}$ of (t', m) in U, such that the restriction

$$\Phi|_{U_{t'}} : U_{t'} \to M ,$$

is a diffeomorphism (biholomorphism) between $U_{t'}$ and its image $\Phi(U_{t'})$.

We refer to such a map Φ as being the *(local) flow* of the vector field $\mathscr{V}$ in a neighborhood of m. The diffeomorphism $\Phi|_{U_{t'}}$ is called the local flow at t' and is usually denoted by $\Phi_{t'}$ (omitting its domain of definition, which is all of M in good cases, for example when M is a compact real manifold).

An important and useful consequence of the (local) existence of the flow of a vector field, is the straightening theorem, which says that the coordinate expression of a vector field on a manifold takes a particularly simple form, at points where the vector field does not vanish.

Theorem B.7 (Straightening theorem). *Let M be a manifold and let $\mathscr{V}$ be a vector field on M. If $m \in M$ is such that $\mathscr{V}(m) \neq 0$, then there exist local coordinates $x_1, \ldots, x_d$ on a neighborhood U of m, such that $\mathscr{V} = \partial / \partial x_1$ on U.*

B.5 The Frobenius Theorem

Instead of having a vector at every point of a manifold M, as is the case of a vector field on M, one may have a one-dimensional subspace of the tangent space to M, at every point of M. This is what is called a 1-dimensional *distribution* on M; a k-dimensional distribution $\mathscr{D}$ on M is then the datum of a k-dimensional subspace $\mathscr{D}(m)$ of T_mM for every $m \in M$. One says that $\mathscr{D}$ is *smooth* (or *holomorphic*) if there exist for every $m \in M$ smooth (or holomorphic) vector fields $\mathscr{V}_1, \ldots, \mathscr{V}_k$, on a neighborhood U of m, such that

$$\mathscr{D}(m) = \operatorname{span}\{(\mathscr{V}_1)_m, \ldots, (\mathscr{V}_k)_m\} ,$$

for every $m \in U$. When it is clear from the context, we often simply say *distribution* for smooth (or holomorphic) distribution. A vector field $\mathscr{V}$, defined on an open subset U of M, is said to be *adapted* to $\mathscr{D}$ on U if $\mathscr{V}_m \in \mathscr{D}(m)$ for every $m \in U$. A distribution $\mathscr{D}$ on M is said to be *involutive* if for every open subset U of M, and for every pair of vector fields on U, which are adapted to $\mathscr{D}$ on U, their Lie bracket is also adapted to $\mathscr{D}$ on U. Involutivity of a distribution is a very strong condition, as is plain from Frobenius' theorem.

Theorem B.8 (Frobenius' theorem). *Let M be a d-dimensional manifold and suppose that $\mathscr{D}$ is a k-dimensional distribution on M. If $\mathscr{D}$ is involutive, then every point $m \in M$ admits a coordinate chart (U, x), with $m \in U$, such that*

$$\mathscr{D}(m') = \operatorname{span}\left\{\left(\frac{\partial}{\partial x_1}\right)_{m'}, \ldots, \left(\frac{\partial}{\partial x_k}\right)_{m'}\right\} , \qquad \textit{for every } m' \in U .$$

For a short and elementary proof, which is immediately adapted to the holomorphic case, we refer to [41] or [138]. It is clear that Frobenius' theorem can be seen as a generalization of the straightening theorem (Theorem B.7).

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