Appendix A
Linear algebra – basics

In this appendix we collect and complete well-known facts concerning projectors and subspaces of $\mathbb{R}^m$ (Section A.1), and generalized inverses (Section A.2). Section A.3 provides material on matrix and projector valued functions with proofs, since these proofs are not easily available. In Section A.4 we introduce $\mathcal{C}^k$-subspaces of $\mathbb{R}^m$ via $\mathcal{C}^k$-projector functions. We show $\mathcal{C}^k$-subspaces to be those which have local $\mathcal{C}^k$ bases.

A.1 Projectors and subspaces

We collect some basic and useful properties of projectors and subspaces.

Definition A.1. (1) A linear mapping $Q \in L(\mathbb{R}^m)$ is called a projector if $Q^2 = Q$.
(2) A projector $Q \in L(\mathbb{R}^m)$ is called a projector onto $S \subseteq \mathbb{R}^m$ if $\operatorname{im} Q = S$.
(3) A projector $Q \in L(\mathbb{R}^m)$ is called a projector along $S \subseteq \mathbb{R}^m$ if $\ker Q = S$.
(4) A projector $Q \in L(\mathbb{R}^m)$ is called an orthogonal projector if $Q = Q^*$.

Example A.2. The m-dimensional matrix $Q = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{bmatrix}$ with arbitrary entries for $*$ becomes a projector onto the one-dimensional subspace spanned by the first column of Q along the $(m-1)$-dimensional subspace $\left\{ v : v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, v_1 = 0 \right\}$. □

Lemma A.3. *Let P and $\bar{P}$ be projectors, and $Q := I - P$, $\bar{Q} := I - \bar{P}$ the complementary projectors. Then the following properties hold:*

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(1) $z \in \operatorname{im} Q \;\Leftrightarrow\; z = Qz.$
(2) *If Q and $\bar{Q}$ project onto the same subspace S, then $\bar{Q} = Q\bar{Q}$ and $Q = \bar{Q}Q$ are valid.*
(3) *If P and $\bar{P}$ project along the same subspace S, then $\bar{P} = \bar{P}P$ and $P = P\bar{P}$ are true.*
(4) *Q projects onto S iff $P := I - Q$ projects along S.*
(5) *Each matrix of the form $I + PZQ$, with arbitrary matrix Z, is nonsingular and its inverse is $I - PZQ$.*
(6) *Each projector P is diagonalizable. Its eigenvalues are 0 and 1. The multiplicity of the eigenvalue 1 is $r = \operatorname{rank} P$.*

Proof. (1) $z = Qy \quad \rightarrow \quad Qz = Q^2 y = Qy = z.$
(2) $\bar{Q}z \in \operatorname{im} \bar{Q} = S = \operatorname{im} Q$, also $\bar{Q}z = Q\bar{Q}z \; \forall z$.
(3) $\bar{P}P = (I - \bar{Q})(I - Q) = I - \bar{Q} - Q + \bar{Q}Q = I - \bar{Q} = \bar{P}.$
(4) $P^2 = P \;\Leftrightarrow\; (I - Q)^2 = I - Q \;\Leftrightarrow\; -Q + Q^2 = 0 \;\Leftrightarrow\; Q^2 = Q$ and $z \in \ker P \;\Leftrightarrow\; Pz = 0 \Leftrightarrow z = Qz \;\Leftrightarrow\; z \in \operatorname{im} Q.$
(5) Multiplying $(I + PZQ)z = 0$ by $Q \Rightarrow Qz = 0$. Now with $(I + PZQ)z = 0$ follows $z = 0$.
$(I + PZQ)(I - PZQ) = I - PZQ + PZQ = I.$
(6) Let $\bar{P}_1$ be a matrix of the r linearly independent columns of P and $\bar{Q}_2$ a matrix of the $m - r$ linearly independent columns of $I - P$. Then by construction $P\begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix} = \begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix}\begin{bmatrix} I & \\ & 0\end{bmatrix}$. Because of the nonsingularity of $\begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix}$ we have the structure $P = \begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix}\begin{bmatrix} I & \\ & 0\end{bmatrix}\begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix}^{-1}$. The columns of $\bar{P}_1$, respectively $\bar{Q}_2$ are the eigenvectors to the eigenvalues 1, respectively 0. □

Lemma A.4. *Let $A \in L(\mathbb{R}^n, \mathbb{R}^k)$, $D \in L(\mathbb{R}^m, \mathbb{R}^n)$ be given, and $r := \operatorname{rank}(AD)$. Then the following two implications are valid:*

(1) $\ker A \cap \operatorname{im} D = 0$, $\operatorname{im}(AD) = \operatorname{im} A \Rightarrow \ker A \oplus \operatorname{im} D = \mathbb{R}^n$.
(2) $\ker A \oplus \operatorname{im} D = \mathbb{R}^n \Rightarrow$

- $\ker A \cap \operatorname{im} D = \{0\}$,
- $\operatorname{im} AD = \operatorname{im} A$,
- $\ker AD = \ker D$,
- $\operatorname{rank} A = \operatorname{rank} D = r$.

Proof. (1) Because of $\operatorname{im}(AD) = \operatorname{im} A$, the matrix A has rank r and $\ker A$ has dimension $n - r$. Moreover, $\operatorname{rank} D \geq r$ must be true. The direct sum $\ker A \oplus \operatorname{im} D$ is well-defined, and it has dimension $n - r + \operatorname{rank} D \leq n$. This means that D has rank r. We are done with (1).

(2) The first relation is an inherent property of the direct sum. Let $R \in L(\mathbb{R}^n)$ denote the projector onto $\operatorname{im} D$ along $\ker A$. By means of suitable generalized inverses D^- and A^- of D and A we may write (Appendix A.2) $R = A^- A = DD^-$, $D = RD$, $A = AR$. This leads to

$$\operatorname{im} AD \subseteq \operatorname{im} A = \operatorname{im} ADD^- \subseteq \operatorname{im} AD,$$
$$\ker AD \subseteq \ker A^-AD = \ker D \subseteq \ker AD.$$

The remaining rank property now follows from (1). □

Lemma A.5. *[94, Ch. 12.4.2]*
Given matrices G, Π, $\mathcal{N}$, $\mathcal{W}$ of suitable sizes such that

$$\ker G = \operatorname{im} \mathcal{N},$$
$$\ker \Pi\mathcal{N} = \operatorname{im} \mathcal{W},$$

then it holds that

$$\ker G \cap \ker \Pi = \ker \mathcal{N}\mathcal{W}.$$

Proof. For $x \in \ker G \cap \ker \Pi$ we find $x = \mathcal{N}y, \Pi x = 0$, further $\Pi\mathcal{N}y = 0$, and hence $y = \mathcal{W}z$, $x = \mathcal{N}\mathcal{W}z \in \operatorname{im} \mathcal{N}\mathcal{W}$.
Conversely, each $x = \mathcal{N}\mathcal{W}z$ obviously belongs to $\ker G$, and $\Pi x = \Pi\mathcal{N}\mathcal{W}z = 0$. □

Lemma A.6. *$N, M \subseteq \mathbb{R}^m$ subspaces $\Rightarrow (N+M)^\perp = N^\perp \cap M^\perp$.*

Proof.

$$\begin{aligned}(N+M)^\perp &= \{z \in \mathbb{R}^m : \forall w \in N+M : \langle z, w\rangle = 0\}\\ &= \{z \in \mathbb{R}^m : \forall w_N \in N, \forall w_M \in M : \langle z, w_N + w_M\rangle = 0\}\\ &= \{z \in \mathbb{R}^m : \forall w_N \in N, \forall w_M \in M : \langle z, w_N\rangle = 0,\ \langle z, w_M\rangle = 0\}\\ &= N^\perp \cap M^\perp.\end{aligned}$$

□

Lemma A.7. (1) *Given two subspaces $N, X \subseteq \mathbb{R}^m$, $N \cap X = \{0\}$, then $\dim N + \dim X \leq m$, and there is a projector $Q \in L(\mathbb{R}^m)$ such that $\operatorname{im} Q = N$, $\ker Q \supseteq X$.*

(2) *Given two subspaces $S, N \subseteq \mathbb{R}^m$. If the decomposition*

$$\mathbb{R}^m = S \oplus N$$

holds true, i.e., S and N are transversal , then there is a uniquely determined projector $P \in L(\mathbb{R}^m)$ such that $\operatorname{im} P = S$, $\ker P = N$.

(3) *An orthoprojector P projects onto $S := \operatorname{im} P$ along $S^\perp = \ker P$.*

(4) *Given the subspaces $K, N \subseteq \mathbb{R}^m$, $\widehat{N} := N \cap K$, if a further subspace $X \subseteq \mathbb{R}^m$ is a complement of $\widehat{N}$ in K, which means $K = \widehat{N} \oplus X$, then there is a projector $Q \in L(\mathbb{R}^m)$ onto N such that*

$$X \subseteq \ker Q. \tag{A.1}$$

Let d_K, d_N, u denote the dimensions of the subspaces $K, N, \widehat{N}$, respectively. Then

$$d_K + d_N \leq m + u \tag{A.2}$$

holds.

(5) *If the subspace K in (4) is the nullspace of a certain projector $\Pi \in L(\mathbb{R}^m)$, that is $K = \ker \Pi = \operatorname{im}(I - \Pi)$, then*

$$\Pi Q(I - \Pi) = 0 \tag{A.3}$$

becomes true.

(6) *Given the two projectors $\Pi, Q \in L(\mathbb{R}^m)$, further $P := I - Q$, $N := \operatorname{im} Q$, $K := \ker \Pi$, then, supposing (A.3) is valid, the products ΠP, ΠQ, $P\Pi P$, $P(I - \Pi)$, $Q(I - \Pi)$ are projectors, too. The relation*

$$\ker \Pi P = \ker P\Pi P = N + K \tag{A.4}$$

holds true, and the subspace $X := \operatorname{im} P(I - \Pi)$ is the complement of $\widehat{N} := N \cap K$ in K, such that $K = \widehat{N} \oplus X$.
Moreover, the decomposition

$$\mathbb{R}^m = (N + K) \oplus \operatorname{im} P\Pi P = N \oplus \underbrace{X \oplus \operatorname{im} P\Pi P}_{\operatorname{im} P}$$

is valid.

(7) *If the projectors Π, Q in (6) are such that $\Pi^* = \Pi$, $(\Pi P)^* = \Pi P$, $(P(I - \Pi))^* = P(I - \Pi)$ and $Q\Pi P = 0$, then it follows that*

$$X = K \cap \widehat{N}^{\perp}, \ \operatorname{im} P = X \oplus (N + K)^{\perp}.$$

Proof. (1) Let $x_1, \ldots, x_r \in \mathbb{R}^m$ and $n_1, \ldots, n_t \in \mathbb{R}^m$ be bases of X and N. Because of $X \cap N = \{0\}$ the matrix

$$F := [x_1 \ldots x_r n_1 \ldots n_t]$$

has full column rank and $r + t = \dim X + \dim N \leq m$. The matrix F^*F is invertible, and

$$Q := F \underset{r\ \ t}{\begin{bmatrix} 0 & \\ & I \end{bmatrix}} (F^*F)^{-1} F^*$$

is a projector we looked for. Namely,

$$Q^2 = F \begin{bmatrix} 0 & \\ & I \end{bmatrix} (F^*F)^{-1} F^* F \begin{bmatrix} 0 & \\ & I \end{bmatrix} (F^*F)^{-1} F^* = Q, \quad \operatorname{im} Q = \operatorname{im} F \begin{bmatrix} 0 & \\ & I \end{bmatrix} = N,$$

and $z \in X$ implies that it has to have the structure $z = F\begin{bmatrix}\alpha\\0\end{bmatrix}\begin{matrix}\}r\\\}t\end{matrix}$, which leads to $Qz = 0$.

(2) For transversal subspaces S and N we apply Assertion (1) with $t = m-r$, i.e., F is square. We have to show that P is unique. Supposing that there are two projectors P, $\bar{P}$ such that $\ker P = \ker\bar{P} = N$, $\operatorname{im} P = \operatorname{im}\bar{P} = S$, we immediately have $P = (\bar{P}+\bar{Q})P = \bar{P}P + \bar{Q}P = \bar{P}P = \bar{P}$.

(3) Let $S := \operatorname{im} P$ and $N := \ker P$. We choose a $v \in N$ and $y \in S$. Lemma A.3 (1) implies $y = Py$, therefore $\langle v,y\rangle = \langle v,Py\rangle = \langle P^*v,y\rangle$. With the symmetry of P we obtain $\langle P^*v,y\rangle = \langle Pv,y\rangle = 0$, i.e., $N = S^\perp$.

(4) X has dimension $d_K - u$. Since the sum space $K+N = X\oplus N \subseteq \mathbb{R}^m$ may have at most dimension m, it results that $\dim(K+N) = \dim X + \dim N = d_K - u + d_N \le m$, and assertion (1) provides Q.

(5) Take an arbitrary $z \in \operatorname{im}(I-\Pi) = K$ and decompose $z = z_{\widehat{N}} + z_X$. It follows that $\Pi Qz = \Pi Qz_{\widehat{N}} + \Pi\underbrace{Qz_X}_{=0} = \Pi z_{\widehat{N}} = 0$, and hence (A.3) is true.

(6) (A.3) means $\Pi Q = \Pi Q\Pi$ and hence

$$
\begin{aligned}
\Pi Q\Pi Q &= \Pi QQ = \Pi Q,\\
\Pi P\Pi P &= \Pi(I-Q)\Pi P = \Pi P - \Pi\underbrace{Q\Pi P}_{=0} = \Pi P,\\
(P\Pi P)^2 &= P\Pi P\Pi P = P\Pi P,\\
(P(I-\Pi))^2 &= P(I-\Pi)(I-Q)(I-\Pi) = P(I-\Pi) - P(I-\Pi)Q(I-\Pi)\\
&= P(I-\Pi) + P\underbrace{\Pi Q(I-\Pi)}_{=0},\\
(Q(I-\Pi))^2 &= Q(I-\Pi) - Q\Pi Q(I-\Pi) = Q(I-\Pi).
\end{aligned}
$$

The representation $I-\Pi = Q(I-\Pi) + P(I-\Pi)$ corresponds to the decomposition $K = \widehat{N}\oplus X$.

Next we verify (A.4). The inclusion $\ker\Pi P \subseteq \ker P\Pi P$ is trivial. On the other hand, $P\Pi Pz = 0$ implies $\Pi P\Pi Pz = 0$ and hence $\Pi Pz = 0$, and it follows that $\ker\Pi P = \ker P\Pi P$. Now it is evident that $K+N \subseteq \ker\Pi P$. Finally, $\Pi Pz = 0$ implies $Pz \in K, z = Qz + Pz \in N+K$.

(7) From $Q\Pi P = 0$ and the symmetry of ΠP we know that $P\Pi P = \Pi P$, $\operatorname{im} P\Pi P = (N+K)^\perp$, $\operatorname{im} P = X\oplus(N+K)^\perp$. Next using Lemma A.6, compute $\widehat{N}^\perp = N^\perp + K^\perp$, and further

$$
\begin{aligned}
K\cap\widehat{N}^\perp &= K\cap(N^\perp + K^\perp)\\
&= \{z\in\mathbb{R}^m : \Pi z = 0, z = z_{N^\perp} + z_{K^\perp}, z_{N^\perp}\in N^\perp, z_{K^\perp}\in K^\perp\}\\
&= \{z\in\mathbb{R}^m : z = (I-\Pi)z_{N^\perp}, z_{N^\perp}\in N^\perp\} = (I-\Pi)N^\perp\\
&= \operatorname{im}(I-\Pi)P^* = \operatorname{im}(P(I-\Pi))^* = \operatorname{im} P(I-\Pi) = X.
\end{aligned}
$$

□

Lemma A.8. *Let $D \in L(\mathbb{R}^m, \mathbb{R}^n)$ be given, and let $M \subseteq \mathbb{R}^m$ be a subspace. Let $D^+ \in L(\mathbb{R}^n, \mathbb{R}^m)$ be the Moore–Penrose inverse of D. Then,*

(1) $\ker D^* = \operatorname{im} D^\perp$, $\operatorname{im} D = \ker D^{*\perp}$, $\ker D = \ker D^{+*}$, $\operatorname{im} D = \operatorname{im} D^{+*}$.
(2) $\ker D \subseteq M \Rightarrow (DM)^\perp = (\operatorname{im} D)^\perp \oplus D^{+*} M^\perp$.
(3) $\ker D \subseteq M \Rightarrow M^\perp = D^*(DM)^\perp$.

Proof. (1) The first two identities are shown in [15] (Theorem 1, p.12).
If $z \in \ker D = \operatorname{im} I - D^+ D$ with Lemma A.3(1) it is valid that $z = (I - D^+ D)z$ or $D^+ Dz = 0$. With (A.11) it holds that $0 = D^+ Dz = (D^+ D)^* z = D^* D^{+*} z \Leftrightarrow D^{+*} z = 0$ because of (A.8) for D^* and we have that $z \in \ker D^{+*}$. We prove $\operatorname{im} D = \operatorname{im} D^{+*}$ analogously.
(2) Let $T \in L(\mathbb{R}^m)$ be the orthoprojector onto M, i.e., $\operatorname{im} T = M$, $\ker T = M^\perp$, $T^* = T$.
$\Rightarrow DM = \operatorname{im} DT$,

$$(DM)^\perp = (\operatorname{im} DT)^\perp = \ker (DT)^* = \ker TD^* = \{z \in \mathbb{R}^n : D^* z \in M^\perp\}$$
$$= \underbrace{\ker D^*}_{= \operatorname{im} D^\perp} \oplus \{v \in \operatorname{im} D : D^* v \in M^\perp\}.$$

It remains to show that

$$\{v \in \operatorname{im} D : D^* v \in M^\perp\} = D^{+*} M^\perp.$$

From $v \in \operatorname{im} D = \operatorname{im} DD^+$ we get with Lemma A.3(1) $v = DD^+ v = (DD^+)^* v = D^{+*} D^* v$. Because of $D^* v \in M^\perp$ it holds that $v \in D^{+*} M^\perp$. Conversely with Lemma A.3(4), $u \in D^{+*} M^\perp = \operatorname{im} D^{+*}(I - T)$ implies $u \in \operatorname{im} D^{+*} = \operatorname{im} D$, and $\exists w : u = D^{+*}(I - T)w$, $D^* u = D^* D^{+*}(I - T)w = D^+ D(I - T)w$. Since $\operatorname{im}(I - T) = M^\perp \subseteq \ker D^\perp = \ker D^+ D^\perp = \operatorname{im}(D^+ D)^* = \operatorname{im} D^+ D$, it holds that $D^+ D(I - T) = I - T$, hence $D^* u = (I - T)w \in M^\perp$.
(3) This is a consequence of (2), because of

$$D^*(DM)^\perp = D^*[(\operatorname{im} D)^\perp \oplus D^{+*} M^\perp] = D^* D^{+*} M^\perp = D^+ DM^\perp = M^\perp.$$

□

Lemma A.9. *[96, Appendix A, Theorem 13]*
Let $A, B \in L(\mathbb{R}^m)$, $\operatorname{rank} A = r < m$, $N := \ker A$, and $S := \{z \in \mathbb{R}^m : Bz \in \operatorname{im} A\}$. The following statements are equivalent:

(1) *Multiplication by a nonsingular $E \in L(\mathbb{R}^m)$ such that*

$$EA = \begin{bmatrix} \bar{A}_1 \\ 0 \end{bmatrix}, \quad EB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \operatorname{rank} \bar{A}_1 = r,$$

yields a nonsingular $\begin{bmatrix} \bar{A}_1 \\ \bar{B}_2 \end{bmatrix}$.

(2) $N \cap S = \{0\}$.
(3) *$A + BQ$ is nonsingular for each projector Q onto N.*
(4) $N \oplus S = \mathbb{R}^m$.
(5) *The pair $\{A, B\}$ is regular with Kronecker index 1.*
(6) *The pair $\{A, B + AW\}$ is regular with Kronecker index 1 for each arbitrary $W \in L(\mathbb{R}^m)$.*

Proof. (1) $\Rightarrow$ (2): With $\bar{N} := \ker \bar{A}_1 = \ker EA = \ker A = N$,

$$\bar{S} := \ker \bar{B}_2 = \{z \in \mathbb{R}^m : EBz \in \operatorname{im} EB\} = S,$$

we have

$$0 = \ker \begin{bmatrix} \bar{A}_1 \\ \bar{B}_2 \end{bmatrix} = \bar{N} \cap \bar{S} = N \cap S.$$

(2) $\Rightarrow$ (3): $(A + BQ)z = 0$ implies $BQz = -Az$, that is $Qz \in N \cap S$, thus $Qz = 0$, $Az = 0$, therefore $z = Qz = 0$.

(3) $\Rightarrow$ (4): Fix any projector $Q \in L(\mathbb{R}^m)$ onto N and introduce $Q_* := Q(A + BQ)^{-1}B$. We show Q_* to be a projector with $\operatorname{im} Q_* = N$, $\ker Q_* = S$ so that the assertion follows. Compute

$$Q_*Q = Q(A + BQ)^{-1}BQ = Q(A + BQ)^{-1}(A + BQ)Q = Q,$$

hence $Q_*^2 = Q_*$, $\operatorname{im} Q_* = N$. Further, $Q_*z = 0$ implies $(A + BQ)^{-1}Bz = (I - Q)(A + BQ)^{-1}Bz$, thus

$$Bz = (A + BQ)(I - Q)(A + BQ)^{-1}Bz = A(A + BQ)^{-1}Bz,$$

that is, $z \in S$. Conversely, $z \in S$ leads to $Bz = Aw$ and

$$Q_*z = Q(A + BQ)^{-1}Bz = Q(A + BQ)^{-1}Aw = Q(A + BQ)^{-1}(A + BQ)(I - Q)w = 0.$$

This proves the relation $\ker Q_* = S$.

(4) $\Rightarrow$ (5): Let Q_* denote the projector onto N along S, $P_* := I - Q_*$. Since $N \cap S = 0$ we know already that $G_* := A + BQ_*$ is nonsingular as well as the representation $Q_* = Q_*G_*^{-1}B$. It holds that

$$\begin{aligned} G_*^{-1}A &= G_*^{-1}(A + BQ_*)P_* = P_*, \\ G_*^{-1}B &= G_*^{-1}BQ_* + G_*^{-1}BP_* = G_*^{-1}(A + BQ_*)Q_* + G_*^{-1}BP_* = Q_* + G_*^{-1}BP_*. \end{aligned}$$

Consider the equation $(\lambda A + B)z = 0$, or the equivalent one $(\lambda G_*^{-1}A + G_*^{-1}B)z = 0$, i.e.,

$$(\lambda P_* + G_*^{-1}BP_* + Q_*)z = 0. \tag{A.5}$$

Multiplying (A.5) by Q_* and taking into account that $Q_*G_*^{-1}BP_* = Q_*P_* = 0$ we find $Q_*z = 0$, $z = P_*z$. Now (A.5) becomes

$$(\lambda I + G_*^{-1}B)z = 0.$$

If λ does not belong to the spectrum of the matrix $-G_*^{-1}B$, then it follows that $z=0$. This means that $\lambda A+B$ is nonsingular except for a finite number of values λ, hence the pair $\{A,B\}$ is regular.
Transform $\{A,B\}$ into Weierstraß–Kronecker canonical form (cf. Section 1.1):

$$\bar{A} := EAF = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \quad \bar{B} := EBF = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}, \quad J^{\mu} = 0,\ J^{\mu-1} \neq 0.$$

We derive further

$$\bar{N} := \ker\bar{A} = F^{-1}\ker A, \quad \bar{S} := \{z \in \mathbb{R}^m : \bar{B}z \in \operatorname{im}\bar{A}\} = F^{-1}S,$$

$$\bar{N}\cap\bar{S} = F^{-1}(N\cap S) = \{0\}, \text{ and}$$

$$\bar{N}\cap\bar{S} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^m : z_1 = 0, Jz_2 = 0, z_2 \in \operatorname{im}J \right\}.$$

Now it follows that $J=0$ must be true since otherwise $\bar{N}\cap\bar{S}$ would be nontrivial.

(5) $\Rightarrow$ (1): This follows from $\bar{A} = EAF = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{B} = EBF = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}$, $\bar{N}\cap\bar{S} = 0$ and $\bar{N}\cap\bar{S} = F^{-1}(N\cap S) = \{0\}$.

(6) $\Rightarrow$ (5) is trivial.

(2) $\Rightarrow$ (6): Set $\tilde{B} := B+AW, \tilde{S} := \{z \in \mathbb{R}^m : \tilde{B}z \in \operatorname{im}A\} = S$. Because of $\tilde{S}\cap N = S\cap N = \{0\}$, and the equivalence of assertion (2) and (5), which is proved already, the pair $\{A,\tilde{B}\}$ is regular with Kronecker index 1. □

Lemma A.10. *Let $A,B \in L(\mathbb{R}^m)$ be given, A singular, $N := \ker A$, $S := \{z \in \mathbb{R}^m : Bz \in \operatorname{im}A\}$, and $N \oplus S = \mathbb{R}^m$. Then the projector Q onto N along S satisfies the relation*

$$Q = Q(A+BQ)^{-1}B. \tag{A.6}$$

Proof. First we notice that Q is uniquely determined. $A+BQ$ is nonsingular due to Lemma A.9. The arguments used in that lemma apply to show $Q(A+BQ)^{-1}B$ to be the projector onto N along S so that (A.6) becomes valid. □

For any matrix $A \in L(\mathbb{R}^m)$ there exists an integer k such that

$$\begin{aligned} \mathbb{R}^m &= \operatorname{im}A^0 \supset \operatorname{im}A \supset \cdots \supset \operatorname{im}A^k = \operatorname{im}A^{k+1} = \cdots, \\ \{0\} &= \ker A^0 \subset \ker A \subset \cdots \subset \ker A^k = \ker A^{k+1} = \cdots, \end{aligned}$$

and $\operatorname{im}A^k \oplus \ker A^k = \mathbb{R}^m$. This integer $k \in \mathbb{N}\cup\{0\}$ is said to be the *index of* A, and we write $k = \operatorname{ind}A$.

Lemma A.11. *[96, Appendix A, Theorem 4]*
Let $A \in L(\mathbb{R}^m)$ be given, $k = \operatorname{ind}A$, $r = \operatorname{rank}A^k$, and let $s_1,\dots,s_r \in \mathbb{R}^m$ and $s_{r+1},\dots,s_m \in \mathbb{R}^m$ be bases of $\operatorname{im}A^k$ *and* $\ker A^k$*, respectively. Then, for $S = [s_1 \dots s_m]$ the product $S^{-1}AS$ has the special structure*

$$S^{-1}AS = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

where $M \in L(\mathbb{R}^r)$ is nonsingular and $N \in L(\mathbb{R}^{m-r})$ is nilpotent, $N^k = 0$, $N^{k-1} \neq 0$.

Proof. For $i \leq r$, it holds that $As_i \in A\,\mathrm{im}A^k = \mathrm{im}A^{k+1} = \mathrm{im}A^k$, therefore $As_i = \sum_{j=1}^{r} s_j m_{ji}$. For $i \geq r+1$, it holds that $As_i \in \ker A^{k+1} = \ker A^k$, thus $As_i = \sum_{j=r+1}^{m} s_j n_{ji}$. This yields the representations $A[s_1 \ldots s_r] = [s_1 \ldots s_r]M$ with $M = (m_{ij})_{i,j=1}^r$, and $A[s_{r+1} \ldots s_m] = [s_{r+1} \ldots s_m]N$, with $N = (n_{ij})_{i,j=r+1}^m$. The block M is nonsingular. Namely, for a $z \in \mathbb{R}^r$ with $Mz = 0$, we have $A[s_1 \ldots s_r]z = 0$, that is,

$$\sum_{j=1}^{r} z_j s_j \in \mathrm{im}A^k \cap \ker A \subseteq \mathrm{im}A^k \cap \ker A^k = \{0\},$$

which shows the matrix M to be nonsingular. It remains to verify the nilpotency of N. We have $AS = S\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$, hence $A^\ell S = S\begin{bmatrix} M^\ell & 0 \\ 0 & N^\ell \end{bmatrix}$. From $A^k s_i = 0$, $i \geq r+1$ it follows that $N^k = 0$ must be valid. It remains to prove the fact that $N^{k-1} \neq 0$. Since $\ker A^{k-1}$ is a proper subspace of $\ker A^k$ there is an index $i_* \geq r+1$ such that the basis element $s_{i_*} \in \ker A^k$ does not belong to $\ker A^{k-1}$. Then, $S\begin{bmatrix} M^{k-1} & 0 \\ 0 & N^{k-1} \end{bmatrix} e_{i_*} = A^{k-1}s_{i_*} \neq 0$, that is, $N^{k-1} \neq 0$. □

A.2 Generalized inverses

In [15] we find a detailed collection of properties of generalized inverses for theory and application. We report here the definitions and relations of generalized inverses we need for our considerations.

Definition A.12. For a matrix $Z \in L(\mathbb{R}^n, \mathbb{R}^m)$, we call the matrix $Z^- \in L(\mathbb{R}^m, \mathbb{R}^n)$ a *reflexive generalized inverse*, if it fulfills

$$ZZ^-Z = Z \quad \text{and} \tag{A.7}$$
$$Z^-ZZ^- = Z^-. \tag{A.8}$$

Z^- is called a $\{1,2\}$-inverse of Z in [15].

The products $ZZ^- \in L(\mathbb{R}^m)$ and $Z^-Z \in L(\mathbb{R}^n)$ are projectors (cf. Appendix A.1). We have $(ZZ^-)^2 = ZZ^-ZZ^- = ZZ^-$ and $(Z^-Z)^2 = Z^-ZZ^-Z = Z^-Z$. We know that the rank of a product of matrices does not exceed the rank of any factor. Let Z have rank r_Z. From (A.7) we obtain rank $r_Z \leq$ rank r_{Z^-} and from (A.8) the opposite, i.e., that both Z and Z^- and also the projectors ZZ^- and Z^-Z have the same rank.

Let $R \in L(\mathbb{R}^n)$ be any projector onto $\mathrm{im}Z$ and $P \in L(\mathbb{R}^m)$ any projector along $\ker Z$.

Lemma A.13. *With (A.7), (A.8) and the conditions*

$$Z^- Z = P \quad \textit{and} \tag{A.9}$$
$$ZZ^- = R \tag{A.10}$$

the reflexive inverse Z^- is uniquely determined.

Proof. Let Y be a further matrix fulfilling (A.7), (A.8), (A.9) and (A.10). Then

$$\begin{aligned} Y &\overset{(A.8)}{=} YZY \overset{(A.7)}{=} YZZ^-ZY \overset{(A.10)}{=} YRZY \\ &\overset{(A.10)}{=} YR \overset{(A.10)}{=} YZZ^- \overset{(A.9)}{=} PZ^- \overset{(A.8)}{=} Z^-. \end{aligned}$$

□

If we choose for the projectors P and R the orthogonal projectors the conditions (A.9) and (A.10) could be replaced by

$$Z^- Z = (Z^- Z)^*, \tag{A.11}$$
$$ZZ^- = (ZZ^-)^*. \tag{A.12}$$

The resulting generalized inverse is called the Moore–Penrose inverse and denoted by Z^+.

To represent the generalized reflexive inverse Z^- we want to use a decomposition of

$$Z = U \begin{bmatrix} S & \\ & 0 \end{bmatrix} V^{-1}$$

with nonsingular matrices U, V and S. Such a decomposition is, e.g., available using an SVD or a Householder decomposition of Z.
A generalized reflexive inverse is given by

$$Z^- = V \begin{bmatrix} S^{-1} & M_2 \\ M_1 & M_1 S M_2 \end{bmatrix} U^{-1} \tag{A.13}$$

with M_1 and M_2 being matrices of free parameters that fulfill

$$P = Z^- Z = V \begin{bmatrix} I & 0 \\ M_1 S & 0 \end{bmatrix} V^{-1}$$

and

$$R = ZZ^- = U \begin{bmatrix} I & SM_2 \\ 0 & 0 \end{bmatrix} U^{-1}$$

(cf. also [219]). There are two ways of looking at the parameter matrices M_1 and M_2. We can compute an arbitrary Z^- with fixed M_1 and M_2. Then also the projectors P and R are fixed by these parameter matrices. Or we provide the projectors P and R, then M_1 and M_2 are given and Z^- is fixed, too.

A.3 Parameter-dependent matrices and projectors

For any two continuously differentiable matrix functions of appropriate size $F : \mathcal{I} \to L(\mathbb{R}^m.\mathbb{R}^k)$ and $G : \mathcal{I} \to L(\mathbb{R}^l.\mathbb{R}^m)$, $\mathcal{I} \subseteq \mathbb{R}$, an interval, the product $FG : \mathcal{I} \to L(\mathbb{R}^l.\mathbb{R}^k)$ is defined pointwise by $(FG)(t) := F(t)G(t)$, $t \in \mathcal{I}$, and the product rule applies to the derivatives, i.e.,

$$(FG)'(t) = F'(t)G(t) + F(t)G'(t).$$

In particular, this is valid for projector valued functions.
Let $P \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^m))$ be a projector valued function and $Q = I - P$ the complementary one. The following three simple rules are useful in computations:

(1) $Q + P = I$, and hence $Q' = -P'$.
(2) $QP = PQ = 0$, and hence $Q'P = -QP'$, $P'Q = -PQ'$.
(3) $PP'P = -PQ'P = PQP' = 0$ and, analogously, $QQ'Q = 0$.

Lemma A.14. (1) *If the matrix function $P \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^m))$ is projector valued, that is, $P(t)^2 = P(t)$, $t \in \mathcal{I}$, then it has constant rank r, and there are r linearly independent functions $\eta_1, \dots, \eta_r \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$ such that* $\operatorname{im} P(t) = \operatorname{span}\{\eta_1(t), \dots, \eta_r(t)\}$, $t \in \mathcal{I}$.
(2) *If a time-dependent subspace $L(t) \subseteq \mathbb{R}^m$, $t \in \mathcal{I}$, with constant dimension r is spanned by functions $\eta_1, \dots, \eta_r \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$, which means $L(t) =$* $\operatorname{span}\{\eta_1(t), \dots, \eta_r(t)\}$, *$t \in \mathcal{I}$, then the orthoprojector function onto this subspace is continuously differentiable.*
(3) *Let the matrix function $A \in \mathcal{C}^k(\mathcal{I}, L(\mathbb{R}^m))$ have constant rank r. Then, there is a matrix function $M \in \mathcal{C}^k(\mathcal{I}, L(\mathbb{R}^m))$ that is pointwise nonsingular such that* $A(t)M(t) = [\underbrace{\tilde{A}(t)}_{r}\ 0], \operatorname{rank} \tilde{A}(t) = r$ *for all $t \in \mathcal{I}$.*

Proof. (1) Denote $Q = I - P$, and let r be the maximal rank of $P(t)$ for $t \in \mathcal{I}$. We fix a value $\bar{t} \in \mathcal{I}$ such that $\operatorname{rank} P(\bar{t}) = r$. Let $\bar{\eta}_1, \dots, \bar{\eta}_r$ be a basis of $\operatorname{im} P(\bar{t})$.
For $i = 1, \dots, r$, the ordinary IVP

$$\eta'(t) = P'(t)\eta(t),\ t \in \mathcal{I}, \quad \eta(\bar{t}) = \bar{\eta}_i,$$

is uniquely solvable. The IVP solutions $\eta_1, \dots, \eta_r$ remain linearly independent on the entire interval $\mathcal{I}$ since they are so at $\bar{t}$.
Moreover, the function values of these functions remain in $\operatorname{im} P$, that is, $\eta_i(t) = P(t)\eta_i(t)$. Namely, multiplying the identity $\eta_i = P'\eta_i$ by Q gives $(Q\eta_i)' = -Q'Q\eta_i$, and because of $Q(\bar{t})\eta_i(\bar{t}) = Q(\bar{t})\bar{\eta}_i = 0$, the function $Q\eta_i$ must vanish identically.
It follows that $\operatorname{span}\{\eta_1(t), \dots, \eta_r(t)\} \subseteq \operatorname{im} P(t)$ for all $t \in \mathcal{I}$, and $r \leq \operatorname{rank} P(t)$, and hence $r = \operatorname{rank} P(t)$ and $\operatorname{span}\{\eta_1(t), \dots, \eta_r(t)\} = \operatorname{im} P(t)$.
(2) The matrix function $\Gamma := [\eta_1\ \ \eta_r]$, the columns of which are the given functions $\eta_1, \dots, \eta_r$, is continuously differentiable and injective, and $\Gamma^*\Gamma$ is invertible. Then $P := \Gamma(\Gamma^*\Gamma)^{-1}\Gamma^*$ is continuously differentiable. The value $P(t)$ is an or-

thoprojector, further $\operatorname{im} P \subseteq \operatorname{im} \Gamma$ by construction, and $P\Gamma = \Gamma$, in consequence $\operatorname{im} P = \operatorname{im} \Gamma = L$.
(3) See [61]. □

For matrix functions depending on several variables we define products pointwise, too. More precisely, for $F : \Omega \to L(\mathbb{R}^m.\mathbb{R}^k)$ and $G : \Omega \to L(\mathbb{R}^l.\mathbb{R}^m)$, $\Omega \subseteq \mathbb{R}^p$, the product $FG : \Omega \to L(\mathbb{R}^l.\mathbb{R}^k)$ is defined pointwise by $(FG)(x) := F(x)G(x)$, $x \in \Omega$.
We speak of a projector function $P : \Omega \to L(\mathbb{R}^l)$, if for all $x \in \Omega$, $P(x)^2 = P(x)$ holds true, and of an orthoprojector function, if, additionally, $P(x)^* = P(x)$. Saying that P is a projector function onto the subspace L we mean that P and L have a common definition domain, say Ω, and $\operatorname{im} P(x) = L(x)$, $x \in \Omega$.

Lemma A.15. *Given a matrix function $A \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m, \mathbb{R}^n))$, $k \in \mathbb{N} \cup \{0\}$, $\Omega \subseteq \mathbb{R}^p$ open, that has constant rank r, then*

(1) *The orthoprojector function onto* $\operatorname{im} A$ *is k times continuously differentiable.*
(2) *The orthoprojector function onto* $\ker A$ *is also k times continuously differentiable.*

Proof. (1) Let $\bar{x} \in \Omega$ be fixed, and $\bar{z}_1, \ldots, \bar{z}_r$ be an orthonormal basis of $\operatorname{im} A(\bar{x})^\perp$. Denote $\bar{u}_i := A(\bar{x})\bar{z}_i$, $i = 1, \ldots, r$. By construction, $\bar{u}_1, \ldots, \bar{u}_r$ are linearly independent.
We form $u_i(x) := A(x)\bar{z}_i$ for $i = 1, \ldots, r$, and then the matrix $U(x) := [u_1(x) \;\; u_r(x)]$, $x \in \Omega$. The matrix $U(\bar{x})$ has full column rank r. Therefore, there is a neighborhood $\mathcal{N}_{\bar{x}}$ of $\bar{x}$ such that $U(x)$ has full column rank r on $\mathcal{N}_{\bar{x}}$. The Gram–Schmidt orthogonalization yields the factorization

$$U(x) = Q(x)R(x), \quad Q(x) \in L(\mathbb{R}^r, \mathbb{R}^n), \quad Q(x)^*Q(x) = I_r, \quad x \in \mathcal{N}_{\bar{x}},$$

with $R(x)$ being upper triangular and nonsingular. It follows that $\operatorname{im} U(x) = \operatorname{im} Q(x)$ is true for $x \in \mathcal{N}_{\bar{x}}$.
Further, $U = A[\bar{z}_1 \;\; \bar{z}_r]$ shows that U is k times continuously differentiable together with A. By construction, Q is as smooth as U. Finally, the matrix function $R_A := Q(Q^*Q)^{-1}Q^*$ is k times continuously differentiable, and it is an orthoprojector function, $\operatorname{im} R_A = \operatorname{im} Q = \operatorname{im} U = \operatorname{im} A$.
(2) This assertion is a consequence of (1). Considering the well-known relation $\ker A^\perp = \operatorname{im} A^*$ we apply (1) and find the orthoprojector function P_A onto $\ker A^\perp$ along $\ker A$ to be k times continuously differentiable, and $I - P_A$ has this property, too. □

Remark A.16. By Lemma A.14 the orthogonal projector function $P \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^m))$, $\mathcal{I} \subseteq \mathbb{R}$ an interval, generates globally on $\mathcal{I}$ defined bases $\eta_1, \ldots, \eta_r \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^m))$, $r = \operatorname{rank} P(t)$, $\operatorname{im} P(t) = \operatorname{im}[\eta_1(t), \ldots, \eta_r(t)]$, $t \in \mathcal{I}$.
In the higher dimensional case, if $P \in \mathcal{C}^1(\Omega, L(\mathbb{R}^m))$, $\Omega \subseteq \mathbb{R}^p$ open, $p > 1$, the situation is different. By Lemma A.20, item (8), there are local bases. However, in general, global bases do not necessarily exist.
For instance, the orthoprojector function onto the nullspace of the matrix function $M(x) = [x_1, x_2, x_3]$, $x \in \mathbb{R}^3 \setminus \{0\}$, reads

$$P(x) = \frac{1}{x_1^2+x_2^2+x_3^2}\begin{bmatrix} x_2^2+x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2+x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_1^2+x_2^2 \end{bmatrix}.$$

This projector function is obviously continuously differentiable. On the other hand, the nullspace $\ker M(x) = \{z \in \mathbb{R}^3 : x_1z_1 + x_2z_2 + x_3z_3 = 0\}$ allows only locally different descriptions by bases, e.g.,

$$\ker M(x) = \operatorname{im} \begin{bmatrix} -\frac{x_2}{x_1} & -\frac{x_3}{x_1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{if } x_1 \neq 0,$$

$$\ker M(x) = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{x_3}{x_2} \\ 0 & 1 \end{bmatrix} \qquad \text{if } x_1 = 0,\ x_2 \neq 0,$$

$$\ker M(x) = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{if } x_1 = 0,\ x_2 = 0,\ x_3 \neq 0.$$

Proposition A.17. *For $k \in \mathbb{N} \cup \{0\}$, let the matrix function $D \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m, \mathbb{R}^n))$ have constant rank on the open set $\Omega \subseteq \mathbb{R}^p$.*

(1) *Then the Moore–Penrose generalized inverse D^+ of D is as smooth as D.*
(2) *Let $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^n))$ be a projector function onto* $\operatorname{im} D$, *and $P \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ be a projector function such that* $\ker P = \ker D$. *Then the four conditions*

$$DD^-D = D, \quad D^-DD^- = D, \quad D^-D = P, \quad DD^- = R,$$

determine uniquely a function D^- that is pointwise a generalized inverse of D, and D^- is k times continuously differentiable.

Proof. The first assertion is well-known, and can be found, e.g., in [49].
The second assertion follows from the first one. We simply show the matrix function $D^- := PD^+R$ to be the required one. By Lemma A.13, the four conditions define pointwise a unique generalized inverse. Taking into account that $\operatorname{im} D = \operatorname{im} R = \operatorname{im} DD^+$ and $\ker D = \ker D^+D = \ker P$ we derive

$$\begin{aligned} D(PD^+R)D &= DD^+R = R, \\ (PD^+R)D(PD^+R) &= PD^+DD^+R = (PD^+R), \\ (PD^+R)D &= PD^+D = P, \\ D(PD^+R) &= DD^+R = R, \end{aligned}$$

so that the four conditions are fulfilled. Obviously, the product PD^+R inherits the smoothness of its factors. □

For what concerns the derivatives, the situation is more difficult, if several variables are involved. We use the symbols $F_x(x,t)$, $F_t(x,t)$ for the partial derivatives and partial Jacobian matrices of the function $F \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^m,\mathbb{R}^k))$ with respect to $x \in \mathbb{R}^p$ and $t \in \mathbb{R}$, taken at the point $(x,t) \in \Omega \times \mathcal{I}$.
For the two functions $F \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^m,\mathbb{R}^k))$ and $G \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^l,\mathbb{R}^m))$, the product $FG \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^l,\mathbb{R}^k))$ is defined pointwise. We have

$$(FG)_x(x,t)z = [F_x(x,t)z]G(x,t) + F(x,t)G_x(x,t)z \quad \text{for all } z \in \mathbb{R}^p.$$

Besides the partial derivatives we apply the *total derivative in jet variables*. For the function $F \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^m,\mathbb{R}^k))$, $\Omega \times \mathcal{I} \subseteq \mathbb{R}^p \times \mathbb{R}$, the function $F' \in \mathcal{C}(\Omega \times \mathcal{I} \times \mathbb{R}^p, L(\mathbb{R}^m,\mathbb{R}^k))$ defined by

$$F'(x,t,x^1) := F_x(x,t)x^1 + F_t(x,t), \quad x \in \Omega,\ t \in \mathcal{I},\ x^1 \in \mathbb{R}^p,$$

is named the total derivative of F in jet variables. For the total derivative, the product rule

$$(FG)' = F'G + FG'$$

is easily checked to be valid.

Lemma A.18. *The total derivatives in jet variables P' and Q' of a continuously differentiable projector function P and its complementary one $Q = I - P$ satisfy the following relations:*

$$\begin{aligned} Q' &= -P', \\ Q'P &= -QP', \\ PP'P &= 0. \end{aligned}$$

Proof. The assertion follows from the identities $Q + P = I$ and $QP = 0$ by regarding the product rule. □

Notice that, for each given function $x_* \in \mathcal{C}^1(\mathcal{I}_*, \mathbb{R}^p)$, $\mathcal{I}_* \subseteq \mathcal{I}$, with values in Ω, the resulting superposition $F(x_*(t),t)$ is continuously differentiable with respect to t on $\mathcal{I}_*$, and it possesses the derivative

$$(F(x_*(t),t))' := (F(x_*(.),.))'(t) = F'(x_*(t),t,x_*'(t)).$$

A.4 Variable subspaces

Definition A.19. Let $\Omega \subseteq \mathbb{R}^P$ be open and connected, and $L(x) \subseteq \mathbb{R}^m$ be a subspace for each $x \in \Omega$. For $k \in \mathbb{N} \cup \{0\}$, L is said to be a $\mathcal{C}^k$*-subspace on* Ω, if there exists a projector function $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ which projects pointwise onto L, i.e., $R(x) = R(x)^2$, $\operatorname{im} R(x) = L(x)$, $x \in \Omega$. We write $\operatorname{im} R = L$.

It should be mentioned at this point that the notion of *smooth subspace* (smooth stands for $\mathcal{C}^1$) is applied in [96], Subsection 1.2.1, to subspaces depending on one real variable ($p = 1$) in the same way.

Lemma A.20. *Let* $k \in \mathbb{N} \cup \{0\}$.

(1) *A* $\mathcal{C}^k$*-subspace on an open connected* Ω *has constant dimension.*
(2) *The orthoprojector function onto a* $\mathcal{C}^k$*-subspace belongs to* $\mathcal{C}^k$.
(3) *If* L *is a* $\mathcal{C}^k$*-subspace, so is* $L^\perp$.
(4) *If* L *and* N *are* $\mathcal{C}^k$*-subspaces, and* $L \cap N$ *has constant dimension, then* $L \cap N$ *is a* $\mathcal{C}^k$*-subspace, too.*
(5) *If* N *and* L *are* $\mathcal{C}^k$*-subspaces, and* $N \oplus L = \mathbb{R}^m$*, then the projector onto* N *along* L *belongs to* $\mathcal{C}^k$.
(6) *If* L *and* N *are* $\mathcal{C}^k$*-subspaces, and* $L \cap N$ *has constant dimension, then there is a* $\mathcal{C}^k$*-subspace* X *such that* $X \subseteq L$*, and*

$$L = X \oplus (N \cap L),$$

as well as a projector $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ *with* $\operatorname{im} R = N$, $\ker R \supseteq X$.
(7) *If* L *and* N *are* $\mathcal{C}^k$*-subspaces, and* $N \cap L = 0$*, then* $L \oplus N$ *is a* C^k*-subspace, too.*
(8) L *is a* $\mathcal{C}^k$*-subspace on* Ω $\Leftrightarrow$ *for each* $\bar{x} \in \Omega$ *there is a neighborhood* $U_{\bar{x}} \subseteq \Omega$ *and a local* $\mathcal{C}^k$*-basis* $\eta_1, \dots, \eta_{r(\bar{x})} \in \mathcal{C}^k(U_{\bar{x}}, \mathbb{R}^m)$ *spanning* L *on* $U_{\bar{x}}$*, i.e.,*

$$\operatorname{span}\{\eta_1(x), \dots, \eta_{r(\bar{x})}(x)\} = L(x), \quad x \in U_{\bar{x}}.$$

Proof. (1) Let $x_0 \in \Omega$, and let the columns of $\xi^0 := [\xi_1^0, \dots, \xi_{r_{x_0}}^0]$ form a basis of $L(x_0)$, i.e., $L(x_0) = \operatorname{im} \xi^0$. $\xi(x) := R(x)\xi^0$ is a $\mathcal{C}^k$ matrix function, and since $\xi(x_0) = R(x_0)\xi^0 = \xi^0$ has full column rank r_{x_0}, there is a neighborhood $U_{x_0} \subset \Omega$ such that $\xi(x)$ has rank r_{x_0} for all $x \in U_{x_0}$. This means $\operatorname{im} \xi(x) \subseteq \operatorname{im} R(x)$,

$$\operatorname{rank} R(x) \geq \operatorname{rank} \xi(x) = r_{x_0}, \quad x \in U_{x_0}.$$

Denote by $r_{\min}$, $r_{\max}$ the minimal and maximal ranks of $R(x)$ on Ω, $0 \leq r_{\min} \leq r_{\max} \leq m$, and by $x_{\min}, x_{\max} \in \Omega$ points with $\operatorname{rank} R(x_{\min}) = r_{\min}$, $\operatorname{rank} R(x_{\max}) = r_{\max}$.
Since Ω is connected, there is a connecting curve of $x_{\min}$ and $x_{\max}$ belonging to Ω. We move on this curve from $x_{\max}$ to $x_{\min}$. If $r_{\min} < r_{\max}$, there must be a x_* on this curve with

$$r_* := \operatorname{rank} R(x_*) < r_{\max},$$

and in each arbitrary neighborhood of x_* there are points $\hat{x}$ with $\operatorname{rank} R(\hat{x}) = r_{\max}$.
At each $x \in \Omega$, as a projector, $R(x)$ has only the eigenvalues 1 and 0 (cf. Lemma A.3(6)). Hence, $R(x_*)$ has eigenvalue 1 with multiplicity r_*, and eigenvalue 0 with multiplicity $m - r_*$, $R(\hat{x})$ has eigenvalue 1 with multiplicity $r_{\max}$ and eigenvalue 0 with multiplicity $m - r_{\max}$.
Since eigenvalues depend continuously on the entries of a matrix, and the entries of

$R(x)$ are $\mathcal{C}^k$-functions in x, the existence of x_* contradicts the continuity of eigenvalues. Therefore, $r_{\min} = r_{\max}$ must be valid.

(2) If L is a $\mathcal{C}^k$-subspace, by definition, there is a projector $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ onto L, and the rank $R(x)$ is constant on Ω. By Lemma A.15, the orthoprojector function onto $\operatorname{im} R = L$ is k times continuously differentiable.

(3) If L is a $\mathcal{C}^k$-subspace, the orthoprojector R onto L belongs to $\mathcal{C}^k$. Then, $I - R$ is a $\mathcal{C}^k$-projector onto $\operatorname{im}(I-R) = L^\perp$.

(4) Suppose L, N are $\mathcal{C}^k$-subspaces in $\mathbb{R}^m$, and R_L, R_N corresponding projectors onto L and N. Then $F := \begin{bmatrix} I - R_L \\ I - R_N \end{bmatrix}$ is a $\mathcal{C}^k$-function, and $\ker F = L \cap N$. Since $L \cap N$ has constant dimension, F has constant rank, and therefore F^+ and F^+F are $\mathcal{C}^k$-functions. F^+F is the orthoprojector onto $\ker F$, thus $\ker F = L \cap N$ is a $\mathcal{C}^k$-subspace.

(5) Let N, L be $\mathcal{C}^k$-subspaces, $N \oplus L = \mathbb{R}^m$. For each arbitrary $x \in \Omega$, $R(x)$ is uniquely determined by $\operatorname{im} R(x) = L(x)$, $\ker R(x) = N(x)$, $R(x)^2 = R(x)$. We have to make sure that R belongs to $\mathcal{C}^k$. To each fixed $x_0 \in \Omega$ we consider bases $\xi_1^0, \ldots, \xi_r^0$ of $L(x_0)$, and $\eta_1^0, \ldots, \eta_{m-r}^0$ of $N(x_0)$, and consider

$$\xi(x) := R_L(x)\xi^0, \quad \eta(x) := R_N(x)\eta^0, \quad x \in \Omega,$$

where

$$\xi^0 = [\xi_1^0, \ldots, \xi_r^0], \quad \eta^0 = [\eta_1^0, \ldots, \eta_{m-r}^0],$$

and R_L, R_N are $\mathcal{C}^k$-projectors according to the $\mathcal{C}^k$-subspaces L and N. There is a neighborhood $U_{x_0} \subset \Omega$ of x_0, such that the columns of $\xi(x)$ and $\eta(x)$, for $x \in U_{x_0}$, are bases of $L(x)$ and $N(x)$, and the matrix $F(x) := [\xi(x), \eta(x)]$ is nonsingular for $x \in U_{x_0}$. Define, for $x \in U_{x_0}$,

$$\tilde{R}(x) := F(x) \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} F(x)^{-1},$$

such that

$$\tilde{R} \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m)), \quad \operatorname{im} \tilde{R}(x) = L(x), \quad \ker \tilde{R}(x) = N(x).$$

Since the projector corresponding to the decomposition $N(x) \oplus L(x) = \mathbb{R}^m$ is unique, we have $R(x) = \tilde{R}(x)$, $x \in U_{x_0}$, and hence R is $\mathcal{C}^k$ on U_{x_0}.

(6) Let L, N be $\mathcal{C}^k$-subspaces, $\dim(N \cap L) = \text{constant} =: u$. By (d), $N \cap L$ is a $\mathcal{C}^k$-subspace. We have $\mathbb{R}^m = (L \cap N) \oplus (L \cap N)^\perp$, $L = (L \cap N) \oplus (L \cap (L \cap N)^\perp)$, and $X := L \cap (L \cap N)^\perp$ is a $\mathcal{C}^k$-subspace, too. Further (cf. Lemma A.6), $(N+L)^\perp = N^\perp \cap L^\perp$ is also a $\mathcal{C}^k$-subspace. With $N + L = N \oplus X$ we find

$$\mathbb{R}^m = (N+L)^\perp \oplus (N+L) = (N+L)^\perp \oplus X \oplus N = S \oplus N, \quad S := (N+L)^\perp \oplus X.$$

Denote by $R^\perp$ and R_X the orthoprojectors onto the $\mathcal{C}^k$-subspaces $(N+L)^\perp$ and X. Due to $X \subseteq N + L$, $(N+L)^\perp \subseteq X^\perp$, hence $\operatorname{im} R_X \subseteq \ker R^\perp$, $\operatorname{im} R^\perp \subseteq \ker R_X$, it holds

that $R_X R^{\perp} = 0$, $R^{\perp} R_X = 0$, hence $R_S := R^{\perp} + R_X$ is a projector and belongs to $\mathcal{C}^k$, $\operatorname{im} R_S = \operatorname{im} R^{\perp} + \operatorname{im} R_X = S$. This makes it clear that S is also a $\mathcal{C}^k$-subspace.
Finally, due to $\mathbb{R}^m = S \oplus N$, there is a projector $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ with $\operatorname{im} R = N$, $\ker R = S \supset X$.

(7) By (6), due to $N \cap L = 0$, there are projectors $R_L, R_N \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ such that $\operatorname{im} R_L = L$, $N \subset \ker R_L$, $\operatorname{im} R_N = N$, $L \subset \ker R_N$, thus $R_L R_N = 0$, $R_N R_L = 0$, and $R := R_L + R_N$ is a $\mathcal{C}^k$-projector, too, and finally $\operatorname{im} R = \operatorname{im} R_L + \operatorname{im} R_N = L \oplus N$.

(8) If L is a $\mathcal{C}^k$-subspace then the orthogonal projector R on L along $L^{\perp}$ is $\mathcal{C}^k$. For each $x_0 \in \Omega$ and a basis $\xi_1^0, \ldots, \xi_r^0$ of $L(x_0)$, the columns of $\xi(x) := R(x)\xi^0$, $\xi = [\xi_1^0, \ldots, \xi_r^0]$, form a $\mathcal{C}^k$-basis of $L(x)$ locally on a neighborhood $U_{x_0} \subset \Omega$ of x_0. Conversely, if there is a local $\mathcal{C}^k$-basis on the neighborhood $U_{\bar{x}}$ of $\bar{x}$, then one can show that the orthoprojector onto $L(x)$, $x \in U_{\bar{x}}$, can be represented by means of this basis. That means, L is $\mathcal{C}^k$ on $U_{\bar{x}}$. □

Corollary A.21. *Any projector function being continuous on an open connected set has constant rank there.*

Proof. The continuous projector function, say $P : \Omega \to L(\mathbb{R}^p)$, defines the $\mathcal{C}$-space $\operatorname{im} P$. Owing to Lemma A.20 item (1), $\operatorname{im} P$ has constant dimension, and hence P has constant rank. □

Appendix B
Technical computations

B.1 Proof of Lemma 2.12

Lemma 2.12
If two projector function sequences $Q_0,\ldots,Q_k$ and $\bar{Q}_0,\ldots,\bar{Q}_k$ are both admissible, then the corresponding matrix functions and subspaces are related by the following properties:

(a) $\ker\bar{\Pi}_j = \bar{N}_0+\cdots+\bar{N}_j = N_0+\cdots+N_j = \ker\Pi_j,\ j=0,\ldots,k,$

(b) $\bar{G}_j = G_j Z_j,$
$\bar{B}_j = B_j - G_j Z_j \bar{D}^-(D\bar{\Pi}_j\bar{D}^-)'D\Pi_j + G_j\sum_{l=0}^{j-1} Q_l\mathfrak{A}_{jl},\ j=1,\ldots,k,$
with nonsingular matrix functions $Z_0,\ldots,Z_{k+1}$ given by
$Z_0 := I,\ Z_{i+1} := Y_{i+1}Z_i,\ i=0,\ldots,k,$

$$Y_1 := I + Q_0(\bar{Q}_0 - Q_0) = I + Q_0\bar{Q}_0P_0,$$
$$Y_{i+1} := I + Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) + \sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_i,\ i=1,\ldots,k,$$

and certain continuous coefficients $\mathfrak{A}_{il}$ that satisfy the condition $\mathfrak{A}_{il} = \mathfrak{A}_{il}\bar{\Pi}_{i-1}$,

(c) $Z_i(\bar{N}_i\cap(\bar{N}_0+\cdots+\bar{N}_{i-1})) = N_i\cap(N_0+\cdots+N_{i-1}),\quad i=1,\ldots,k,$

(d) $\bar{G}_{k+1} = G_{k+1}Z_{k+1},\quad \bar{N}_0+\cdots+\bar{N}_{k+1} = N_0+\cdots+N_{k+1},$
$Z_{k+1}(\bar{N}_{k+1}\cap(\bar{N}_0+\cdots+\bar{N}_k)) = N_{k+1}\cap(N_0+\cdots+N_k).$

Proof. We have $G_0 = AD = \bar{G}_0,\ B_0 = B = \bar{B}_0,\ \ker P_0 = N_0 = \bar{N}_0 = \ker\bar{P}_0$, hence $P_0 = P_0\bar{P}_0,\ \bar{P}_0 = \bar{P}_0P_0$.
The generalized inverses D^- and $\bar{D}^-$ of D satisfy the properties $DD^- = D\bar{D}^- = R$, $D^-D = P_0,\ \bar{D}^-D = \bar{P}_0$, and therefore $\bar{D}^- = \bar{D}^-D\bar{D}^- = \bar{D}^-DD^- = \bar{P}_0D^-$, $D^- = P_0\bar{D}^-$.
Compare $G_1 = G_0 + B_0Q_0$ and

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$$\begin{aligned}\bar{G}_1 &= \bar{G}_0+\bar{B}_0\bar{Q}_0 = G_0+B_0\bar{Q}_0 = G_0+B_0Q_0\bar{Q}_0\\ &= (G_0+B_0Q_0)(P_0+\bar{Q}_0) = G_1Z_1,\end{aligned}$$

where $Z_1 := Y_1 := P_0+\bar{Q}_0 = I+Q_0\bar{Q}_0P_0 = I+Q_0(\bar{Q}_0-Q_0)$. Z_1 is invertible, and it has inverse $Z_1^{-1} = I-Q_0\bar{Q}_0P_0$.
The nullspaces N_1 and $\bar{N}_1$ are, due to $\bar{G}_1 = G_1Z_1$, related by $\bar{N}_1 = Z_1^{-1}N_1 \subseteq N_0+N_1$. This implies $\bar{N}_0+\bar{N}_1 = N_0+(Z_1^{-1}N_1) \subseteq N_0+N_1$. From $N_1 = Z_1\bar{N}_1 \subseteq N_0+\bar{N}_1 = \bar{N}_0+\bar{N}_1$, we obtain $\bar{N}_0+\bar{N}_1 = N_0+N_1$.
Since the projectors $\Pi_1 = P_0P_1$ and $\bar{\Pi}_1 = \bar{P}_0\bar{P}_1$ have the common nullspace $N_0+N_1 = \bar{N}_0+\bar{N}_1$, we may now derive

$$\begin{aligned}D\bar{P}_0\bar{P}_1\bar{D}^- &= D\bar{P}_0\bar{P}_1P_0P_1\bar{P}_0D^- = D\bar{P}_0\bar{P}_1P_0P_1D^- = D\bar{P}_0\bar{P}_1\bar{D}^-DP_0P_1D^-,\\ DP_0P_1D^- &= DP_0P_1D^-D\bar{P}_0\bar{P}_1\bar{D}^-.\end{aligned}$$

Next we compute

$$\begin{aligned}\bar{B}_1 &= \bar{B}_0\bar{P}_0-\bar{G}_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-)'D\bar{P}_0\\ &= B_0(P_0+Q_0)\bar{P}_0-G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-DP_0P_1D^-)'D\\ &= B_0P_0+B_0Q_0\bar{P}_0-G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-)'DP_0P_1-G_1Z_1\bar{P}_0\bar{P}_1\bar{D}^-(DP_0P_1D^-)'D\\ &= B_1+G_1D^-(DP_0P_1D^-)'D-G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-)'DP_0P_1\\ &\qquad -G_1Z_1\bar{P}_0\bar{P}_1D^-(DP_0P_1D^-)'D+B_0Q_0\bar{P}_0\\ &= B_1-G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-)'DP_0P_1+\mathfrak{B}_1\end{aligned}$$

with $\mathfrak{B}_1 := G_1Q_0\bar{P}_0+G_1(I-Z_1\bar{\Pi}_1)D^-(D\Pi_1D^-)'D$.
The identity $0 = \bar{G}_1\bar{Q}_1 = G_1Z_1\bar{Q}_1 = G_1\bar{Q}_1+G_1(Z_1-I)\bar{Q}_1$ leads to $G_1\bar{Q}_1 = -G_1(Z_1-I)\bar{Q}_1$ and further to

$$\begin{aligned}G_1(I-Z_1\bar{\Pi}_1) &= G_1(I-\bar{\Pi}_1-(Z_1-I)\bar{\Pi}_1) = G_1(\bar{Q}_1+\bar{Q}_0\bar{P}_1-Q_0\bar{Q}_0P_0\bar{\Pi}_1)\\ &= G_1(-Q_0\bar{Q}_0P_0\bar{Q}_1+\bar{Q}_0\bar{P}_1-Q_0\bar{Q}_0P_0\bar{P}_1) = G_1(-Q_0\bar{Q}_0P_0+\bar{Q}_0\bar{P}_1)\\ &= G_1(-Q_0\bar{Q}_0+Q_0+Q_0\bar{Q}_0\bar{P}_1) = G_1(-Q_0\bar{Q}_0\bar{Q}_1+Q_0).\end{aligned}$$

Inserting into the expression for $\mathfrak{B}_1$ yields $\mathfrak{B}_1 = G_1Q_0\bar{P}_0 - G_1Q_0\bar{Q}_0\bar{Q}_1D^-(D\Pi_1D^-)'D = G_1Q_0\mathfrak{A}_{10}$ with $\mathfrak{A}_{10} := \bar{P}_0-\bar{Q}_0\bar{Q}_1D^-(D\Pi_1D^-)'D$ and $\mathfrak{A}_{10} = \mathfrak{A}_{10}\bar{P}_0$.
In order to verify assertions (a) and (b) by induction, we assume the relations

$$\begin{aligned}\bar{N}_0+\cdots+\bar{N}_j &= N_0+\cdots+N_j,\\ \bar{G}_j &= G_jZ_j,\\ \bar{B}_j &= B_j-G_jZ_j\bar{D}^-(D\bar{\Pi}_j\bar{D}^-)'D\Pi_j+G_j\sum_{l=0}^{j-1}Q_l\mathfrak{A}_{jl}\end{aligned}\tag{B.1}$$

to be valid for $j=1,\ldots,i$, $i<k$, with nonsingular Z_i as described above.
By construction, Z_i is of the form $Z_j = Y_jZ_{j-1} = Y_jY_{j-1}\cdots Y_1$. By carrying out the

multiplication and rearranging the terms we find the expression

$$Z_j - I = \sum_{l=0}^{j-1} Q_l \mathcal{C}_{jl} \tag{B.2}$$

with continuous coefficients $\mathcal{C}_{jl}$.
It holds that $Y_1 - I = Q_0\bar{Q}_0P_0$ and

$$Y_j - I = (Y_j - I)\Pi_{j-2},\ j = 2,\dots,i, \tag{B.3}$$

such that $(Y_j - I)(Z_{j-1} - I) = 0$ must be true. From this it follows that $Y_j(Z_{j-1} - I) = Z_{j-1} - I$, and $Z_j = Y_jZ_{j-1} = Y_j + Y_j(Z_{j-1} - I) = Y_j + Z_{j-1} - I = Y_j - I + Z_{j-1}$, i.e.,

$$\begin{aligned} Z_j &= Y_j - I + \dots + Y_1 - I + Z_0, \\ Z_j - Z_0 &= Z_j - I = \sum_{l=1}^{j} (Y_l - I). \end{aligned} \tag{B.4}$$

From (B.4) one can obtain special formulas for the coefficients $\mathcal{C}_{jl}$ in (B.2), but in our context there is no need for these special descriptions.
Now we compare $\bar{G}_{i+1}$ and G_{i+1}. We have

$$\bar{G}_{i+1} = \bar{G}_i + \bar{B}_i\bar{Q}_i = G_iZ_i + \bar{B}_i\bar{Q}_i.$$

Because of $\bar{B}_i = \bar{B}_i\bar{\Pi}_{i-1}$ we may write

$$\bar{B}_i\bar{Q}_i(Z_i - I) = \bar{B}_i\bar{\Pi}_{i-1}\bar{Q}_i(Z_i - I) = \bar{B}_i\bar{\Pi}_{i-1}\bar{Q}_i\bar{\Pi}_{i-1}(Z_i - I)$$

and using (B.2) and $Q_i = \bar{Q}_iQ_i$ we obtain $\bar{B}_i\bar{Q}_i(Z_i - I) = 0$, i.e., $\bar{B}_i\bar{Q}_i = \bar{B}_i\bar{Q}_iZ_i$. This yields

$$\bar{G}_{i+1} = (G_i + \bar{B}_i\bar{Q}_i)Z_i.$$

Derive further

$$\bar{G}_{i+1}Z_i^{-1} = G_i + \bar{B}_i\bar{Q}_i = G_{i+1} + (\bar{B}_i\bar{Q}_i - B_iQ_i)$$

and using (B.1) and $\bar{Q}_i = Q_i\bar{Q}_i$ we obtain

$$\begin{aligned} &= G_{i+1} + B_i(\bar{Q}_i - Q_i) + G_i\sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_i \\ &= G_{i+1} + B_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) + G_{i+1}\sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_i \\ &= G_{i+1} + B_iQ_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) + G_{i+1}\sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_i \\ &= G_{i+1}Y_{i+1}, \end{aligned}$$

and $\bar{G}_{i+1} = G_{i+1}Y_{i+1}Z_i = G_{i+1}Z_{i+1}$, that is, $\bar{G}_{i+1}$ and G_{i+1} are related as demanded. Next we show the invertibility of Y_{i+1} and compute the inverse. Consider the linear equation $Y_{i+1}z = w$, i.e.,

$$z + Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i)z + \sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_iz = w.$$

Because of (B.3) we immediately realize that

$$\Pi_iz = \Pi_iw, \quad z = w - (Y_{i+1} - I)\Pi_{i-1}z,$$

and

$$\Pi_{i-1}z + \Pi_{i-1}Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i)z = \Pi_{i-1}w.$$

Taking into account that

$$\begin{aligned} \Pi_{i-1}Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) &= \Pi_{i-1}Q_i\bar{Q}_i - \Pi_{i-1}Q_i = -\Pi_{i-1}Q_i\bar{P}_i \\ &= -\Pi_{i-1}Q_i\bar{\Pi}_{i-1}\bar{P}_i = -\Pi_{i-1}Q_i\bar{P}_i\Pi_i \end{aligned}$$

we conclude

$$\Pi_{i-1}z = \Pi_{i-1}w - \Pi_{i-1}Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i)w$$

and

$$\begin{aligned} z &= w - (Y_{i+1} - I)(I - Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i))w, \\ Y_{i+1}^{-1} &= I - (Y_{i+1} - I)(I - Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i)). \end{aligned}$$

The inverse $Z_{i+1}^{-1} = (Y_{i+1}\cdots Y_1)^{-1} = Y_1^{-1}\cdots Y_{i+1}^{-1}$ may be expressed as

$$Z_{i+1}^{-1} = I + \sum_{l=0}^{i} Q_l\mathfrak{E}_{i+1,l}$$

with certain continuous coefficients $\mathfrak{E}_{i+1,l}$. We have

$$\begin{aligned} \bar{N}_{i+1} &= Z_{i+1}^{-1}N_{i+1} \subseteq N_0 + \cdots + N_{i+1}, \\ \bar{N}_0 + \cdots + \bar{N}_{i+1} &= N_0 + \cdots + N_i + \bar{N}_{i+1} \subseteq N_0 + \cdots + N_{i+1}, \\ N_0 + \cdots + N_{i+1} &= N_0 + \cdots + N_i + (Z_{i+1}\bar{N}_{i+1}) \\ &\subseteq N_0 + \cdots + N_i + \bar{N}_{i+1} = \bar{N}_0 + \cdots + \bar{N}_{i+1}, \end{aligned}$$

thus $\bar{N}_0 + \cdots + \bar{N}_{i+1} = N_0 + \cdots + N_{i+1}$. It follows that

$$D\bar{\Pi}_{i+1}\bar{D}^- = D\bar{\Pi}_{i+1}\bar{D}^-D\Pi_{i+1}D^-.$$

Now we consider the terms $\bar{B}_{i+1}$ and B_{i+1}. We have

$$\begin{aligned}
\bar{B}_{i+1} &= \bar{B}_i\bar{P}_i - \bar{G}_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\bar{\Pi}_i \\
&= \bar{B}_i\bar{P}_i - \bar{G}_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-D\Pi_{i+1}D^-)'D\bar{\Pi}_i \\
&= \bar{B}_i\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} - G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-(D\Pi_{i+1}D^-)'D\bar{\Pi}_i \\
&= \bar{B}_i\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} \\
&\qquad -G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-\{(D\Pi_{i+1}D^-)'D\Pi_i - D\Pi_{i+1}D^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i\} \\
&= \bar{B}_i\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} \\
&\qquad -G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i + G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i.
\end{aligned}$$

Taking into account the given result for $\bar{B}_i$ we obtain

$$\begin{aligned}
\bar{B}_{i+1} &= \{B_i - G_iZ_i\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i + G_i\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\}(P_i+Q_i)\bar{P}_i \\
&\quad - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} - G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i \\
&\quad + G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i \\
&= B_iP_i - G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i + G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i + B_iQ_i\bar{P}_i \\
&\quad - G_iZ_i\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i + G_i\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} \\
&\quad - G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i + G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i,
\end{aligned}$$

hence

$$\bar{B}_{i+1} = B_{i+1} - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} + \mathfrak{B}_{i+1}$$

with

$$\begin{aligned}
\mathfrak{B}_{i+1} &= B_iQ_i\bar{P}_i + G_i\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{P}_i + G_{i+1}(I - Z_{i+1}\bar{\Pi}_{i+1})D^-(D\Pi_{i+1}D^-)'D\Pi_i \\
&\qquad -G_{i+1}(P_iZ_i - Z_{i+1}\bar{\Pi}_{i+1})\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i.
\end{aligned}$$

It remains to show that $\mathfrak{B}_{i+1}$ can be expressed as $G_{i+1}\sum_{l=0}^{i}Q_l\mathfrak{A}_{i+1\,l}$. For this purpose we rewrite

$$\begin{aligned}
\mathfrak{B}_{i+1} &= G_{i+1}Q_i\bar{P}_i + G_{i+1}\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{P}_i \\
&\qquad +G_{i+1}(I - \bar{\Pi}_{i+1} - (Z_{i+1}-I)\bar{\Pi}_{i+1})D^-(D\Pi_{i+1}D^-)'D\Pi_i \\
&\qquad -G_{i+1}(Z_i - I - Q_iZ_i + I - \bar{\Pi}_{i+1} - (Z_{i+1}-I)\bar{\Pi}_{i+1})\bar{D}^-(D\bar{\Pi}_iD^-)'D\Pi_i.
\end{aligned}$$

Take a closer look at the term $G_{i+1}(I-\bar{\Pi}_{i+1}) = G_{i+1}(\bar{Q}_{i+1} + (I-\bar{\Pi}_i)\bar{P}_{i+1})$. By means of the identity $0 = \bar{G}_{i+1}\bar{Q}_{i+1} = G_{i+1}Z_{i+1}\bar{Q}_{i+1} = G_{i+1}\bar{Q}_{i+1}+$

$G_{i+1}(Z_{i+1}-I)\bar{Q}_{i+1}$ we obtain the relation

$$G_{i+1}\bar{Q}_{i+1} = -G_{i+1}(Z_{i+1}-I)\bar{Q}_{i+1}$$

and hence

$$G_{i+1}(I-\bar{\Pi}_{i+1}) = G_{i+1}(-(Z_{i+1}-I)\bar{Q}_{i+1} + (I-\bar{\Pi}_i)\bar{P}_{i+1}).$$

This yields

$$\begin{aligned}\mathfrak{B}_{i+1} = G_{i+1}Q_i\bar{P}_i + G_{i+1}\sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{P}_i \\ + G_{i+1}\{-(Z_{i+1}-I)\bar{Q}_{i+1} + (I-\bar{\Pi}_i)\bar{P}_{i+1}) - (Z_{i+1}-I)\bar{\Pi}_{i+1}\}\times \\ \times D^-(D\Pi_{i+1}D^-)'D\Pi_i - G_{i+1}\{Z_i - I - Q_iZ_i - (Z_{i+1}-I)\bar{Q}_{i+1} \\ + (I-\bar{\Pi}_i)\bar{P}_{i+1} - (Z_{i+1}-I)\bar{\Pi}_{i+1}\}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i.\end{aligned}$$

With

$$\begin{aligned}Z_{i+1}-I = \sum_{l=0}^{i} Q_l\mathfrak{C}_{i+1\,l},\; Z_i - I = \sum_{l=0}^{i-1} Q_l\mathfrak{C}_{il}, \\ I-\bar{\Pi}_i = (I-\Pi_i)(I-\bar{\Pi}_i) = Q_i + Q_{i-1}P_i + \cdots + Q_0P_1\cdots P_i)(I-\bar{\Pi}_i),\end{aligned}$$

by rearranging the terms we arrive at

$$\mathfrak{B}_{i+1} = G_{i+1}\sum_{l=0}^{i} Q_l\mathfrak{A}_{i+1\,l},$$

e.g., with

$$\begin{aligned}\mathfrak{A}_{i+1\,i} := \bar{P}_i + \{-\mathfrak{C}_{i+1\,i}(\bar{Q}_{i+1}+\bar{\Pi}_{i+1}) + (I-\bar{\Pi}_i)\bar{P}_{i+1}\}D^-(D\Pi_{i+1}D^-)'D\Pi_i \\ - \{-Z_i - \mathfrak{C}_{i+1\,i}(\bar{Q}_{i+1}+\bar{\Pi}_{i+1}) + (I-\bar{\Pi}_i)\bar{P}_{i+1}\}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i.\end{aligned}$$

It is evident that all coefficients have the required property $\mathfrak{A}_{i+1\,l} = \mathfrak{A}_{i+1\,l}\bar{\Pi}_i$.
Finally, we are done with assertions (a), (b). At the same time, we have proved the first two relations in (d).
Assertion (c) is a consequence of (a), (b) and the special form (B.2) of the nonsingular matrix function Z_i. Namely, we have $Z_i(N_0+\cdots+N_{i-1}) = N_0+\cdots+N_{i-1}$, $Z_i\bar{N}_i = N_i$, thus

$$\begin{aligned}Z_i(\bar{N}_i\cap(\bar{N}_0+\cdots+\bar{N}_{i-1})) = (Z_i\bar{N}_i)\cap(Z_i(\bar{N}_0+\cdots+N_{i-1})) = \\ N_i\cap(Z_i(N_0+\cdots+N_{i-1})) = N_i\cap(N_0+\cdots+N_{i-1}).\end{aligned}$$

The same arguments apply for obtaining the third relation in (d). □

B.2 Proof of Lemma 2.41

Lemma 2.41 *Let the DAE (2.44) with sufficiently smooth coefficients be regular with tractability index $\mu \geq 3$, and let $Q_0, \ldots, Q_{\mu-1}$ be admissible projector functions. Let $k \in \{1, \ldots, \mu-2\}$ be fixed, and let $\bar{Q}_k$ be an additional continuous projector function onto $N_k = \ker G_k$ such that $D\Pi_{k-1}\bar{Q}_k D^-$ is continuously differentiable and the inclusion $N_0 + \cdots + N_{k-1} \subseteq \ker \bar{Q}_k$ is valid. Then the following becomes true:*

(1) *The projector function sequence*

$$\begin{aligned}&\bar{Q}_0 := Q_0, \ldots, \bar{Q}_{k-1} := Q_{k-1},\\ &\bar{Q}_k,\\ &\bar{Q}_{k+1} := Z_{k+1}^{-1} Q_{k+1} Z_{k+1}, \ldots, \bar{Q}_{\mu-1} := Z_{\mu-1}^{-1} Q_{\mu-1} Z_{\mu-1},\end{aligned}$$

is also admissible with the continuous nonsingular matrix functions $Z_{k+1}, \ldots, Z_{\mu-1}$, determined below.

(2) *If, additionally, the projector functions $Q_0, \ldots, Q_{\mu-1}$ provide an advanced decoupling in the sense that the conditions (cf. Lemma 2.31)*

$$Q_{\mu-1*}\Pi_{\mu-1} = 0, \ldots, \; Q_{k+1*}\Pi_{\mu-1} = 0$$

are given, then also the relations

$$\bar{Q}_{\mu-1*}\bar{\Pi}_{\mu-1} = 0, \ldots, \; \bar{Q}_{k+1*}\bar{\Pi}_{\mu-1} = 0, \tag{B.5}$$

are valid, and further

$$\bar{Q}_{k*}\bar{\Pi}_{\mu-1} = (Q_{k*} - \bar{Q}_k)\Pi_{\mu-1}. \tag{B.6}$$

The matrix functions Z_i are consistent with those given in Lemma 2.12, however, for easier reading we do not access this general lemma in the proof below. In the special case given here, Lemma 2.12 yields simply $Z_0 = I, Y_1 = Z_1 = I, \ldots, Y_k = Z_k = I$, and further

$$\begin{aligned}&Y_{k+1} = I + Q_k(\bar{Q}_k - Q_k) + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{kl} \bar{Q}_k = (I + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{kl} Q_k)(I + Q_k(\bar{Q}_k - Q_k)),\\ &Z_{k+1} = Y_{k+1},\\ &\quad Y_j = I + \sum_{l=0}^{j-2} Q_l \mathfrak{A}_{j-1\,l} Q_{j-1}, \quad Z_j = Y_j Z_{j-1}, \quad j = k+2, \ldots, \mu.\end{aligned}$$

Besides the general property $\ker \bar{\Pi}_j = \ker \Pi_j$, $j = 0, \ldots, \mu-1$, which follows from Lemma 2.12, now it additionally holds that

$$\operatorname{im} \bar{Q}_k = \operatorname{im} Q_k, \quad \text{but} \quad \ker \bar{Q}_j = \ker Q_j, \quad j = k+1, \ldots, \mu-1.$$

Proof (of Lemma 2.41). **(1)** Put $\bar{Q}_i = Q_i$ for $i = 0, \dots, k-1$ such that $\bar{Q}_0, \dots, \bar{Q}_k$ are admissible by the assumptions and the following relations are valid:

$$\begin{aligned}
\Pi_k &= \Pi_k \bar{\Pi}_k,\ \bar{\Pi}_k = \bar{\Pi}_k \Pi_k, \\
\bar{Q}_k P_k &= \bar{Q}_k \Pi_k, \\
Q_k \bar{P}_k &= Q_k(I - \bar{Q}_k) = Q_k - \bar{Q}_k = \bar{Q}_k Q_k - \bar{Q}_k = -\bar{Q}_k P_k, \\
\bar{\Pi}_k &= \Pi_{k-1}(P_k + Q_k)\bar{P}_k = \Pi_k + \Pi_{k-1} Q_k \bar{P}_k = (I - \Pi_{k-1}\bar{Q}_k)\Pi_k.
\end{aligned}$$

We verify the assertion level by level by induction. Set $\bar{G}_i = G_i, Z_i = I, \bar{B}_i = B_i$, for $i = 0, \dots, k-1$, $\bar{G}_k = G_k$, $Z_k = I$, and derive

$$\begin{aligned}
\bar{B}_k &= B_{k-1}P_{k-1} - G_k D^-(D\bar{\Pi}_k D^-)' D\Pi_{k-1} \\
&= B_{k-1}P_{k-1} - G_k D^-\{D\bar{\Pi}_k D^-(D\Pi_k D^-)' + (D\bar{\Pi}_k D^-)' D\Pi_k D^-\} D\Pi_{k-1} \\
&= B_{k-1}P_{k-1} - G_k \bar{\Pi}_k D^-(D\Pi_k D^-)' D\Pi_{k-1} - G_k D^-(D\bar{\Pi}_k D^-)' D\Pi_k \\
&= B_k + G_k(I - \bar{\Pi}_k) D^-(D\Pi_k D^-)' D\Pi_{k-1} - G_k D^-(D\bar{\Pi}_k D^-)' D\Pi_k \\
&= B_k + G_k \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} - G_k D^-(D\bar{\Pi}_k D^-)' D\Pi_k,
\end{aligned}$$

where we have used $G_k\bar{Q}_k = 0$ and $I - \bar{\Pi}_k = \bar{Q}_k + Q_{k-1}\bar{P}_k + \cdots + Q_0 P_1 \cdots P_{k-1}\bar{P}_k$ and with coefficients

$$\mathfrak{A}_{k,l} = Q_l P_{l+1} \cdots P_{k-1} \bar{P}_k D^-(D\bar{\Pi}_k D^-)' D\Pi_{k-1}.$$

Next we compute

$$\begin{aligned}
\bar{G}_{k+1} = G_k + \bar{B}_k \bar{Q}_k &= G_k + B_k \bar{Q}_k + G_k \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} \bar{Q}_k \\
&= G_{k+1} + B_k(\bar{Q}_k - Q_k) + G_k \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} \bar{Q}_k = G_{k+1} Z_{k+1},
\end{aligned}$$

$$Z_{k+1} = I + Q_k(\bar{Q}_k - Q_k) + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} \bar{Q}_k = (I + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} Q_k)(I + Q_k(\bar{Q}_k - Q_k)),$$

$$Z_{k+1}^{-1} = (I - Q_k(\bar{Q}_k - Q_k))(I - \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} Q_k) = I - Q_k(\bar{Q}_k - Q_k) - \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} Q_k.$$

Put $\bar{Q}_{k+1} = Z_{k+1}^{-1} Q_{k+1} Z_{k+1} = Z_{k+1}^{-1} Q_{k+1}$ such that

$$\bar{Q}_{k+1} P_{k+1} = 0,\ \bar{Q}_{k+1} = \bar{Q}_{k+1}\Pi_{k-1},\ \Pi_k \bar{Q}_{k+1} = \Pi_k Q_{k+1},$$

$\bar{\Pi}_k\bar{Q}_{k+1} = \bar{\Pi}_k\Pi_k Q_{k+1}$ is continuous and $D\bar{\Pi}_k\bar{Q}_{k+1}D^- = D\bar{\Pi}_k D^- D\Pi_k Q_{k+1}D^-$ is continuously differentiable, and hence $\bar{Q}_0,\ldots,\bar{Q}_k,\bar{Q}_{k+1}$ are admissible. It holds that

$$\Pi_{k+1} = \Pi_{k+1}\bar{\Pi}_{k+1}, \quad \bar{\Pi}_{k+1} = \bar{\Pi}_{k+1}\Pi_{k+1}, \quad \bar{\Pi}_{k+1} = (I-\Pi_{k-1}\bar{Q}_k)\Pi_{k+1}.$$

We obtain the expression

$$\bar{B}_{k+1} = B_{k+1} - \bar{G}_{k+1}D^-(D\bar{\Pi}_{k+1}D^-)'D\Pi_{k+1} + G_{k+1}\sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l},$$

with continuous coefficients $\mathfrak{A}_{k+1,l} = \mathfrak{A}_{k+1,l}\Pi_k = \mathfrak{A}_{k+1,l}\bar{\Pi}_k$, and then

$$\begin{aligned}\bar{G}_{k+2} &= \bar{G}_{k+1} + \bar{B}_{k+1}\bar{Q}_{k+1} = (G_{k+1} + \bar{B}_{k+1}Q_{k+1})Z_{k+1}\\ &= (G_{k+1} + B_{k+1}Q_{k+1} + G_{k+1}\sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1})Z_{k+1}\\ &= G_{k+2}(I + \sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1})Z_{k+1} =: G_{k+2}Z_{k+2},\end{aligned}$$

with the nonsingular matrix function

$$\begin{aligned}Z_{k+2} &= (I + \sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1})Z_{k+1}\\ &= I + Q_k(\bar{Q}_k - Q_k) + \sum_{l=0}^{k-1} Q_l\mathfrak{A}_{k,l}\bar{Q}_k + \sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1}\end{aligned}$$

such that

$$Z_{k+1}Z_{k+2}^{-1} = I - \sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1}.$$

Letting $\bar{Q}_{k+2} = Z_{k+2}^{-1}Q_{k+2}Z_{k+2} = Z_{k+2}^{-1}Q_{k+2}$ we find

$$\begin{aligned}&Q_{k+2}\bar{Q}_{k+2} = Q_{k+2}, \quad \bar{Q}_{k+2}Q_{k+2} = \bar{Q}_{k+2}, \quad \bar{Q}_{k+2} = \bar{Q}_{k+2}\Pi_{k+1} = \bar{Q}_{k+2}\bar{\Pi}_{k+1},\\ &\bar{\Pi}_{k+1}\bar{Q}_{k+2} = \bar{\Pi}_{k+1}\Pi_{k+1}Q_{k+2}, \quad D\bar{\Pi}_{k+1}\bar{Q}_{k+2}D^- = D\bar{\Pi}_{k+1}D^-D\Pi_{k+1}Q_{k+2}D^-,\end{aligned}$$

so that $\bar{Q}_0,\ldots,\bar{Q}_{k+2}$ are known to be admissible.
Further, we apply induction. For a certain $\kappa \geq k+2$, let, the projector functions $\bar{Q}_0,\ldots,\bar{Q}_\kappa$ be already shown to be admissible and, for $i = k+2,\ldots,\kappa$,

$$
\begin{aligned}
\bar{B}_{i-1} &= B_{i-1} - \bar{G}_{i-1}D^-(D\bar{\Pi}_{i-1}D^-)'D\Pi_{i-1} + G_{i-1}\sum_{l=0}^{i-2} Q_l\mathfrak{A}_{i-1,l},\\
\mathfrak{A}_{i-1,l} &= \mathfrak{A}_{i-1,l}\Pi_{i-2},\\
\bar{G}_i &= G_iZ_i, \quad Z_i = (I + \sum_{l=0}^{i-2} Q_l\mathfrak{A}_{i-1,l}Q_{i-1})Z_{i-1},\\
\bar{Q}_i &= Z_i^{-1}Q_iZ_i = Z_i^{-1}Q_i, \quad \bar{\Pi}_i = (I - \Pi_{k-1}\bar{Q}_k)\Pi_i.
\end{aligned}
$$

Now we consider

$$
\begin{aligned}
\bar{B}_\kappa &= \bar{B}_{\kappa-1}\bar{P}_{\kappa-1} - \bar{G}_\kappa D^-(D\bar{\Pi}_\kappa D^-)'D\bar{\Pi}_{\kappa-1}\\
&= \bar{B}_{\kappa-1}P_{\kappa-1} - \bar{G}_\kappa D^-(D\bar{\Pi}_\kappa D^-)'D\Pi_\kappa - \bar{G}_\kappa\bar{\Pi}_\kappa D^-(D\Pi_\kappa D^-)'D\bar{\Pi}_{\kappa-1}\\
&= B_\kappa - \bar{G}_\kappa D^-(D\bar{\Pi}_\kappa D^-)'D\Pi_\kappa + G_\kappa\sum_{l=0}^{\kappa-2} Q_l\mathfrak{A}_{\kappa-1,l}P_{\kappa-1} + \mathfrak{C}_\kappa,
\end{aligned}
$$

with

$$
\begin{aligned}
\mathfrak{C}_\kappa :=& G_\kappa D^-(D\Pi_\kappa D^-)'D\Pi_{\kappa-1} - \bar{G}_\kappa\bar{\Pi}_\kappa D^-(D\Pi_\kappa D^-)'D\bar{\Pi}_{\kappa-1}\\
&- \bar{G}_{\kappa-1}D^-(D\bar{\Pi}_{\kappa-1}D^-)'D\Pi_{\kappa-1}\\
=& G_\kappa D^-(D\Pi_\kappa D^-)'D\Pi_{\kappa-1} - \bar{G}_\kappa\bar{\Pi}_\kappa D^-\{(D\Pi_\kappa D^-)' - D\Pi_\kappa D^-(D\bar{\Pi}_{\kappa-1}D^-)'\}\times\\
&\times D\Pi_{\kappa-1} - \bar{G}_{\kappa-1}D^-(D\bar{\Pi}_{\kappa-1}D^-)'D\Pi_{\kappa-1}\\
=& G_\kappa(I - Z_\kappa\bar{\Pi}_\kappa)D^-(D\Pi_\kappa D^-)'D\Pi_{\kappa-1}\\
&- G_\kappa(P_{\kappa-1}Z_{\kappa-1} - Z_\kappa\bar{\Pi}_\kappa)D^-(D\bar{\Pi}_{\kappa-1}D^-)'D\Pi_{\kappa-1}.
\end{aligned}
$$

Regarding the relations $\Pi_\kappa Z_\kappa = \Pi_\kappa$ and $\Pi_\kappa Z_{\kappa-1} = \Pi_\kappa$ we observe that

$$\Pi_\kappa(I - Z_\kappa\bar{\Pi}_\kappa) = 0, \quad \Pi_\kappa(P_{\kappa-1}Z_{\kappa-1} - Z_\kappa\bar{\Pi}_\kappa) = 0.$$

The representation $I - \Pi_\kappa = Q_\kappa + Q_{\kappa-1}P_\kappa + \cdots + Q_0P_1\cdots P_\kappa$ admits of the expressions

$$I - Z_\kappa\bar{\Pi}_\kappa = \sum_{l=0}^{\kappa} Q_l\mathfrak{E}_{\kappa,l}, \quad P_{\kappa-1}Z_{\kappa-1} - Z_\kappa\bar{\Pi}_\kappa = \sum_{l=0}^{\kappa} Q_l\mathfrak{F}_{\kappa,l}.$$

Considering $G_\kappa Q_\kappa = 0$, this leads to the representations

$$\mathfrak{C}_\kappa = \sum_{l=0}^{\kappa-1} Q_l\{\mathfrak{E}_{\kappa,l}D^-(D\Pi_\kappa D^-)'D\Pi_{\kappa-1} - \mathfrak{F}_{\kappa,l}D^-(D\bar{\Pi}_{\kappa-1}D^-)'D\Pi_{\kappa-1}\},$$

and hence

$$\bar{B}_\kappa = B_\kappa - \bar{G}_\kappa D^-(D\bar{\Pi}_\kappa D^-)'D\Pi_\kappa + G_\kappa\sum_{l=0}^{\kappa-1} Q_l\mathfrak{A}_{\kappa,l},$$

with continuous coefficients

$$\mathfrak{A}_{\kappa,l} = \mathfrak{A}_{\kappa,l}\Pi_{\kappa-1},\ l = 0,\ldots,\kappa-1.$$

It follows that

$$\begin{aligned}\bar{G}_{\kappa+1} &= \bar{G}_\kappa + \bar{B}_\kappa\bar{Q}_\kappa = G_\kappa Z_\kappa + \bar{B}_{\kappa+1}Z_\kappa^{-1}Q_\kappa Z_\kappa\\ &= \{G_\kappa + B_\kappa Q_\kappa + G_\kappa\sum_{l=0}^{\kappa-1} Q_l\mathfrak{A}_{\kappa,l}Q_\kappa\}Z_\kappa\\ &= G_{\kappa+1}\{I + \sum_{l=0}^{\kappa-1} Q_l\mathfrak{A}_{\kappa,l}Q_\kappa\}Z_\kappa =: G_{\kappa+1}Z_{\kappa+1}.\end{aligned}$$

Letting $\bar{Q}_{\kappa+1} = Z_{\kappa+1}^{-1}Q_{\kappa+1}Z_{\kappa+1} = Z_{\kappa+1}^{-1}Q_{\kappa+1}$ we find

$$\begin{aligned}\bar{Q}_{\kappa+1} &= \bar{Q}_{\kappa+1}\Pi_\kappa = \bar{Q}_{\kappa+1}\Pi_\kappa\bar{\Pi}_\kappa = \bar{Q}_{\kappa+1}\bar{\Pi}_\kappa,\\ \bar{\Pi}_\kappa\bar{Q}_{\kappa+1} &= \bar{\Pi}_\kappa\Pi_\kappa Q_{\kappa+1} \quad D\bar{\Pi}_\kappa\bar{Q}_{\kappa+1}D^- = D\bar{\Pi}_\kappa D^- D\Pi_\kappa Q_{\kappa+1}D^-,\end{aligned}$$

which shows the sequence $\bar{Q}_0,\ldots,\bar{Q}_{\kappa+1}$ to be admissible and all required relations to be valid. We are done with Assertion (1).

(2) Owing to Lemma 2.31, the functions

$$\begin{aligned}&Q_{\mu-1*} = Q_{\mu-1}G_\mu^{-1}B_{\mu-1},\\ &\quad Q_{i*} = Q_iP_{i+1}\cdots P_{\mu-1}G_\mu^{-1}\underbrace{\{B_i + G_iD^-(D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\}}_{=:\mathfrak{B}_i},\quad i = 1,\ldots,\mu-2,\end{aligned}$$

are continuous projector-valued functions such that

$$\operatorname{im} Q_{i*} = \operatorname{im} Q_i = \ker G_i,\quad Q_{i*} = Q_{i*}\Pi_{i-1},\quad i = 1,\ldots,\mu-1.$$

Since $Q_0,\ldots,Q_{\mu-1}$ are admissible, for $j = 1,\ldots,\mu-2$, it holds that

$$\begin{aligned}Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}G_j &= Q_jP_{j+1}\cdots P_{\mu-1}P_{\mu-1}\cdots P_j = Q_jP_{j+1}\cdots P_{\mu-1}P_j\\ &= Q_jP_{j+1}\cdots P_{\mu-1} - Q_j = -Q_j(I - P_{j+1}\cdots P_{\mu-1})\\ &= -Q_j\{Q_{j+1} + P_{j+1}Q_{j+2} + \cdots + P_{j+1}\cdots P_{\mu-2}Q_{\mu-1}\}.\end{aligned}\tag{B.7}$$

Property (B.7) immediately implies

$$Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}G_j = Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}G_j\Pi_j,\tag{B.8}$$

$$Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}G_j\Pi_{\mu-1} = 0,\tag{B.9}$$

$$Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}G_i = Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}G_j \quad \text{for} \quad i < j.\tag{B.10}$$

Analogous relations are valid also for the new sequence $\bar{Q}_0,\ldots,\bar{Q}_{\mu-1}$, and, additionally,

$$\bar{Q}_j\bar{P}_{j+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_j = \bar{Q}_j\bar{P}_{j+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}G_j, \tag{B.11}$$

$$\bar{Q}_j\bar{P}_{j+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_j = \bar{Q}_j\bar{P}_{j+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_j\Pi_j. \tag{B.12}$$

Noting that $\bar{Q}_l = \bar{Q}_l Q_l$, $Q_l = Q_l\bar{Q}_l$ for $l \geq k+1$, we have further

$$\bar{Q}_j\bar{P}_{j+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_j\Pi_{\mu-1} = 0, \quad \text{for} \quad j \geq k. \tag{B.13}$$

Now, assume the projector function sequence $Q_0,\ldots,Q_{\mu-1}$ provides an already advanced decoupling such that

$$Q_{\mu-1*}\Pi_{\mu-1} = 0,\ldots,Q_{k+1*}\Pi_{\mu-1} = 0.$$

Recall that $k \leq \mu-2$. Taking into account the relation $Q_{\mu-1}G_\mu^{-1}G_{\mu-1} = Q_{\mu-1}P_{\mu-1} = 0$, we immediately conclude

$$\begin{aligned}\bar{Q}_{\mu-1*}\bar{\Pi}_{\mu-1} &= \bar{Q}_{\mu-1}\bar{G}_\mu^{-1}\bar{B}_{\mu-1}\bar{\Pi}_{\mu-1} = \bar{Q}_{\mu-1}\underbrace{Q_{\mu-1}Z_\mu^{-1}}_{=Q_{\mu-1}}G_\mu^{-1}\bar{B}_{\mu-1}\underbrace{\Pi_{\mu-2}\bar{\Pi}_{\mu-1}}_{=\Pi_{\mu-1}}\\ &= \bar{Q}_{\mu-1}Q_{\mu-1}G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} = \bar{Q}_{\mu-1}Q_{\mu-1*}\Pi_{\mu-1} = 0.\end{aligned}$$

Next, for $k \leq i \leq \mu-2$, we investigate the terms

$$\begin{aligned}\bar{Q}_{i*}\bar{\Pi}_{\mu-1} &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{\mathfrak{B}}_i\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1} + \mathfrak{D}_i,\end{aligned}$$

with $\mathfrak{D}_i := \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\{\bar{\mathfrak{B}}_i - \mathfrak{B}_i\}\bar{\Pi}_{\mu-1}$. First we show that $\mathfrak{D}_i = 0$ thanks to (B.11)–(B.13). Namely, we have by definition

$$\begin{aligned}\mathfrak{D}_i &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\{\bar{B}_i + \bar{G}_iD^-(D\bar{\Pi}_{\mu-1}D^-)'D\bar{\Pi}_{i-1} - B_i\\ &\quad - G_iD^-(D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\}\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\{-\bar{G}_iD^-(D\bar{\Pi}_iD^-)'D\Pi_i + G_i\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{i,l}\\ &\quad + \bar{G}_iD^-(D\bar{\Pi}_{\mu-1}D^-)'D\bar{\Pi}_{i-1} - G_iD^-(D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\}\bar{\Pi}_{\mu-1},\end{aligned}$$

yielding

$$\begin{aligned}\mathfrak{D}_i &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_iD^-\{-(D\Pi_iD^- - D\Pi_{k-1}\bar{Q}_kD^-D\Pi_iD^-)'D\Pi_i\\ &\quad + (D\Pi_{\mu-1}D^- - D\Pi_{k-1}\bar{Q}_kD^-D\Pi_{\mu-1}D^-)'(D\Pi_{i-1}D^- - D\Pi_{k-1}\bar{Q}_kD^-D\Pi_{i-1})\\ &\quad - (D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\}\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_iD^-\{(D\Pi_{k-1}\bar{Q}_kD^-)'D\Pi_i + (D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\\ &\quad - (D\Pi_{\mu-1}D^-)'D\Pi_{k-1}\bar{Q}_kD^-D\Pi_{i-1} - (D\Pi_{k-1}\bar{Q}_kD^-)'D\Pi_{\mu-1}\\ &\quad - (D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\}\bar{\Pi}_{\mu-1}.\end{aligned}$$

Due to $\Pi_i\bar{\Pi}_{\mu-1} = \Pi_{\mu-1}$ we arrive at

$$\begin{aligned}\mathfrak{D}_i &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_iD^-\{-(D\Pi_{\mu-1}D^-)'D\Pi_{k-1}\bar{Q}_kD^-D\Pi_{i-1}\}\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_iD^-D\Pi_{\mu-1}D^-(D\Pi_{k-1}\bar{Q}_kD^-)'D\Pi_{i-1}\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_{\mu-1}D^-(D\Pi_{k-1}\bar{Q}_kD^-)'D\Pi_{i-1}\bar{\Pi}_{\mu-1} = 0,\end{aligned}$$

which proves the relation

$$\bar{Q}_{i*}\bar{\Pi}_{\mu-1} = \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1} \tag{B.14}$$

for $k \le i \le \mu-2$. By means of the formula

$$Z_jZ_{j+1}^{-1} = I - \sum_{l=0}^{j-1} Q_l\mathfrak{A}_{j,l}Q_j$$

being available for $j = k+1,\ldots,\mu-1$, we rearrange the terms in (B.14) as

$$\begin{aligned}\bar{Q}_{i*}\bar{\Pi}_{\mu-1} &= \bar{Q}_iZ_{i+1}^{-1}P_{i+1}Z_{i+1}Z_{i+2}^{-1}P_{i+2}\cdots Z_{\mu-1}^{-1}P_{\mu-1}Z_{\mu-1}Z_\mu^{-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_iZ_{i+1}^{-1}P_{i+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1}\\ &\quad+\sum_{j=i+1}^{\mu-2}\mathfrak{C}_{i,j}Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1}+\mathfrak{C}_{i,\mu-1}Q_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1}.\end{aligned}$$

The very last term in this formula disappears because of

$$\begin{aligned}Q_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1} &= Q_{\mu-1}G_\mu^{-1}B_i\bar{\Pi}_{\mu-1} = Q_{\mu-1}G_\mu^{-1}B_{\mu-1}\bar{\Pi}_{\mu-1}\\ &= Q_{\mu-1*}(I-\Pi_{k-1}Q_k)\Pi_{\mu-1} = Q_{\mu-1*}\Pi_{\mu-1} = 0.\end{aligned}$$

Next we prove the involved sum also vanishes. For this aim we consider the relation

$$(B_j-B_i)\Pi_{\mu-1} = -\sum_{l=i+1}^{j} G_lD^-(D\Pi_lD^-)'D\Pi_{\mu-1}, \quad \text{for} \quad j \ge i+1. \tag{B.15}$$

We first assume $i > k$ leading to $\mathfrak{B}_i\bar{\Pi}_{\mu-1} = \mathfrak{B}_i\Pi_{i-1}\Pi_{\mu-1} = \mathfrak{B}_i\Pi_{\mu-1}$ and further

$$\begin{aligned}&Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\Pi_{\mu-1}\\ &\quad= \underbrace{Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_j\Pi_{\mu-1}}_{=Q_{j*}\Pi_{\mu-1}=0}+Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}(\mathfrak{B}_i-\mathfrak{B}_j)\Pi_{\mu-1}\\ &\quad= Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\{\sum_{l=i+1}^{j} G_lD^-(D\Pi_lD^-)'D\Pi_{\mu-1}\\ &\qquad\qquad\qquad\qquad+(G_j-G_i)(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\}.\end{aligned}$$

Applying once more the properties (B.8) and (B.10), we derive

$$
\begin{aligned}
&Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\Pi_{\mu-1}\\
&\quad= Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\{\sum_{l=i+1}^{j} G_lD^-(D\Pi_lD^-)'D\Pi_{\mu-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + (G_j-G_i)(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\}\\
&\quad= Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}G_j\Pi_jD^-\sum_{l=i+1}^{j}(D\Pi_lD^-)'D\Pi_{\mu-1}=0.
\end{aligned}
$$

Now, for $i>k$, it results that

$$
\begin{aligned}
\bar{Q}_{i*}\Pi_{\mu-1} &= \bar{Q}_iZ_{i+1}^{-1}P_{i+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\Pi_{\mu-1} = \bar{Q}_iQ_iP_{i+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\Pi_{\mu-1}\\
&= \bar{Q}_iQ_{i*}\Pi_{\mu-1}=0,
\end{aligned}
$$

which verifies property (B.5). By the same means one obtains

$$
\begin{aligned}
\bar{Q}_{k*}\Pi_{\mu-1} &= \underbrace{\bar{Q}_kZ_{k+1}^{-1}}_{=Q_k}P_{k+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_k\Pi_{\mu-1} = Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_k\Pi_{\mu-1}\\
&= Q_{k*}\Pi_{\mu-1}.
\end{aligned}
$$

Finally, it remains to investigate the expression $\bar{Q}_{k*}\bar{\Pi}_{\mu-1}$. Since $\bar{Q}_{k*}$ also projects onto $\operatorname{im}\bar{Q}_k=\ker G_k$, it follows that $\bar{Q}_{k*}\bar{Q}_k=\bar{Q}_k$. This proves property (B.6), namely

$$
\begin{aligned}
\bar{Q}_{k*}\bar{\Pi}_{\mu-1} &= \bar{Q}_{k*}(I-\Pi_{k-1}\bar{Q}_k)\Pi_{\mu-1} = \bar{Q}_{k*}\Pi_{\mu-1}-\bar{Q}_{k*}\Pi_{k-1}\bar{Q}_k\Pi_{\mu-1}\\
&= Q_{k*}\Pi_{\mu-1}-\bar{Q}_k\Pi_{\mu-1} = (Q_{k*}-\bar{Q}_k)\Pi_{\mu-1}.
\end{aligned}
$$

□

B.3 Admissible projectors for $Nx'+x=r$

In this part, admissible projectors are generated for the DAE (B.16) with a nilpotent matrix function N typical for the normal form in the framework of strangeness index (cf. [130]). Our admissible projectors are given explicitly by formulas (B.26) below; they have upper block triangular form corresponding to the strict upper block triangular form of N.

Roughly speaking Lemma B.1 below is the technical key when proving that any DAE which has a well-defined regular strangeness index is at the same time regular in the tractability-index framework, and, in particular, the constant-rank requirements associated to the strangeness index are sufficient for the constant-rank conditions associated to the tractability index.

We deal with the special DAE

$$
Nx'+x=r, \tag{B.16}
$$

given by a matrix function $N \in C(\mathcal{I}, L(\mathbb{R}^m))$, $\mathcal{I} \subseteq \mathbb{R}$ an interval, that has strict upper block triangular structure uniform on $\mathcal{I}$

$$N = \begin{bmatrix} 0 & N_{12} & \dots & N_{1\mu} \\ & 0 & \ddots & \vdots \\ & & \ddots & \vdots \\ & & 0 & N_{\mu-1\mu} \\ & & & 0 \end{bmatrix} \begin{matrix} \}\ \ell_1 \\ \\ \\ \}\ \ell_{\mu-1} \\ \}\ \ell_\mu \end{matrix},$$

$1 \le \ell_1 \le \dots \le \ell_\mu$, $\ell_1 + \dots + \ell_\mu = m$, $\mu \ge 2$. The blocks N_{ii+1}, $i = 1, \dots, \mu-1$, are supposed to have full row rank each, i.e.,

$$\operatorname{rank} N_{ii+1} = \ell_i, \quad i = 1, \dots, \mu-1. \tag{B.17}$$

This implies that all powers of N have constant rank, namely

$$\begin{aligned} \operatorname{rank} N &= \ell_1 + \dots + \ell_{\mu-1}, \\ \operatorname{rank} N^k &= \ell_1 + \dots + \ell_{\mu-k}, \quad k = 1, \dots, \mu-1, \\ \operatorname{rank} N^\mu &= 0. \end{aligned} \tag{B.18}$$

N is nilpotent with index μ, i.e., $N^{\mu-1} \neq 0$, $N^\mu = 0$. For $i = 1, \dots, \mu-1$, we introduce projectors $\mathcal{V}^{[1]}_{i+1,i+1} \in C(\mathcal{I}, L(\mathbb{R}^{\ell_{i+1}}))$ onto the continuous subspace $\ker N_{i,i+1}$, and $\mathcal{U}^{[1]}_{i+1,i+1} := I_{\ell_{i+1}} - \mathcal{V}^{[1]}_{i+1,i+1}$. $\mathcal{V}^{[1]}_{i+1,i+1}$ and $\mathcal{U}^{[1]}_{i+1,i+1}$ have constant rank $\ell_{i+1} - \ell_i$ and ℓ_i, respectively. Exploiting the structure of N we build a projector $\mathcal{V}^{[1]} \in C(\mathcal{I}, L(\mathbb{R}^m))$ onto the continuous subspace $\ker N$, which has a corresponding upper block triangular structure

$$\mathcal{V}^{[1]} = \begin{bmatrix} I & & & & \\ & \mathcal{V}^{[1]}_{22} & * & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & * \\ & & & & \mathcal{V}^{[1]}_{\mu\mu} \end{bmatrix} \begin{matrix} \}\ \ell_1 \\ \\ \\ \}\ \ell_{\mu-1} \\ \}\ \ell_\mu \end{matrix}. \tag{B.19}$$

The entries indicated by "$*$" are uniquely determined by the entries of N and generalized inverses $N^-_{i,i+1}$ with

$$N^-_{i,i+1} N_{i,i+1} = \mathcal{V}^{[1]}_{i+1,i+1}, \quad N_{i,i+1} N^-_{i,i+1} = I_{\ell_i}, \quad i = 1, \dots, \mu-1.$$

In the following, we assume the nullspace $\ker N$ to be just a C^1 subspace, and the projector $\mathcal{V}^{[1]}$ to be continuously differentiable. Obviously, the property $N \in C^1(\mathcal{I}, L(\mathbb{R}^m))$ is sufficient for that but might be too generous. For this reason, we do not specify further smoothness conditions in terms of N but in terms of projectors

and subspaces.
Making use of $N = N\mathcal{U}^{[1]}, \mathcal{U}^{[1]} := I - \mathcal{V}^{[1]}$, we reformulate the DAE (B.16) as

$$N(\mathcal{U}^{[1]}x)' + (I - N\mathcal{U}^{[1]\prime})x = r. \tag{B.20}$$

The matrix function $N\mathcal{U}^{[1]\prime}$ is again strictly upper block triangular, and $I - N\mathcal{U}^{[1]\prime}$ is nonsingular, upper block triangular with identity diagonal blocks.

$$M_0 := (I - N\mathcal{U}^{[1]\prime})^{-1}N = \sum_{\ell=0}^{\mu-1} (N\mathcal{U}^{[1]\prime})^{\ell} N$$

has the same strict upper block triangular structure as N, the same nullspace, and entries $(M_0)_{i,i+1} = N_{i,i+1}$, $i = 1,\dots,\mu-1$. Scaling equation (B.20) by $(I - N\mathcal{U}^{[1]\prime})^{-1}$ yields

$$M_0(\mathcal{U}^{[1]}x)' + x = q, \tag{B.21}$$

where $q := (I - N\mathcal{U}^{[1]\prime})r$. By construction, the DAE (B.21) has a properly stated leading term (cf. Definition 2.1). Written as a general linear DAE

$$A(Dx)' + Bx = q$$

with $A = M_0$, $D = \mathcal{U}^{[1]}$, $B = I$, we have $\ker A = \ker M_0 = \ker N = \ker \mathcal{U}^{[1]}$, $\operatorname{im} D = \operatorname{im} \mathcal{U}^{[1]}$, $R = \mathcal{U}^{[1]}$.
Next we choose $D^- = \mathcal{U}^{[1]}$, and, correspondingly $P_0 = \mathcal{U}^{[1]}$, $Q_0 = \mathcal{V}^{[1]}$. With these projectors, $\Pi_0 = P_0$, and $G_0 = AD = M_0\mathcal{U}^{[1]} = M_0$, $B_0 = I$, we form a matrix function sequence and admissible projectors $Q_0,\dots,Q_\kappa$ for the DAE (B.21) as described in Section 2.2.2. In particular, we shall prove this DAE to be regular with tractability index μ.
The first matrix function (cf. Section 2.2.2) G_1 is

$$G_1 = M_0 + Q_0,$$

and $G_1 z = 0$, i.e., $(M_0 + Q_0)z = 0$, leads to $P_0 M_0 z = 0$, $Q_0 z = -Q_0 M_0 P_0 z$, $z = (I - Q_0 M_0)P_0 z$, $z \in \ker P_0 M$. Because of $P_0 M_0 = M_0^- M_0 M_0$, $M_0^2 = M_0 P_0 M_0$ the nullspaces of $P_0 M_0$ and M_0^2 coincide. The inclusion $\ker M_0 \subset \ker M_0^2 = \ker P_0 M_0$ allows for the decomposition $\ker M_0^2 = \ker M_0 \oplus P_0 \ker M_0^2$. If $\mathcal{V}^{[2]}$ denotes a projector onto $\ker M_0^2$, $\mathcal{U}^{[2]} := I - \mathcal{V}^{[2]}$, then it follows that

$$\begin{aligned}
\operatorname{im} \mathcal{V}^{[2]} &= \operatorname{im} \mathcal{V}^{[1]} \oplus \operatorname{im} \mathcal{U}^{[1]} \mathcal{V}^{[2]}, \\
\mathcal{V}^{[2]}\mathcal{V}^{[1]} &= \mathcal{V}^{[1]}, \quad (\mathcal{U}^{[1]}\mathcal{U}^{[2]})^2 = \mathcal{U}^{[1]}\mathcal{U}^{[2]}, \\
(\Pi_0 \mathcal{V}^{[2]})^2 &= \Pi_0 \mathcal{V}^{[2]}, \\
\operatorname{rank} \mathcal{U}^{[2]} &= \operatorname{rank} M_0^2 = \ell_1 + \cdots + \ell_{\mu-2}, \\
\operatorname{rank} \mathcal{V}^{[2]} &= \ell_{\mu-1} + \ell_\mu, \\
\operatorname{rank} \Pi_0 \mathcal{V}^{[2]} &= \operatorname{rank} \mathcal{V}^{[2]} - \operatorname{rank} \mathcal{V}^{[1]} = \ell_{\mu-1}.
\end{aligned}$$

The matrix function

$$Q_1 := (I - Q_0 M_0)\Pi_0 \mathcal{V}^{[2]} \tag{B.22}$$

has the properties

$$Q_1 Q_0 = (I - Q_0 M_0)\Pi_0 \mathcal{V}^{[2]}\mathcal{V}^{[1]} = (I - Q_0 M_0)\Pi_0 \mathcal{V}^{[1]} = (I - Q_0 M_0)\Pi_0 Q_0 = 0,$$

hence $Q_1 \cdot Q_1 = Q_1$, and

$$\begin{aligned} G_1 Q_1 &= (M_0 + Q_0)(I - Q_0 M_0)\Pi_0 \mathcal{V}^{[2]} = (M_0 - Q_0 M_0 + Q_0)\Pi_0 \mathcal{V}^{[2]} \\ &= P_0 M_0 \Pi_0 \mathcal{V}^{[2]} = P_0 M_0 \mathcal{V}^{[2]} = 0. \end{aligned}$$

It becomes clear that Q_1 is actually the required projector onto $\ker G_1$, if $\operatorname{rank} Q_1 = m - \operatorname{rank} G_1$. $I - Q_0 M_0$ is nonsingular, and Q_1 has the same rank as $\Pi_0 \mathcal{V}^{[2]}$, that is, $\operatorname{rank} Q_1 = \ell_{\mu-1}$. Proposition 2.5(3) allows for an easy rank determination of the matrix function G_1. With

$$\mathcal{W}_0 := \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & I \end{bmatrix} \} \ell_\mu$$

we find $\operatorname{im} G_1 = \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0 B_0 Q_0 = \operatorname{im} M_0 \oplus \operatorname{im} \mathcal{W}_0 Q_0$, thus $r_1 = r_0 + \operatorname{rank} \mathcal{V}^{[1]}_{\mu\mu} = m - \ell_\mu + \ell_\mu - \ell_{\mu-1} = m - \ell_{\mu-1}$. It turns out that Q_0, Q_1 are admissible, supposing $\pi_1 = \mathcal{U}^{[1]}\mathcal{U}^{[2]}$ is continuously differentiable.
Next, due to the structure of M_0^2, the projector $\mathcal{V}^{[2]}$ can be chosen to be upper block triangular,

$$\mathcal{V}^{[2]} = \begin{bmatrix} I & & & & \\ & I & & & \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix}, \quad \mathcal{U}^{[2]} = I - \mathcal{V}^{[2]} = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix}.$$

The entries in the lower right corners play their role in rank calculations. They are

$$\mathcal{V}^{[2]}_{\mu\mu} = I - \mathcal{U}^{[2]}_{\mu\mu}, \quad \mathcal{U}^{[2]}_{\mu\mu} = (N_{\mu-2,\mu-1}N_{\mu-1,\mu})^{-}N_{\mu-2,\mu-1}N_{\mu-1,\mu}.$$

To realize this we just remember that the entry $(\mu-2,\mu)$ of M_0^2 is $[M_0^2]_{\mu-2,\mu} = N_{\mu-2,\mu-1}N_{\mu-1,\mu}$. Both $N_{\mu-2,\mu-1}$ and $N_{\mu-1,\mu}$ have full row rank $\ell_{\mu-2}$, respectively $\ell_{\mu-1}$. Therefore, the product $N_{\mu-2,\mu-1}N_{\mu-1,\mu}$ has full row rank equal to $\ell_{\mu-2}$. From this it follows that

$$\operatorname{rank} \mathcal{V}^{[2]}_{\mu\mu} = \dim\ker N_{\mu-2,\mu-1}N_{\mu-1,\mu} = \ell_\mu - \ell_{\mu-2}.$$

Taking into account the inclusion

$$\operatorname{im}\mathcal{V}_{\mu\mu}^{[1]} = \ker N_{\mu-1,\mu} \subseteq \ker N_{\mu-2,\mu-1}N_{\mu-1,\mu} = \operatorname{im}\mathcal{V}_{\mu\mu}^{[2]}$$

we find

$$\operatorname{rank}\mathcal{U}_{\mu\mu}^{[1]}\mathcal{V}_{\mu\mu}^{[2]} = \operatorname{rank}\mathcal{V}_{\mu\mu}^{[2]} - \operatorname{rank}\mathcal{V}_{\mu\mu}^{[1]} = \ell_{\mu-1} - \ell_{\mu-2}.$$

By Proposition 2.5(3), with the projector along $\operatorname{im} G_1$

$$\mathcal{W}_1 := \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \mathcal{U}_{\mu\mu}^{[1]} \end{bmatrix}, \quad \mathcal{W}_1 = \mathcal{W}_0\mathcal{U}^{[1]},$$

we compute (before knowing G_2 in detail)

$$\operatorname{im} G_2 = \operatorname{im} G_1 \oplus \operatorname{im}\mathcal{W}_1 Q_1, \quad \mathcal{W}_1 Q_1 = \mathcal{W}_0\mathcal{U}^{[1]}\mathcal{V}^{[2]},$$
$$r_2 = r_1 + \operatorname{rank}\mathcal{W}_1 Q_1 = r_1 + \operatorname{rank}\mathcal{U}_{\mu\mu}^{[1]}\mathcal{V}_{\mu\mu}^{[2]} = m - \ell_{\mu-1} + \ell_{\mu-1} - \ell_{\mu-2} = m - \ell_{\mu-2}.$$

We compute $G_2 = G_1 + (B_0\Pi_0 - G_1 D^-(D\Pi_1 D^-)'D\Pi_0)Q_1$ (cf. Section 2.2.2) itself as

$$\begin{aligned} G_2 &= M_0 + Q_0 + \Pi_0 Q_1 - (M_0 + Q_0)P_0\Pi_1'\Pi_0 Q_1 \\ &= M_0 + Q_0 + \Pi_0 Q_1 - M_0 F_1 \Pi_0 Q_1, \end{aligned}$$

where $F_1 := P_0\Pi_1'\Pi_0 Q_1$ is upper block triangular as are all its factors. It follows that

$$G_2 = M_0 + Q_0 + (I - M_0F_1)P_0(I - \Pi_1),$$

and G_2 is upper block triangular. Due to the nonsingularity of $I - M_0F_1$, as well as the simple property $(I - M_0F_1)Q_0 = Q_0$, we may use the description

$$G_2 = (I - M_0F_1)^{-1}\{M_1 + I - \Pi_1\},$$

where $M_1 := (I - M_0F_1)^{-1}M_0$ again has the strict upper block triangular structure of N, and entries $[M_1]_{ii+1} = N_{ii+1}$, $i = 1, \ldots, \mu - 1$. From the representation

$$\begin{aligned} \Pi_1 M_1 &= \Pi_1 P_0 M_1 = \Pi_1 P_0(I + M_0F_1 + \cdots + (M_0F_0)^{\mu-1})M_0 \\ &= \Pi_1(I + M_0F_1 + \cdots + (M_0F_1)^{\mu-1})P_0M_0 \end{aligned}$$

we know the inclusion $\ker\Pi_0 M_0 \subseteq \ker\Pi_1 M_1$ to be valid. Furthermore, we have $\ker M_0^2 M_1 = \ker\Pi_1 M_1$ because of the representations $\ker\mathcal{U}^{[2]} = \ker M_0^2 = \ker P_0 M_0$, $\Pi_1 M_1 = P_0\mathcal{U}^{[2]}M_1 = P_0(M_0^2)^- M_0^2 M_1$, and $M_0^2 M_1 = M_0^2\mathcal{U}^{[2]}M_1 = M_0^2 P_0\mathcal{U}^{[2]}M_1 = M_0^2\Pi_1 M_1$.

The next lemma shows that we may proceed further in this way to construct admissible projectors for the DAE (B.21). We shall use certain auxiliary continuous matrix functions which are determined from level to level as

$$
\begin{aligned}
F_0 &:= 0,\\
F_i &:= F_{i-1} + \sum_{\ell=1}^{i} P_0 \Pi'_\ell \Pi_{i-1} Q_i = \sum_{j=1}^{i}\sum_{\ell=1}^{j} P_0 \Pi'_\ell \Pi_{j-1} Q_i, \quad i \geq 1, \qquad \text{(B.23)}\\
H_2 &:= H_1 := H_0 := 0,\\
H_i &:= H_{i-1} + \sum_{\ell=2}^{i-1} (I - H_{\ell-1}) P_0 (I - \Pi_{\ell-1}) \Pi'_\ell \Pi_{i-1} Q_i\\
&= \sum_{j=3}^{i}\sum_{\ell=2}^{j-1} (I - H_{\ell-1}) P_0 (I - \Pi_{\ell-1}) \Pi'_\ell \Pi_{j-1} Q_j, \quad i \geq 3. \qquad \text{(B.24)}
\end{aligned}
$$

These matrix functions inherit the upper block triangular structure. They disappear if the projectors $\Pi_1, \dots, \Pi_i$ do not vary with time (what is given at least in the constant coefficient case).
It holds that $F_i = F_i P_0$, $H_i = H_i P_0$. The products $F_i M_0$ are strictly upper block triangular so that $I - M_0 F_i$ is nonsingular, and

$$M_i := (I - M_0 F_i)^{-1} M_0 \qquad \text{(B.25)}$$

again has strict upper block triangular structure. The entries $(j, j+1)$ of M_i coincide with those of N, i.e.,

$$[M_i]_{j,j+1} = N_{j,j+1}. \qquad \text{(B.26)}$$

If the projectors $\Pi_0, \dots, \Pi_i$ are constant, then we simply have $M_i = M_0 = N$.

Lemma B.1. *Let N be sufficiently smooth so that the continuous projectors Π_i arising below are even continuously differentiable. Let $k \in \mathbb{N}$, $k \leq \mu - 1$, and let $Q_0 := \mathcal{V}^{[1]}$ be given by* (B.19)*, and, for $i = 1, \dots, k$,*

$$Q_i := \Big(I - \sum_{j=0}^{i-1} Q_j (I - H_{i-1})^{-1} M_{i-1}\Big) \Pi_{i-1} \mathcal{V}^{[i+1]}, \qquad \text{(B.27)}$$

$\mathcal{V}^{[i+1]} \in C(\mathcal{I}, L(\mathbb{R}^m))$ an upper block triangular projector onto $\ker M_0^2 M_1 \cdots M_{i-1}$, $\mathcal{U}^{[i+1]} := I - \mathcal{V}^{[i+1]}$. Then, the matrix functions $Q_0, \dots, Q_k$ are admissible projectors for the DAE (B.21) *on $\mathcal{I}$, and, for $i = 1, \dots, k$, it holds that*

$$
\begin{aligned}
&\Pi_{i-1} Q_i = \Pi_{i-1} \mathcal{V}^{[i+1]}, \quad \Pi_i = \mathcal{U}^{[1]} \cdots \mathcal{U}^{[i+1]}, && \text{(B.28)}\\
&\ker \Pi_{i-1} M_{i-1} \subset \ker \Pi_i M_i, && \text{(B.29)}\\
&\ker \Pi_i M_i = \ker M_0^2 M_1 \dots M_i, && \text{(B.30)}\\
&G_{i+1} = M_0 + Q_0 + (I - M_0 F_i)(I - H_i) P_0 (I - \Pi_i), && \text{(B.31)}\\
&r_{i+1} = \operatorname{rank} G_{i+1} = m - \ell_{\mu-i-1}, \quad \operatorname{im} G_{i+1} = \operatorname{im} G_i \oplus \operatorname{im} \mathcal{W}_0 \Pi_{i-1} Q_i,
\end{aligned}
$$

and $I-H_i$ is nonsingular.

Before we turn to the proof of Lemma B.1 we realize that it provides admissible projectors $Q_0,\dots,Q_{\mu-1}$ and characteristics $r_0=m-\ell_\mu,\dots,r_{\mu-1}=m-\ell_1<m$. Because of the strict upper block triangular structure of $M_0,\dots,M_{\mu-2}$, the product $M_0^2M_1\cdots M_{\mu-2}$ disappears (as N^μ does). This leads to $\mathcal{V}^{[\mu]}=I$, $\mathcal{U}^{[\mu]}=0$, thus $\Pi_{\mu-1}=0$, and

$$\begin{aligned} G_\mu &= M_0+Q_0+(I-M_0F_{\mu-1})(I-H_{\mu-1})P_0(I-\Pi_{\mu-1})\\ &= M_0+Q_0+(I-M_0F_{\mu-1})(I-H_{\mu-1})P_0\\ &= (I-M_0F_{\mu-1})(I-H_{\mu-1})\{(I-H_{\mu-1})^{-1}M_{\mu-1}+I\}. \end{aligned}$$

The factors $I-M_0F_{\mu-1}$ and $I-H_{\mu-1}$ are already known to be nonsingular. $(I-H_{\mu-1})^{-1}M_{\mu-1}$ inherits the strict upper block triangular structure from $M_{\mu-1}$, but then $I+(I-H_{\mu-1})^{-1}M_{\mu-1}$ is nonsingular, and so is G_μ. Hence we have proved an important consequence of Lemma B.1:

Proposition B.2. *Let N be sufficiently smooth to make the continuous projectors $\Pi_0,\dots,\Pi_{\mu-2}$ even continuously differentiable. Then the DAE (B.21) is on $\mathcal{I}$ regular with tractability index μ and characteristic values*

$$r_i=m-\ell_{\mu-i},\quad i=0,\dots,\mu-1,\quad r_\mu=m.$$

It holds that $\Pi_{\mu-1}=0$, and there is no inherent regular ODE within the DAE.

To prepare the proof of Lemma B.1 we give the following lemma

Lemma B.3. *Let $\mathcal{V}_i\in L(\mathbb{R}^m)$ be idempotent, $\mathcal{U}_i:=I-\mathcal{V}_i$, $L_i:=\operatorname{im}\mathcal{V}_i$, $\nu_i:=\operatorname{rank}\mathcal{V}_i$, $i=1,\dots,k$, and $L_i\subseteq L_{i+1}$, $i=1,\dots,k-1$.*
Then the products $\mathcal{U}_1\mathcal{V}_2,\dots,\mathcal{U}_1\cdots\mathcal{U}_{k-1}\mathcal{V}_k,\mathcal{U}_1\mathcal{U}_2,\dots,\mathcal{U}_1\cdots\mathcal{U}_k$ are projectors, too, and it holds that

$$\begin{aligned} &\mathcal{U}_1\cdots\mathcal{U}_i\mathcal{V}_{i+1}\mathcal{V}_j=0,\quad 1\le j\le i,\quad i=1,\dots,k-1,\\ &\ker\mathcal{U}_1\cdots\mathcal{U}_i=L_i,\quad i=1,\dots,k,\\ &L_k=L_1\oplus\mathcal{U}_1L_2\oplus\cdots\oplus\mathcal{U}_1\cdots\mathcal{U}_{k-1}L_k,\\ &\dim\mathcal{U}_1\cdots\mathcal{U}_{k-1}L_k=\nu_k-\nu_{k-1}. \end{aligned}\tag{B.32}$$

Proof. The inclusions $L_1\subseteq L_2\subseteq\cdots\subseteq L_{i+1}$ lead to $\mathcal{V}_{i+1}\mathcal{V}_j=\mathcal{V}_j$, for $j=1,\dots,i$. Compute

$$\begin{aligned} \mathcal{U}_1\mathcal{V}_2\mathcal{U}_1\mathcal{V}_2&=\mathcal{U}_1\mathcal{V}_2(I-\mathcal{V}_1)\mathcal{V}_2=\mathcal{U}_1\mathcal{V}_2-\mathcal{U}_1\mathcal{V}_1\mathcal{V}_2=\mathcal{U}_1\mathcal{V}_2,\\ \mathcal{U}_1\mathcal{U}_2\mathcal{U}_1\mathcal{U}_2&=\mathcal{U}_1(I-\mathcal{V}_2)(I-\mathcal{V}_1)\mathcal{U}_2=\mathcal{U}_1(I-\mathcal{V}_1-\mathcal{V}_2+\mathcal{V}_1)\mathcal{U}_2=\mathcal{U}_1\mathcal{U}_2. \end{aligned}$$

$L_2=\operatorname{im}\mathcal{V}_2\subseteq\ker\mathcal{U}_1\mathcal{U}_2$ holds trivially. $z\in\ker\mathcal{U}_1\mathcal{U}_2$ means $(I-\mathcal{V}_1)(I-\mathcal{V}_2)z=0$, hence $z=\mathcal{V}_1z+\mathcal{V}_2z-\mathcal{V}_1\mathcal{V}_2z\in L_2$, so that $\ker\mathcal{U}_1\mathcal{U}_2=L_2$ is true.
By induction, if $\mathcal{U}_1\cdots\mathcal{U}_{i-1}Q_i$, $\mathcal{U}_1\cdots\mathcal{U}_i$ are projectors, $\ker\mathcal{U}_1\cdots\mathcal{U}_i=L_i$, then these properties remain valid for $i+1$ instead of i. Namely,

$$\begin{aligned}\mathcal{U}_1\cdots\mathcal{U}_{i+1}\mathcal{U}_1\cdots\mathcal{U}_{i+1} &= \mathcal{U}_1\cdots\mathcal{U}_i(I-\mathcal{V}_{i+1})\mathcal{U}_1\cdots\mathcal{U}_{i+1}\\ &= \mathcal{U}_1\cdots\mathcal{U}_i\mathcal{U}_1\cdots\mathcal{U}_{i+1} = \mathcal{U}_1\cdots\mathcal{U}_{i+1},\end{aligned}$$

$$\begin{aligned}\mathcal{U}_1\cdots\mathcal{U}_i\mathcal{V}_{i+1}\mathcal{U}_1\cdots\mathcal{U}_i\mathcal{V}_{i+1} &= \mathcal{U}_1\cdots\mathcal{U}_i\mathcal{V}_{i+1},\\ L_{i+1} = \ker\mathcal{U}_{i+1} &\subseteq \ker\mathcal{U}_1\cdots\mathcal{U}_{i+1},\end{aligned}$$

and $z \in \ker\mathcal{U}_1\cdots\mathcal{U}_{i+1}$ implies $\mathcal{U}_{i+1}z \in \operatorname{im}\mathcal{U}_1\cdots\mathcal{U}_i = L_i$, $z - \mathcal{V}_{i+1}z \in L_i$, hence $z \in L_i + L_{i+1} = L_{i+1}$. Now we can decompose

$$\begin{aligned}L_2 &= L_1 \oplus \mathcal{U}_1 L_2,\\ L_3 &= L_1 \oplus \mathcal{U}_1 L_2 \oplus \mathcal{U}_1\mathcal{U}_2 L_3 = L_2 \oplus \mathcal{U}_1\mathcal{U}_2 L_3,\\ L_{i+1} &= \underbrace{L_1 \oplus \mathcal{U}_1 L_2 \oplus \cdots \oplus}_{=L_i}\mathcal{U}_1\cdots\mathcal{U}_i L_{i+1} = L_i \oplus \mathcal{U}_1\cdots\mathcal{U}_i L_{i+1},\end{aligned}$$

and it follows that $\dim\mathcal{U}_1\cdots\mathcal{U}_i L_{i+1} = v_{i+1} - v_i$, $i = 1,\dots,k-1$. □

Proof (of Lemma B.1). We apply induction. For $k = 1$ the assertion is already proved, and the corresponding projector Q_1 is given by (B.22).
Let the assertion be true up to level k. We are going to show its validity for level $k+1$. We stress once more that we are dealing with structured triangular matrices.
We already know that $Q_0,\dots,Q_k$ are admissible, and, in particular, it holds that $Q_iQ_j = 0$, for $0 \le j < i \le k$. A closer look at the auxiliary matrix functions H_i (cf. (B.24)) shows that $H_iQ_1 = 0$, $H_iQ_2 = 0$, further $H_i\Pi_i = 0$, and $\Pi_{i-2}H_i = 0$.
Namely, $\Pi_1 H_3 = \Pi_1 P_0(I-\Pi_1)\Pi_2'\Pi_1 Q_2 = 0$, and $\Pi_{j-3}H_{j-1} = 0$, for $j \le i$, implies $\Pi_{i-2}H_i = 0$ (due to $\Pi_{i-2}H_\ell = 0$, $\Pi_{i-2}P_0(I-\Pi_{\ell-1}) = 0$, $\ell = 1,\dots,i-1$).
The functions $F_1,\dots,F_k$ (cf. (B.23)) are well-defined, and they have the properties

$$(F_k - F_j)\Pi_k = 0, \quad (F_k - F_j)\Pi_j = F_k - F_j, \quad \text{for } j = 1,\dots,k. \tag{B.33}$$

It follows that, for $j = 1,\dots,k$,

$$(I - M_0F_k)^{-1}(I - M_0F_j) = I + (I - M_0F_k)^{-1}M_0(F_k - F_j)\Pi_j.$$

Next we verify the property

$$\Pi_{j-1}M_kQ_j = 0, \quad j = 0,\dots,k. \tag{B.34}$$

From $G_jQ_j = 0$, $j = 0,\dots,k$, we know

$$M_0Q_j + Q_0Q_j + (I - M_0F_{j-1})(I - H_{j-1})P_0(I - \Pi_{j-1})Q_j = 0. \tag{B.35}$$

Multiplication by $(I - M_0F_k)^{-1}$ leads to

$$M_kQ_j + Q_0Q_j + \{I + (I - M_0F_k)^{-1}M_0(F_k - F_{j-1})\Pi_{j-1}\}(I - H_{j-1})P_0(I - \Pi_{j-1})Q_j = 0,$$

and further, taking account of $\Pi_{j-1}H_{j-1} = 0$, $\Pi_{j-1}P_0(I - \Pi_{j-1}) = 0$,

$$M_kQ_j + Q_0Q_j + (I-H_{j-1})P_0(I-\Pi_{j-1})Q_j = 0, \tag{B.36}$$

and hence $\Pi_{j-1}M_kQ_j = 0$, i.e., (B.34). Now it follows that $\Pi_kM_kQ_j = 0$, for $j = 0,\dots,k$, hence

$$\Pi_kM_k = \Pi_kM_k\Pi_k, \tag{B.37}$$

a property that will appear to be very helpful.
Recall that we already have a nonsingular $I - H_k$, as well as

$$\begin{aligned} G_{k+1} &= M_0 + Q_0 + (I-M_0F_k)(I-H_k)P_0(I-\Pi_k) \\ &= (I-M_0F_k)(I-H_k)\{(I-H_k)^{-1}M_k + I - \Pi_k\}, \end{aligned} \tag{B.38}$$

and G_{k+1} has rank $r_{k+1} = m - \ell_{\mu-k-1}$. We have to show the matrix function

$$Q_{k+1} := \Big(I - \sum_{j=0}^{k} Q_j(I-H_k)^{-1}M_k\Big)\Pi_k\mathcal{V}^{[k+2]}$$

to be a suitable projector. We check first whether $G_{k+1}Q_{k+1} = 0$ is satisfied. Derive (cf. (B.38))

$$\begin{aligned} G_{k+1}Q_{k+1} &= (I-M_0F_k)\{M_k + (I-H_k)(I-\Pi_k)\}\Big(I - \sum_{j=0}^{k} Q_j(I-H_k)^{-1}M_k\Big)\Pi_k\mathcal{V}^{[k+2]} \\ &= (I-M_0F_k)\Big\{M_k - \sum_{j=1}^{k} M_kQ_j(I-H_k)^{-1}M_k \\ &\qquad\qquad - (I-H_k)\sum_{j=0}^{k} Q_i(I-H_k)^{-1}M_k\Big\}\Pi_k\mathcal{V}^{[k+2]} \\ &= (I-M_0F_k)\Big\{I - H_k - \sum_{j=1}^{k} M_kQ_j - (I-H_k)\sum_{j=0}^{k} Q_j\Big\}\times \\ &\qquad\qquad \times (I-H_k)^{-1}M_k\Pi_k\mathcal{V}^{[k+2]}. \end{aligned} \tag{B.39}$$

From (B.36) we obtain, for $j = 1,\dots,k$,

$$\begin{aligned} M_kQ_j + (I-H_k)Q_j &= -Q_0Q_j - (I-H_{j-1})P_0(I-\Pi_{j-1})Q_j + (I-H_k)Q_j \\ &= P_0Q_j - H_kQ_j - (I-H_{j-1})(I-\Pi_{j-1})P_0Q_j \\ &= P_0Q_j - H_kQ_j - (I-H_{j-1})P_0Q_j + \Pi_{j-1}Q_j \\ &= -(H_k - H_{j-1})P_0Q_j + \Pi_{j-1}Q_j \end{aligned}$$

and, therefore,

$$\sum_{j=1}^{k}(M_kQ_j + (I-H_k)Q_j) = \sum_{j=1}^{k}\Pi_{j-1}Q_j - \sum_{j=1}^{k}(H_k - H_{j-1})P_0Q_j$$
$$= \sum_{j=1}^{k}\Pi_{j-1}Q_j - H_k.$$

The last relation becomes true because of $(H_k - H_0)Q_1 = 0$, $(H_k - H_1)Q_2 = 0$, and the construction of H_i (cf. (B.24)),

$$\sum_{j=1}^{k}(H_k - H_{j-1})P_0Q_j = \sum_{j=3}^{k}(H_k - H_{j-1})P_0Q_j$$
$$= \sum_{j=3}^{k}\Big[\sum_{\nu=j}^{k}\sum_{\ell=2}^{\nu-1}(I-H_{\ell-1})P_0(I-\Pi_{\ell-1})\Pi'_\ell\Pi_{\nu-1}Q_\nu\Big]P_0Q_j$$
$$= \sum_{j=3}^{k}\sum_{\ell=2}^{j-1}(I-H_{\ell-1})P_0(I-\Pi_{\ell-1})\Pi'_\ell\Pi_{j-1}Q_j = H_k.$$

Together with (B.39) this yields

$$G_{k+1}Q_{k+1} = (I-M_0F_k)\Big\{I - H_k - \Big(\sum_{j=1}^{k}\Pi_{j-1}Q_j - H_k\Big) - Q_0\Big\}(I-H_k)^{-1}M_k\Pi_k\mathcal{V}^{[k+2]}$$
$$= (I-M_0F_k)\Big\{I - Q_0 - \sum_{j=1}^{k}\Pi_{j-1}Q_j\Big\}(I-H_k)^{-1}M_k\Pi_k\mathcal{V}^{[k+2]}$$
$$= (I-M_0F_k)\Pi_k(I-H_k)^{-1}M_k\Pi_k\mathcal{V}^{[k+2]}. \tag{B.40}$$

For more specific information on $(I-H_k)^{-1}$ we consider the equation $(I-H_k)z = w$, i.e. (cf. (B.24))

$$(I-H_{k-1})z - \sum_{\ell=2}^{k-1}(I-H_{\ell-1})P_0(I-\Pi_{\ell-1})\Pi'_\ell\Pi_{k-1}Q_kz = w. \tag{B.41}$$

Because of $\Pi_{k-1}H_{k-1} = 0$, $\Pi_{k-1}H_{\ell-1} = 0$, $\Pi_{k-2}P_0(I-\Pi_{\ell-1}) = 0$, multiplication of (B.41) by $\Pi_{k-1}Q_k = \Pi_{k-1}Q_k\Pi_{k-1}$ yields $\Pi_{k-1}Q_kz = \Pi_{k-1}Q_kw$, such that

$$z = (I-H_{k-1})^{-1}\Big\{w + \sum_{\ell=2}^{k-1}(I-H_{\ell-1})P_0(I-\Pi_{\ell-1})\Pi'_\ell\Pi_{k-1}Q_kw\Big\}$$

results, and further,

$$(I-H_k)^{-1} = (I-H_{k-1})^{-1}\Big(I - \sum_{\ell=2}^{k-1}(I-H_{\ell-1})P_0(I-\Pi_{\ell-1})\Pi'_\ell\Pi_{k-1}Q_k\Big)$$

$$= (I-H_3)^{-1}\Big(I+\sum_{\ell=2}^{3}(I-H_{\ell-1})P_0(I-\Pi_{\ell-1})\Pi'_\ell\Pi_3Q_4\Big)\times\cdots$$
$$\cdots\times\Big(I+\sum_{\ell=2}^{k-1}(I-H_{\ell-1})P_0(I-\Pi_{\ell-1})\Pi'_\ell\Pi_{k-1}Q_k\Big)$$
$$= (I+P_0Q_1\Pi'_2\Pi_2Q_3)\times\cdots\times\Big(I+\sum_{\ell=2}^{k-1}(I-H_{\ell-1})P_0(I-\Pi_{\ell-1})\Pi'_\ell\Pi_{k-1}Q_k\Big).$$

This shows that $\Pi_k(I-H_k)^{-1}=\Pi_k$ holds true. On the other hand $F_k\Pi_k=0$ is also given, which leads to
$$G_{k+1}Q_{k+1}=\Pi_kM_k\Pi_k\mathcal{V}^{[k+2]}.$$

With the help of (B.37), and taking into account that $\ker\Pi_kM_k=\ker M_0^2M_1\cdots M_k$, we arrive at
$$G_{k+1}Q_{k+1}=\Pi_kM_k\mathcal{V}^{[k+2]}=0,$$

that is, the matrix function Q_{k+1} satisfies the condition $\operatorname{im}Q_{k+1}\subseteq\ker G_{k+1}$. The inclusions (cf. (B.29), (B.30))
$$\ker\Pi_{i-1}M_{i-1}=\ker M_0^2M_1\cdots M_{i-1}\subset\ker\Pi_iM_i=\ker M_0^2M_1\cdots M_i$$

are valid for $i=1,\dots,k$. This leads to
$$\operatorname{im}\mathcal{V}^{[1]}\subset\operatorname{im}\mathcal{V}^{[2]}\subset\cdots\subset\mathcal{V}^{[\kappa+2]}$$

which allows an application of Lemma B.3. We make use of the structural properties
$$\operatorname{rank}M_0^2M_1\cdots M_i=\operatorname{rank}N^{i+2}=\ell_1+\cdots+\ell_{\mu-i-2},$$
$$\operatorname{rank}\mathcal{V}^{[i+2]}=m-(\ell_1+\cdots+\ell_{\mu-i-2})=\ell_{\mu-i-1}+\cdots+\ell_\mu,$$

so that Lemma B.3 yields
$$\operatorname{rank}\mathcal{U}^{[1]}\cdots\mathcal{U}^{[k+1]}\mathcal{V}^{[k+2]}=\operatorname{rank}\mathcal{V}^{[k+2]}-\operatorname{rank}\mathcal{V}^{[k+1]}=\ell_{\mu-k-1}.$$

Writing Q_{k+1} in the form
$$Q_{k+1}=\Big(I-\sum_{j=0}^{k}Q_j(I-H_k)^{-1}M_k\Pi_k\Big)\Pi_k\mathcal{V}^{[k+2]},$$

and realizing that the first factor is nonsingular, we conclude
$$\operatorname{rank}Q_{k+1}=\operatorname{rank}\Pi_k\mathcal{V}^{[k+2]}=\ell_{\mu-k-1}=m-\operatorname{rank}G_{k+1}.$$

Applying Lemma B.3 again we derive, for $j=0,\dots,k$,

$$Q_{k+1}Q_j = \Big(I - \sum_{j=0}^{k} Q_j(I-H_k)^{-1}M_k\Big)\Pi_k\mathcal{V}^{[k+2]}Q_j,$$

$$\begin{aligned}\Pi_k\mathcal{V}^{[k+2]}Q_j &= \mathcal{U}^{[1]}\cdots\mathcal{U}^{[k+1]}\mathcal{V}^{[k+2]}Q_j\\ &= \mathcal{U}^{[1]}\cdots\mathcal{U}^{[k+1]}\mathcal{V}^{[k+2]}\mathcal{U}^{[1]}\cdots\mathcal{U}^{[j]}Q_j\\ &= \mathcal{U}^{[1]}\cdots\mathcal{U}^{[k+1]}\mathcal{V}^{[k+2]}\mathcal{U}^{[1]}\cdots\mathcal{U}^{[j]}\mathcal{V}^{[j+1]} = 0,\end{aligned}$$

such that $Q_{k+1}Q_j = 0,\ j = 0,\dots,k$, and furthermore $Q_{k+1}Q_{k+1} = Q_{k+1}$. This completes the proof that Q_{k+1} is a suitable projector function, and that $Q_0,\dots,Q_k,Q_{k+1}$ are admissible.
It remains to verify (B.29)–(B.31) for $i = k+1$, to consider the rank of G_{k+2} as well as to show the nonsingularity of $I - H_{k+1}$.
First we consider the rank of G_{k+2}. Following Proposition 2.5(3) it holds that

$$\operatorname{im} G_{k+2} = \operatorname{im} G_{k+1} \oplus \operatorname{im} \mathcal{W}_{k+1}\Pi_k Q_{k+1},$$

with a projector $\mathcal{W}_{k+1}$ such that $\ker \mathcal{W}_{k+1} = \operatorname{im} G_{k+1}$. Because of

$$\begin{aligned}\operatorname{im} G_{k+1} &= \operatorname{im} G_k \oplus \operatorname{im} \mathcal{W}_0\Pi_{k-1}Q_k\\ &= \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0 Q_0 \oplus \cdots \oplus \operatorname{im} \mathcal{W}_0\Pi_{k-1}Q_k\\ &= \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0(Q_0 + \cdots + \Pi_{k-1}Q_k)\\ &= \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0(I - \Pi_k)\end{aligned}$$

we may choose the projector

$$\mathcal{W}_{k+1} = \mathcal{W}_0\Pi_k = \mathcal{W}_0\Pi_k\mathcal{W}_0.$$

This leads to

$$\operatorname{im} G_{k+2} = \operatorname{im} G_{k+1} \oplus \operatorname{im} \mathcal{W}_0\Pi_k Q_{k+1},$$

as well as to

$$\begin{aligned}r_{k+2} &= r_{k+1} + \operatorname{rank} \mathcal{W}_0\Pi_k Q_{k+1} = r_{k+1} + \operatorname{rank}\,[\Pi_k Q_{k+1}]_{\mu\mu}\\ &= r_{k+1} + \operatorname{rank} \mathcal{U}^{[1]}_{\mu\mu}\cdots\mathcal{U}^{[k+1]}_{\mu\mu}\mathcal{V}^{[k+2]}_{\mu\mu} = m - \ell_{\mu-k-1} + (\ell_{\mu-k-1} - \ell_{\mu-k-2})\\ &= m - \ell_{\mu-k-2}.\end{aligned}$$

Therefore, to show that $\operatorname{rank} \mathcal{U}^{[1]}_{\mu\mu}\cdots\mathcal{U}^{[k+1]}_{\mu\mu}\mathcal{V}^{[k+2]}_{\mu\mu} = \ell_{\mu-k-1} - \ell_{\mu-k-2}$ we recall that

$$\begin{aligned}&\mathcal{V}^{[1]}_{\mu\mu} \text{ projects onto } \ker N_{\mu-1,\mu},\\ &\mathcal{V}^{[2]}_{\mu\mu} \text{ projects onto } \ker N_{\mu-2,\mu-1}N_{\mu-1,\mu},\\ &\dots\\ &\mathcal{V}^{[k+1]}_{\mu\mu} \text{ projects onto } \ker N_{\mu-k-1,\mu-k}\cdots N_{\mu-1,\mu}\end{aligned}$$

and

$$\mathcal{V}_{\mu\mu}^{[k+2]} \text{ projects onto } \ker N_{\mu-k-2,\mu-k-1}\cdots N_{\mu-1,\mu},$$

and

$$\operatorname{im}\mathcal{V}_{\mu\mu}^{[1]} \subset \operatorname{im}\mathcal{V}_{\mu\mu}^{[2]} \subset \cdots \subset \operatorname{im}\mathcal{V}_{\mu\mu}^{[k+2]},$$
$$\operatorname{rank}\mathcal{V}_{\mu\mu}^{[i]} = \ell_\mu - \ell_{\mu-i}, \quad i = 1,\ldots,k+2.$$

Here, Lemma B.3 applies again, and it follows that

$$\begin{aligned}&\operatorname{rank}\mathcal{U}_{\mu\mu}^{[1]}\cdots\mathcal{U}_{\mu\mu}^{[k+1]}\mathcal{V}_{\mu\mu}^{[k+2]} = \operatorname{rank}\mathcal{V}_{\mu\mu}^{[k+2]} - \operatorname{rank}\mathcal{V}_{\mu\mu}^{[k+1]}\\ &= \ell_\mu - \ell_{\mu-k-2} - (\ell_\mu - \ell_{\mu-k-1}) = \ell_{\mu-k-1} - \ell_{\mu-k-2}.\end{aligned}$$

So we are done with the range and rank of G_{k+2}.
In the next step we provide G_{k+2} itself (cf. Section 2.2.2). Compute

$$\begin{aligned}G_{k+2} &= G_{k+1} + \Pi_k Q_{k+1} - \sum_{j=1}^{k+1} G_j P_0 \Pi_j' \Pi_k Q_{k+1}\\
&= M_0 + Q_0 + (I - M_0F_k)(I - H_k)P_0(I - \Pi_k) + \Pi_k Q_{k+1} - M_0\Pi_1'\Pi_k Q_{k+1}\\
&\quad - \sum_{j=2}^{k+1}\{M_0 + (I - M_0F_{j-1})(I - H_{j-1})P_0(I - \Pi_{j-1})\}\Pi_j'\Pi_k Q_{k+1}\\
&= M_0 + Q_0 + (I - M_0F_k)P_0(I - \Pi_k) - (I - M_0F_k)H_k + \Pi_k Q_{k+1}\\
&\quad - \sum_{j=1}^{k+1} M_0\Pi_j'\Pi_k Q_{k+1} - \sum_{j=2}^{k+1}(I - M_0F_{j-1})(I - H_{j-1})P_0(I - \Pi_{j-1})\Pi_j'\Pi_k Q_{k+1}\end{aligned}$$

and rearrange (cf. (B.23), (B.24)) certain terms to

$$(I - M_0F_k)P_0(I - \Pi_k) + \Pi_k Q_{k+1} - M_0\sum_{j=1}^{k+1} P_0\Pi_j'\Pi_k Q_{k+1} = (I - M_0F_{k+1})P_0(I - \Pi_{k+1})$$

and

$$\begin{aligned}&(I - M_0F_k)H_k + \sum_{j=2}^{k}(I - M_0F_{j-1})(I - H_{j-1})P_0(I - \Pi_{j-1})\Pi_j'\Pi_k Q_{k+1}\\
&= (I - M_0F_k)\Big\{H_k + \sum_{j=2}^{k}(I - M_0F_k)^{-1}(I - M_0F_{j-1})(I - H_{j-1}) \times\\
&\qquad\qquad \times P_0(I - \Pi_{j-1})\Pi_j'\Pi_k Q_{k+1}\Big\}\\
&= (I - M_0F_k)\Big\{H_k + \sum_{j=2}^{k}(I - H_{j-1})P_0(I - \Pi_{j-1})\Pi_j'\Pi_k Q_{k+1}\Big\}\\
&= (I - M_0F_k)H_{k+1} = (I - M_0F_{k+1})(I - M_0F_{k+1})^{-1}(I - M_0F_k)H_{k+1}\end{aligned}$$

$$= (I - M_0 F_{k+1}) H_{k+1} = (I - M_0 F_{k+1}) H_{k+1} P_0 (I - \Pi_{k+1}),$$

which leads to

$$\begin{aligned} G_{k+2} &= M_0 + Q_0 + (I - M_0 F_{k+1}) P_0 (I - \Pi_{k+1}) - (I - M_0 F_{k+1}) H_{k+1} P_0 (I - \Pi_{k+1}) \\ &= M_0 + Q_0 + (I - M_0 F_{k+1})(I - H_{k+1}) P_0 (I - \Pi_{k+1}), \end{aligned}$$

and we are done with G_{k+2} (cf. (B.31)).
Next, $I - H_{k+1}$ is nonsingular, since $(I - H_{k+1})z = 0$ implies $\Pi_k Q_{k+1} z = 0$, thus $(I - H_k)z = 0$, and, finally $z = 0$ due to the nonsingularity of $(I - H_k)$.
To complete the proof of Lemma B.1 we have to verify (B.29) and (B.30) for $i = k+1$, supposing $\ker \Pi_{k-1} M_{k-1} \subseteq \ker \Pi_k M_k$, $\ker \Pi_k M_k = \ker M_0^2 M_1 \cdots M_k$ are valid. From $\Pi_k M_k = \Pi_k M_k \Pi_k$ (cf. (B.37)) and $\ker M_0^2 M_1 \cdots M_k = \ker \Pi_k M_k = \ker \mathcal{U}^{[k+2]}$ we obtain the relations

$$\begin{aligned} \Pi_{k+1} M_{k+1} &= \Pi_k \mathcal{U}^{[k+2]} M_{k+1} = \Pi_k (M_0^2 M_1 \cdots M_k)^- M_0^2 M_1 \cdots M_k M_{k+1}, \\ M_0^2 M_1 \cdots M_{k+1} &= M_0^2 M_1 \cdots M_k \mathcal{U}^{[k+2]} M_{k+1} \\ &= M_0^2 M_1 \cdots M_k (\Pi_k M_k)^- \Pi_k M_k \mathcal{U}^{[k+2]} M_{k+1} \\ &= M_0^2 M_1 \cdots M_k (\Pi_k M_k)^- \Pi_k M_k \Pi_k \mathcal{U}^{[k+2]} M_{k+1} \\ &= M_0^2 M_1 \cdots M_k (\Pi_k M_k)^- \Pi_k M_k \Pi_{k+1} M_{k+1}, \end{aligned}$$

hence $\ker \Pi_{k+1} M_{k+1} = \ker M_0^2 M_1 \cdots M_{k+1}$ holds true. Additionally, from

$$\begin{aligned} \Pi_{k+1} M_{k+1} &= \Pi_{k+1} (I - M_0 F_{k+1})^{-1} (I - M_0 F_k) M_k \\ &= \Pi_{k+1} [I + (I - M_0 F_{k+1})^{-1} M_0 (F_{k+1} - F_k) \Pi_k] M_k \\ &= \Pi_{k+1} [I + (I - M_0 F_{k+1})^{-1} M_0 (F_{k+1} - F_k) \Pi_k] \Pi_k M_k \end{aligned}$$

we conclude the inclusion

$$\ker \Pi_k M_k \subseteq \ker \Pi_{k+1} M_{k+1}.$$

□

Appendix C
Analysis

C.1 A representation result

Proposition C.1. *Let the function $d : \Omega \times \mathcal{I} \to \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^m$ open, $\mathcal{I} \subseteq \mathbb{R}$ an interval, be continuously differentiable, and let the partial Jacobian $d_x(x,t)$ have constant rank r on $\Omega \times \mathcal{I}$. Let $x_* : \mathcal{I}_* \to \mathbb{R}^m$, $\mathcal{I}_* \subseteq \mathcal{I}$, be a continuous function with values in Ω, i.e., $x_*(t) \in \Omega$, $t \in \mathcal{I}_*$. Put $u_*(t) := d(x_*(t),t)$, $t \in \mathcal{I}_*$.*
Then, if u_* is continuously differentiable the inclusion

$$u_*'(t) - d_t(x_*(t),t) \in \operatorname{im} d_x(x_*(t),t), \quad t \in \mathcal{I}_* \tag{C.1}$$

is valid, and there exists a continuous function $w_* : \mathcal{I}_* \to \mathbb{R}^m$ such that

$$u_*'(t) - d_t(x_*(t),t) = d_x(x_*(t),t) w_*(t), \quad t \in \mathcal{I}_*. \tag{C.2}$$

If $d_x(x_*(t),t)$ is injective, then $w_*(t)$ is uniquely determined by (C.2).

Proof. Derive, for $t \in \mathcal{I}_*$,

$$\begin{aligned} u_*'(t) &= \lim_{\tau\to 0} \frac{1}{\tau}\big(d(x_*(t+\tau),t+\tau) - d(x_*(t),t)\big) \\ &= \lim_{\tau\to 0} \Big\{ \frac{1}{\tau}\big(d(x_*(t+\tau),t+\tau) - d(x_*(t),t+\tau)\big) \\ &\qquad\qquad + \frac{1}{\tau}\big(d(x_*(t),t+\tau) - d(x_*(t),t)\big) \Big\}. \end{aligned}$$

Since the expression $\frac{1}{\tau}(d(x_*(t),t+\tau) - d(x_*(t),t))$ has, for $\tau \to 0$, the limit $d_t(x_*(t),t)$, the other difference must possess a limit, too, i.e.

$$e_*(t) := \lim_{\tau\to 0} \frac{1}{\tau}\big(d(x_*(t+\tau),t+\tau) - d(x_*(t),t+\tau)\big)$$

is well-defined, and $u_*'(t) = e_*(t) + d_t(x_*(t),t)$. Rewrite, for fixed $t \in \mathcal{I}_*$,

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$$\begin{aligned} e_*(t) &= \lim_{\tau\to 0}\int_0^1 d_x(x_*(t)+s(x_*(t+\tau)-x_*(t)),t+\tau)ds\frac{1}{\tau}(x_*(t+\tau)-x_*(t)) \\ &=: \lim_{\tau\to 0} E(\tau)g(\tau), \end{aligned}$$

with

$$\begin{aligned} E(\tau) &:= \int_0^1 d_x(x_*(t)+s(x_*(t+\tau)-x_*(t)),t+\tau))ds, \\ g(\tau) &:= \frac{1}{\tau}(x_*(t+\tau)-x_*(t)), \quad \tau\in(-\rho,\rho),\ \rho>0 \text{ small.} \end{aligned}$$

Recall that $g(\tau)$ has not necessarily a limit for $\tau\to 0$, but we can make use of the existing limits $\lim_{\tau\to 0} E(\tau)g(\tau)=e_*(t)$ and $\lim_{\tau\to 0} E(\tau)=d_x(x_*(t),t)=E(0)$. The matrix $E(\tau)\in L(\mathbb{R}^m,\mathbb{R}^n)$ depends continuously on τ, and $E(0)$ has rank r, so that, at least for all sufficiently small τ, it holds that $\operatorname{rank} E(\tau)\geq r$. On the other hand, for all sufficiently small τ and $z\in\mathbb{R}^m$, the decomposition

$$[\operatorname{im} d_x(x_*(t),t)]^\perp \oplus \operatorname{im} d_x(x_*(t)+z,t+\tau)=\mathbb{R}^n \tag{C.3}$$

is valid. If $\mathcal{V}_*\in L(\mathbb{R}^n)$ denotes the projector onto $[\operatorname{im} d_x(x_*(t),t)]^\perp$ according to the decomposition (C.3), then we have, for all sufficiently small τ, that

$$\mathcal{V}_*E(\tau)=\int_0^1 \mathcal{V}_* d_x(x_*(t)+s(x_*(t+\tau)-x_*(t)),t+\tau)ds=0.$$

$\mathcal{V}_*$ has rank $n-r$, hence $\mathcal{V}_*E(\tau)=0$ implies $\operatorname{rank} E(\tau)\leq r$ for all τ being sufficiently small.
Now, $E(\tau)$ is, for small τ, a constant-rank matrix, so that $E(\tau)^+$ and $\mathcal{U}(\tau):=E(\tau)E(\tau)^+$ are also continuous in τ. This leads to

$$\begin{aligned} e_*(t) &= \lim_{\tau\to 0} E(\tau)g(\tau)=\lim_{\tau\to 0}\mathcal{U}(\tau)E(\tau)g(\tau) \\ &= \lim_{\tau\to 0}\mathcal{U}(\tau)\cdot\lim_{\tau\to 0} E(\tau)g(\tau)=\mathcal{U}(0)\cdot e_*(t), \end{aligned}$$

which means $e_*(t)\in\operatorname{im} d_x(x_*(t),t)$, or, equivalently,

$$u_*'(t)-d_t(x_*(t),t)\in\operatorname{im} d_x(x_*(t),t), \quad t\in\mathcal{I}_*,$$

that is, we are done with the inclusion (C.1). Next, taking any continuous reflexive generalized inverse $d_x(x,t)^-$ to $d_x(x,t)$, the function $w_*:\mathcal{I}_*\to\mathbb{R}^m$,

$$w_*(t):=d_x(x_*(t),t)^-(u_*'(t)-d_t(x_*(t),t)), \quad t\in\mathcal{I}^*,$$

is continuous and satisfies

$$\begin{aligned} d_x(x_*(t),t)w_*(t) &= d_x(x_*(t),t)d_x(x_*(t),t)^-(u_*'(t)-d_t(x_*(t),t)) \\ &= u_*'(t)-d_t(x_*(t),t), \end{aligned}$$

since (C.1) is valid, and $d_x(x_*(t),t)d_x(x_*(t),t)^-$ is a projector onto $\operatorname{im} d_x(x_*(t),t)$. Finally, (C.2) together with (C.1) define $w_*(t)$ uniquely, if $d_x(x_*(t),t)$ is injective, since then $d_x(x_*(t),t)^- d_x(x_*(t),t) = I$, independently of the special choice of the generalized inverse. □

Notice that one can also choose $w_*(t) = x_*'(t)$ to satisfy (C.2) supposing x_* is known to be continuously differentiable.

C.2 ODEs

Proposition C.2. *Let the function $g \in C(\mathcal{I}, \mathbb{R}^m)$, $\mathcal{I} = [0,\infty)$, satisfy the one-sided Lipschitz condition*

$$\langle g(x,t) - g(\bar{x},t), x - \bar{x} \rangle \leq \gamma |x - \bar{x}|^2,\ x, \bar{x} \in \mathbb{R}^m,\ t \in \mathcal{I}, \tag{C.4}$$

with a constant $\gamma \leq 0$.
Then the ODE

$$x'(t) = g(x(t),t) \tag{C.5}$$

has the following properties:

(1) *The IVP for (C.5) with the initial condition*

$$x(t_0) = x_0,\ t_0 \in \mathcal{I},\ x_0 \in \mathbb{R}^m,$$

is uniquely solvable, and the solution is defined on $\mathcal{I}$.

(2) *Each pair of solutions $x(.)$, $\bar{x}(.)$ satisfies the inequality*

$$|x(t) - \bar{x}(t)| \leq e^{\gamma t} |x(0) - \bar{x}(0)|,\ t \in \mathcal{I}.$$

(3) *The ODE has at most one stationary solution.*

Proof. (1), (2): Let $x(.)$, $\bar{x}(.)$ be arbitrary solutions defined on $[0,\tau)$, with $\tau > 0$. Derive for $t \in [0,\tau)$:

$$\begin{aligned}\frac{d}{dt}|x(t) - \bar{x}(t)|^2 &= 2\langle g(x(t),t) - g(\bar{x}(t),t), x(t) - \bar{x}(t) \rangle \\ &\leq 2\gamma |x(t) - \bar{x}(t)|^2.\end{aligned}$$

By means of Gronwall's lemma we find

$$|x(t) - \bar{x}(t)| \leq e^{\gamma t} |x(0) - \bar{x}(0)|,\ t \in [0,\tau). \tag{C.6}$$

The inequality

$$|x(t)| - |x(0) - \bar{x}(0)| \leq |\bar{x}(t)| \leq |x(t)| + |x(0) - \bar{x}(0)|,\ t \in [0,\tau)$$

is a particular consequence of (C.6). It shows that $x(t)$ grows unboundedly for $t \to \tau$ if $\bar{x}(t)$ does, and vice versa.
This means that all solutions of the ODE can simultaneously be continued through τ or not.
Assume that $\tau_* > 0$ exists such that all IVPs for (C.5) and $x(0) = x_0$ have solutions $x(.,x_0)$ defined on $[0,\tau_*)$, but $x(t,x_0)$ grows unboundedly, if $t \to \tau_*$. Fix $x_* \in \mathbb{R}^m$ and put $x_{**} := x(\frac{1}{2}\tau_*, x_*)$.
The solution $x(.,x_{**})$ is also defined on $[0,\tau_*)$, in particular at $t = \frac{1}{2}\tau_*$. However, this contradicts the property $x(\frac{1}{2}\tau_*, x_{**}) = x(\tau_*, x_*)$. In consequence, such a value τ_* does not exist, and all solutions can be continued on the infinite interval.
(3): For two stationary solutions c and $\bar{c}$, (2) implies

$$|c - \bar{c}| \le e^{\gamma t}|c - \bar{c}| \to 0 \quad (t \to \infty),$$

and hence $c = \bar{c}$. □

Lemma C.3. *Given a real valued $m \times m$ matrix C, then:*

(1) *If all eigenvalues of C have strictly negative real parts, then there exist a constant $\beta < 0$ and an inner product $\langle \cdot, \cdot \rangle$ for $\mathbb{R}^m$, such that*

$$\langle Cz, z \rangle \le \beta |z|^2, \text{ for all } z \in \mathbb{R}^m, \tag{C.7}$$

and vice versa.

(2) *If all eigenvalues of C have nonpositive real parts, and the eigenvalues on the imaginary axis are nondefective, then there is an inner product $\langle \cdot, \cdot \rangle$ for $\mathbb{R}^m$, such that*

$$\langle Cz, z \rangle \le 0, \text{ for all } z \in \mathbb{R}^m, \tag{C.8}$$

and vice versa.

Proof. Let $\sigma_1, \ldots, \sigma_m \in \mathbb{C}$ denote the eigenvalues of C, and let $T \in L(\mathbb{C}^m)$ be the transformation into Jordan canonical form J such that, with entries $\delta_1, \ldots, \delta_{m-1}$ being 0 or 1,

$$J = T^{-1}CT = \begin{bmatrix} \sigma_1 & \delta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \delta_{m-1} \\ & & & \sigma_m \end{bmatrix}$$

is given. For $\varepsilon > 0$, we form further

$$J_\varepsilon = D_\varepsilon^{-1} J D_\varepsilon = \begin{bmatrix} \sigma_1 & \varepsilon\delta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon\delta_{m-1} \\ & & & \sigma_m \end{bmatrix}, \quad D_\varepsilon = \begin{bmatrix} \varepsilon & & & \\ & \varepsilon^2 & & \\ & & \ddots & \\ & & & \varepsilon^m \end{bmatrix}.$$

J_ε and C are similar, $J_\varepsilon = D_\varepsilon^{-1}T^{-1}CTD_\varepsilon = (TD_\varepsilon)^{-1}C(TD_\varepsilon)$.
By $\langle z,y\rangle_\varepsilon := \langle (TD_\varepsilon)^{-1}z,(TD_\varepsilon)^{-1}y\rangle_2$ and $|z|_\varepsilon := |(TD_\varepsilon)^{-1}z|_2,\ z,y \in \mathbb{C}^m$, we introduce an inner product and the corresponding norm for C^m. Moreover, the expression

$$a_\varepsilon(u,v) := \operatorname{Re}\langle (TD_\varepsilon)^{-1}u,(TD_\varepsilon)^{-1}v\rangle_2,\ u,v \in \mathbb{R}^m,$$

defines an inner product for $\mathbb{R}^m$.
Recall that the relation

$$\operatorname{Re}\langle Mz,z\rangle_2 = \langle \frac{1}{2}(M+M^*)z,z\rangle_2 \le \lambda_{max}(\frac{1}{2}(M+M^*))|z|_2^2,\ z \in \mathbb{C}^m,$$

is valid for each arbitrary matrix $M \in L(\mathbb{C}^m)$, and, in particular,

$$\operatorname{Re}\langle J_\varepsilon z,z\rangle_2 \le \lambda_{max}(\frac{1}{2}(J_\varepsilon + J_\varepsilon^*))|z|_2^2,\ z \in \mathbb{R}^m.$$

We have

$$\frac{1}{2}(J_\varepsilon + J_\varepsilon^*) = \begin{bmatrix} \operatorname{Re}\sigma_1 & \frac{\varepsilon}{2}\delta_1 & & \\ \frac{\varepsilon}{2}\delta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \frac{\varepsilon}{2}\delta_{m-1} \\ & & \frac{\varepsilon}{2}\delta_{m-1} & \operatorname{Re}\sigma_m \end{bmatrix} \xrightarrow[\varepsilon\to 0]{} \begin{bmatrix} \operatorname{Re}\sigma_1 & & \\ & \ddots & \\ & & \ddots & \\ & & & \operatorname{Re}\sigma_m \end{bmatrix}$$

and $\lambda_{max}(\frac{1}{2}(J_\varepsilon + J_\varepsilon^*)) \xrightarrow[\varepsilon\to 0]{} \max_{i=1,\dots,m} \operatorname{Re}\sigma_i$.
If all eigenvalues of C have strictly negative real parts, that is, $\max_{i=1,\dots,m} \operatorname{Re}\sigma_i =: 2\beta < 0$, then choose a value $\varepsilon > 0$ such that $\lambda_{max}(\frac{1}{2}(J_\varepsilon + J_\varepsilon^*)) \le \beta < 0$.
If the eigenvalues $\sigma_1,\dots,\sigma_m$ have zero real part, but these eigenvalues are nondefective, and $\operatorname{Re}\sigma_j < 0,\ j = s,\dots,m$, then it holds that

$$\frac{1}{2}(J_\varepsilon + J_\varepsilon^*) = \left[\begin{array}{ccc|c} 0 & & & \\ & \ddots & & \\ & & 0 & \\ \hline & & & \frac{1}{2}(\check{J}_\varepsilon + \check{J}_\varepsilon^*) \end{array}\right] \xrightarrow[\varepsilon\to 0]{} \left[\begin{array}{ccc|ccc} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ \hline & & & \operatorname{Re}\sigma_s & & \\ & & & & \ddots & \\ & & & & & \operatorname{Re}\sigma_m \end{array}\right],$$

$$\check{J}_\varepsilon = \begin{bmatrix} \sigma_s & \delta_s & & \\ & \ddots & \ddots & \\ & & \ddots & \delta_{m-1} \\ & & & \sigma_m \end{bmatrix} \in L(\mathbb{C}^{m-s+1}).$$

Now we fix an $\varepsilon > 0$ such that $\lambda_{max}(\frac{1}{2}(\check{J}_\varepsilon + \check{J}_\varepsilon^*)) \le 0,\ \lambda_{max}(\frac{1}{2}(J_\varepsilon + J_\varepsilon^*)) = 0$.
In both cases it results that

$$
\begin{aligned}
a_\varepsilon(Cx,x) &= \operatorname{Re}\langle (TD_\varepsilon)^{-1}Cx, (TD_\varepsilon)^{-1}x\rangle_2 \\
&= \operatorname{Re}\langle \underbrace{(TD_\varepsilon)^{-1}C(TD_\varepsilon)}_{J_\varepsilon}(TD_\varepsilon)^{-1}x, (TD_\varepsilon)^{-1}x\rangle_2 \\
&\leq \lambda_{max}(\frac{1}{2}(J_\varepsilon + J_\varepsilon^*))|(TD_\varepsilon)^{-1}x|_2^2 \\
&= \lambda_{max}(\frac{1}{2}(J_\varepsilon + J_\varepsilon^*))|x|_\varepsilon^2,
\end{aligned}
$$

and hence the inequalities (C.7) and (C.8) are proved to follow from the given properties of C.
The converse assertions become evident if one considers the homogeneous ODE $x'(t) = Cx(t)$. All its solutions satisfy

$$\frac{d}{dt}|x(t)|^2 = 2\langle Cx(t), x(t)\rangle \leq 2\beta|x(t)|^2,\ t \geq 0,$$

thus $|x(t)| \leq e^{\beta t}|x(0)|$, $t \geq 0$ in the first case, and

$$\frac{d}{dt}|x(t)|^2 = 2\langle Cx(t), x(t)\rangle \leq 0,\ t \geq 0,$$

thus $|x(t)| \leq |x(0)|$, $t \geq 0$ in the second case. □

C.3 Basics for evolution equations

This section summarizes basic spaces and their properties for the treatment of evolution equations (see, e.g., [217]).

1. Dual space. Let V be a real Banach space. Then, V^* denotes the set of all linear continuous functionals on V, i.e., the set of all linear continuous maps $f: V \to \mathbb{R}$. Furthermore,
$$\langle f, v\rangle := f(v) \qquad \text{for all} \quad v \in V$$
and
$$\|f\|_{V^*} := \sup_{\|v\|_V \leq 1} |\langle f, v\rangle|.$$
In this way, V^* becomes a real Banach space. It is called the dual space to V.
2. Reflexive Banach space. Let V be a real Banach space. Then, V is called *reflexive* if $V = V^{**}$.
3. Evolution triple. The spaces $V \subseteq H \subseteq V^*$ are called an evolution triple if

   (i) V is a real, separable, and reflexive Banach space,
   (ii) H is a real, separable Hilbert space,
   (iii) the embedding $V \subseteq H$ is continuous, i.e.,

$$\|v\|_H \leq \text{const}\|v\|_V \qquad \text{for all} \quad v \in V,$$

and V is dense in H.

Below, Proposition C.5 explains how the inclusion $H \subseteq V^*$ is to be understood.

4. The Lebesgue space $L_p(t_0,T;V)$ of vector valued functions. Let V be a Banach space, $1 < p < \infty$, and $t_0 < T < \infty$. The space $L_p(t_0,T;V)$ consists of all measurable functions $v: (t_0,T) \to V$ for which

$$\|v\|_p := \left(\int_{t_0}^{T} \|v(t)\|_V^p \,\mathrm{d}t\right)^{\frac{1}{p}} < \infty.$$

The dual space of $L_p(t_0,T;V)$ is given by $L_q(t_0,T;V^*)$ where $p^{-1}+q^{-1}=1$.

5. Generalized derivatives. Let X and Y be Banach spaces. Furthermore, let $u \in L_1(t_0,T;X)$ and $w \in L_1(t_0,T;Y)$. Then, the function w is called the generalized derivative of the function u on (t_0,T) if

$$\int_{t_0}^{T} \varphi'(t)u(t)\,\mathrm{d}t = -\int_{t_0}^{T} \varphi(t)w(t)\,\mathrm{d}t \qquad \text{for all} \quad \varphi \in C_0^\infty(t_0,T).$$

The last equation includes the requirement that the integrals on both sides belong to $X \cap Y$.

6. The Sobolev space $W_2^1(t_0,T;V,H)$. Let $V \subseteq H \subseteq V^*$ be an evolution triple and $t_0 < T < \infty$. Then, the Sobolev space

$$W_2^1(t_0,T;V,H) := \{u \in L_2(t_0,T;V) : \; u' \in L_2(t_0,T;V^*)\}$$

forms a Banach space with the norm

$$\|u\|_{W_2^1} = \|u\|_{L_2(t_0,T;V)} + \|u'\|_{L_2(t_0,T;V^*)}.$$

The following proposition is a consequence of the Riesz theorem.

Proposition C.4. *Let H be a Hilbert space. Then for each $u \in H$, there is a unique linear continuous functional Ju on V with*

$$\langle Ju, v\rangle = (u|v) \qquad \textit{for all} \quad u,v \in V,$$

where $(\cdot|\cdot)$ denotes the scalar product of H. The operator $J: V \to V^*$ is linear, bijective, and norm isomorphic, i.e.,

$$\|Ju\|_{V^*} = \|u\|_V \qquad \textit{for all} \quad u \in V.$$

Therefore, one can identify Ju with u for all $u \in V$. This way we get $H = H^*$ and

$$\langle u, v\rangle = (u|v) \qquad \text{for all} \quad u,v \in V.$$

The next proposition explains how the relation $H \subseteq V^*$ is to be understood.

Proposition C.5. *Let $V \subseteq H \subseteq V^*$ be an evolution triple. Then, the following is satisfied*

(i) *To each $u \in H$, there corresponds a linear continuous functional $\bar{u} \in V^*$ with*

$$\langle \bar{u}, v \rangle_V = (u|v)_H \quad \text{for all} \quad v \in V.$$

(ii) *The mapping $u \mapsto \bar{u}$ from H into V^* is linear, injective, and continuous.*

Proof. (i) Let $u \in H$. Then:

$$|(u|v)_H| \leq \|u\|_H \|v\|_H \leq \text{const}\|u\|_H \|v\|_V$$

is fulfilled for all $v \in V$. Therefore, there exists a $\bar{u} \in V^*$ with

$$\langle \bar{u}, v \rangle_V = (u|v)_H \qquad \text{and} \qquad \|\bar{u}\|_{V^*} \leq \text{const}\|u\|_H.$$

(ii) The mapping $u \mapsto \bar{u}$ is obviously linear and continuous. In order to show injectivity, we assume that $\bar{u} = 0$. This implies

$$(u|v)_H = 0 \qquad \text{for all} \quad v \in V.$$

Since V is dense in H, we get $u = 0$. □

This allows us to identify $\bar{u}$ with u such that

$$\begin{aligned} \langle u, v \rangle_V &= (u|v)_H \qquad \text{for all} \quad u \in H,\ v \in V, \\ \|u\|_{V^*} &\leq \text{const}\|u\|_H \qquad \text{for all} \quad u \in H. \end{aligned}$$

In this sense, the relation $H \subseteq V^*$ is to be understood. Obviously, this embedding is continuous.

The next theorem extends the solvability results for linear systems from Chapter 2 to distributions on the right-hand side. We consider DAEs of the form

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \tag{C.9}$$

$$D(t_0)x_0 = z_0 \in \operatorname{im} D(t_0). \tag{C.10}$$

Theorem C.6. *If $q \in L^2(t_0, T; \mathbb{R}^n)$, then the index-1 IVP (C.9)–(C.10) has a unique solution x in*

$$L^2_D(t_0, T; \mathbb{R}^n) := \{x \in L^2(t_0, T; \mathbb{R}^n) : \ Dx \in C([t_0, T], \mathbb{R}^m)\}.$$

Equation (C.9) holds for almost all $t \in [t_0, T]$. Furthermore, Dx is differentiable for almost all $t \in [t_0, T]$ and there is a constant $C > 0$ such that

$$\|x\|_{L^2(t_0,T;\mathbb{R}^n)} + \|Dx\|_{C([t_0,T],\mathbb{R}^m)} + \|(Dx)'\|_{L^2(t_0,T;\mathbb{R}^m)} \leq C\left(\|z_0\| + \|q\|_{L^2(t_0,T;\mathbb{R}^n)}\right).$$

For continuous solutions, the right-hand side belonging to the nondynamical part has to be continuous. The next theorem describes this more precisely.

Theorem C.7. *If $q \in L^2(t_0,T;\mathbb{R}^n)$ and $Q_0G_1^{-1}q \in C([t_0,T];\mathbb{R}^n)$, then the solution x of the index-1 IVP (C.9)–(C.10) belongs to $C([t_0,T];\mathbb{R}^n)$ and we find a constant $C>0$ such that*

$$\|x\|_{C([t_0,T],\mathbb{R}^n)} + \|(Dx)'\|_{L^2(t_0,T;\mathbb{R}^m)} \le C\left(\|z_0\| + \|q\|_{L^2(t_0,T;\mathbb{R}^n)} + \|Q_0G_1^{-1}q\|_{C([t_0,T],\mathbb{R}^n)}\right).$$

Proof (of Theorems C.6 and C.7). The proof is straightforward. We simply have to combine standard techniques from DAE and Volterra operator theory. Due to the index-1 assumption, the matrix

$$G_1(t) = A(t)D(t) + B(t)Q_0(t)$$

is nonsingular for all $t \in [t_0,T]$. Recall that $Q_0(t)$ is a projector onto $\ker A(t)D(t)$. Multiplying (C.9) by $D(t)G^{-1}(t)$ and $Q_0(t)G^{-1}(t)$, respectively, we obtain the system

$$(Dx)'(t) - R'(t)(Dx)(t) + (DG_1^{-1}BD^-)(t)(Dx)(t) = (DG_1^{-1}r)(t), \tag{C.11}$$
$$(Q_0x)(t) + (Q_0G_1^{-1}BD^-)(t)(Dx)(t) = (Q_0G_1^{-1}r)(t), \tag{C.12}$$

which is equivalent to (C.9). Here, we have used the properties

$$(DG_1^{-1}A)(t) = R(t), \qquad (G_1^{-1}BQ_0)(t) = Q_0(t)$$

for all $t \in [t_0,T]$. Recall that $R(t) = D(t)D^-(t)$ is a continuously differentiable projector onto $\operatorname{im} D(t)$ along $\ker A(t)$ and $D^-(t)$ is a generalized inverse that satisfies $D^-(t)D(t) = P_0(t)$.
For $z := Dx$, equation (C.11) together with (C.10) represents an ordinary initial value problem of the form

$$z'(t) = \hat{A}(t)z(t) + b(t), \qquad z(t_0) = z_0 \tag{C.13}$$

with $\hat{A} \in C([t_0,T],L(\mathbb{R}^m,\mathbb{R}^m))$ and $b \in L^2(t_0,T;\mathbb{R}^m)$. Since $\hat{A}$ is linear and continuous, the map

$$x \mapsto \hat{A}(t)x$$

is Lipschitz continuous as a map from $L^2(t_0,T;\mathbb{R}^m)$ into $L^2(t_0,T;\mathbb{R}^m)$ with a Lipschitz constant that is independent of t. Consequently (see, e.g., [79], pp. 166–167), the IVP (C.13) has a unique solution $z \in C([t_0,T],\mathbb{R}^m)$ with $z' \in L^2(t_0,T;\mathbb{R}^m)$. The solution z satisfies (C.13) for almost all $t \in [t_0,T]$ and it is differentiable for almost all $t \in [t_0,T]$. Furthermore, there is a constant $C_1 > 0$ such that

$$\|z\|_{C([t_0,T],\mathbb{R}^m)} + \|z'\|_{L^2(t_0,T;\mathbb{R}^m)} \le C_1\left(\|z_0\| + \|b\|_{L^2(t_0,T;\mathbb{R}^m)}\right). \tag{C.14}$$

In [79], this was proven not only for maps into the finite-dimensional space $\mathbb{R}^m$ but also for maps into any Banach space. In the finite-dimensional case, the unique solvability of (C.13) and the validity of the estimation (C.14) follow also from the theorem of Carathéodory (see, e.g., [218], [121]), an a priori estimate and the gen-

eralized Gronwall lemma (see, e.g., [216]). For convenience, we omit an extended explanation of the second way.
Multiplying (C.11) by $I-R(t)$, we obtain that

$$((I-R)z)'(t) = -R'(t)((I-R)z)(t)$$

for the solution z and almost all $t \in [t_0,T]$. Since z_0 belongs to $\operatorname{im} D(t_0)$, we get

$$((I-R)z)(t_0) = 0.$$

Using again the unique solvability, we obtain that

$$((I-R)z)(t) = 0 \qquad \text{for almost all } t \in [t_0,T]. \tag{C.15}$$

From (C.12), we see that all solutions of (C.9)–(C.10) are given by

$$x(t) = D^-(t)z(t) - (Q_0G_1^{-1}BD^-)(t)z(t) + (Q_0G_1^{-1}r)(t), \tag{C.16}$$

where z is the unique solution of (C.13). Obviously, $Dx = z$ belongs to $C([t_0,T],\mathbb{R}^m)$. Since D, R and P_0 are continuous on $[t_0,T]$, the generalized inverse D^- is continuous. This implies $x \in L^2(t_0,T;\mathbb{R}^n)$ since $r \in L^2(t_0,T;\mathbb{R}^n)$. Recall that G_1 is continuous due to the index-1 assumption. If, additionally, $Q_0G^{-1}r$ is continuous on $[t_0,T]$, then the whole solution x belongs to $C([t_0,T],\mathbb{R}^n)$. The estimations of Theorems C.6 and C.7 are a simple conclusion of the solution representation (C.16) and the estimation (C.14). □

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