Appendix A

Transformation of the Rotating to Inertial Coordinate Systems

To calculate the moment acting upon the rotor by using the angular momentum theorem, the product rule of differentiation is applied. The resulting moment in the rotating coordinate system (x',y',z') is transformed into the inertial coordinate system (x,y,z). Therefore, the relations between the unit vectors of two coordinate systems are needed.

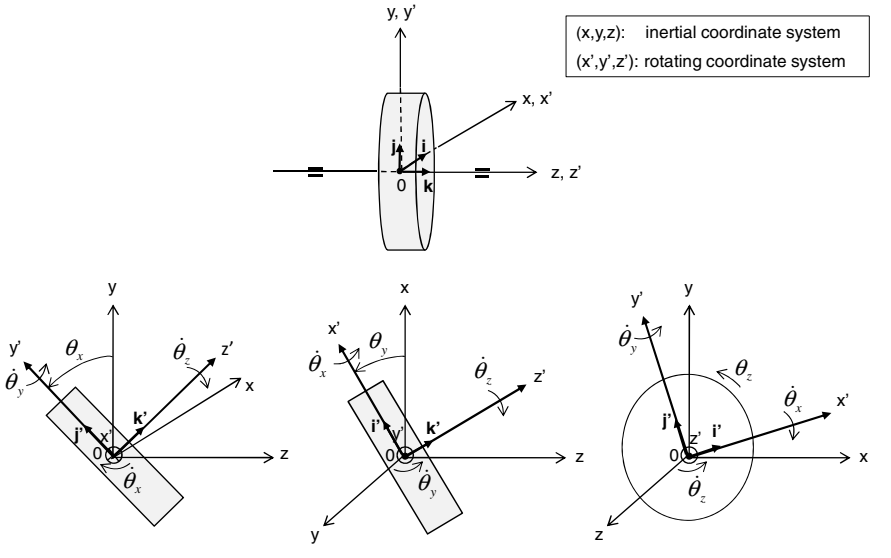

Fig. 1 Transformation between the rotating and inertial coordinate systems

Two-dimensional angular position vector $\boldsymbol{\theta}$ is written in the rotating coordinate system (x',y',z') with $\theta_z = 0$.

$$\boldsymbol{\theta} = \theta_x \mathbf{i}' + \theta_y \mathbf{j}' \tag{1}$$

where

θ_x, θ_y, and θ_z are the proper Euler angles in the directions x', y', and z'.

The relation between the unit vectors (**i'**, **j'**, **k'**) and (**i**, **j**, **k**) of the rotating and inertial coordinate systems is derived [1].

$$\begin{pmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{pmatrix} = \mathbf{I}\mathbf{R}_{\theta x}\mathbf{R}_{\theta y}\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \mathbf{R}\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \tag{2}$$

where
I is the unit matrix;

$\mathbf{R}_{\theta x}$ and $\mathbf{R}_{\theta y}$ are the transform matrices in functions of θ_x and θ_y, respectively.

$$\mathbf{R}_{\theta x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix};\ \mathbf{R}_{\theta y} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \tag{3}$$

The transform matrix **R** is calculated from **I**, $\mathbf{R}_{\theta x}$ and $\mathbf{R}_{\theta y}$ as follows:

$$\mathbf{R} \equiv \mathbf{I}\mathbf{R}_{\theta x}\mathbf{R}_{\theta y} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ \sin\theta_x \sin\theta_y & \cos\theta_x & \sin\theta_x \cos\theta_y \\ \cos\theta_x \sin\theta_y & -\sin\theta_x & \cos\theta_x \cos\theta_y \end{pmatrix} \tag{4}$$

Therefore,

$$\begin{cases} \mathbf{i}' = \cos\theta_y\,\mathbf{i} - \sin\theta_y\,\mathbf{k} \\ \mathbf{j}' = (\sin\theta_x \sin\theta_y)\,\mathbf{i} + \cos\theta_x\,\mathbf{j} + (\sin\theta_x \cos\theta_y)\mathbf{k} \\ \mathbf{k}' = (\cos\theta_x \sin\theta_y)\,\mathbf{i} - \sin\theta_x\,\mathbf{j} + (\cos\theta_x \cos\theta_y)\mathbf{k} \end{cases} \tag{5}$$

At small angles θ_x, $\theta_y << 1$ with $\cos\theta \approx 1-\theta^2/2$ and $\sin\theta \approx \theta$, one obtains

$$\begin{cases} \mathbf{i}' \approx \mathbf{i} - \theta_y\mathbf{k} \approx \mathbf{i} \\ \mathbf{j}' \approx \mathbf{j} + \theta_x\mathbf{k} \approx \mathbf{j} \\ \mathbf{k}' \approx \theta_y\mathbf{i} - \theta_x\mathbf{j} + \mathbf{k} \approx \mathbf{k} \end{cases} \tag{6}$$

Generally, the three-dimensional angular position vector $\boldsymbol{\theta}$ is written in the rotating coordinate system (x',y',z').

$$\boldsymbol{\theta} = \theta_x\mathbf{i}' + \theta_y\mathbf{j}' + \theta_z\mathbf{k}' \tag{7}$$

The relations of the unit vectors are formulated by the transform matrices.

$$\begin{pmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{pmatrix} = \mathbf{R}_{\theta z}\mathbf{R}_{\theta x}\mathbf{R}_{\theta y}\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \equiv \mathbf{R}^{*}\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \tag{8}$$

with
the transform matrices $\mathbf{R}_{\theta x}$ and $\mathbf{R}_{\theta y}$ given in Eq. (3) and

$$\mathbf{R}_{\theta z} = \begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

The transformation matrix R* is calculated from $\mathbf{R}_{\theta z}$, $\mathbf{R}_{\theta x}$ and $\mathbf{R}_{\theta y}$.

$$\mathbf{R}^{*} = \begin{pmatrix} \cos\theta_z\cos\theta_y + \sin\theta_z\sin\theta_x\sin\theta_y & \sin\theta_z\cos\theta_x & -\cos\theta_z\sin\theta_y + \sin\theta_z\sin\theta_x\cos\theta_y \\ -\sin\theta_z\cos\theta_y + \cos\theta_z\sin\theta_x\sin\theta_y & \cos\theta_z\cos\theta_x & \sin\theta_z\sin\theta_y + \cos\theta_z\sin\theta_x\cos\theta_y \\ \cos\theta_x\sin\theta_y & -\sin\theta_x & \cos\theta_x\cos\theta_y \end{pmatrix} \tag{10}$$

At small angles $\theta_x, \theta_y, \theta_z \ll 1$ with $\cos\theta \approx 1-\theta^2/2$ and $\sin\theta \approx \theta$, one obtains

$$\begin{cases} \mathbf{i}' \approx \mathbf{i} + \theta_z\mathbf{j} - \theta_y\mathbf{k} \approx \mathbf{i} \\ \mathbf{j}' \approx -\theta_z\mathbf{i} + \mathbf{j} + \theta_x\mathbf{k} \approx \mathbf{j} \\ \mathbf{k}' \approx \theta_y\mathbf{i} - \theta_x\mathbf{j} + \mathbf{k} \approx \mathbf{k} \end{cases} \tag{11}$$

Appendix B

Calculating Value x from X in the Log_{10} Scale

The value x at an arbitrary ratio ξ is interpolated from the measured values X at an arbitrary ratio ξ in the $\log_{10}$ scale, as shown Fig. 1.

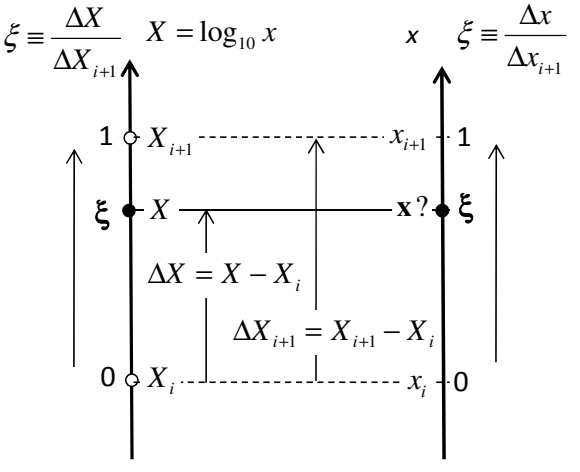

Fig. 1 Calculating value x from X in the $\log_{10}$ scale

The intervals ΔX and ΔX_{i+1} in the $\log_{10}$ scale are calculated.

$$\Delta X = X - X_i = \log_{10}\left(\frac{x}{x_i}\right) \tag{1}$$

$$\Delta X_{i+1} = X_{i+1} - X_i = \log_{10}\left(\frac{x_{i+1}}{x_i}\right) \tag{2}$$

Having divided Eq. (1) to Eq. (2), one obtains

$$\log_{10}\left(\frac{x}{x_i}\right) = \left(\frac{\Delta X}{\Delta X_{i+1}}\right)\log_{10}\left(\frac{x_{i+1}}{x_i}\right) = \log_{10}\left(\frac{x_{i+1}}{x_i}\right)^{\left(\frac{\Delta X}{\Delta X_{i+1}}\right)} \tag{3}$$

Thus,

$$\left(\frac{x}{x_i}\right)=\left(\frac{x_{i+1}}{x_i}\right)^{\left(\frac{\Delta X}{\Delta X_{i+1}}\right)}=\left(\frac{x_{i+1}}{x_i}\right)^{\xi}=10^{\xi} \tag{4}$$

where

$$\xi \equiv \frac{\Delta X}{\Delta X_{i+1}}=\frac{X-X_i}{X_{i+1}-X_i};\ 0<\xi<1 \tag{5}$$

The corresponding value x of X in the $\log_{10}$ scale is computed from the ratio ξ according to Eq. (4) where ξ is the relative distance of X between X_{i+1} and X_i. The values x are plotted for various values ξ in case $x_i = 10^1$ in Figure 2. The results help us to calculate the value x at the ratio ξ by interpolating the measured values at the ratio ξ in the $\log_{10}$ scale.

Table 1 Computed values x from ξ of the $\log_{10}$ scale

ξ	x/x_i	$x = x_i.10^{\xi}$ for		
		$x_i = 10^0$	$x_i = 10^1$	$x_i = 10^2$
0,00	**1**	1,00	10,00	100,00
0,05	**1,12**	1,12	11,22	112,20
0,10	**1,26**	1,26	12,59	125,89
0,15	**1,41**	1,41	14,13	141,25
0,20	**1,58**	1,58	15,85	158,49
0,25	**1,78**	1,78	17,78	177,83
0,30	**2,00**	2,00	19,95	199,53
0,35	**2,24**	2,24	22,39	223,87
0,40	**2,51**	2,51	25,12	251,19
0,45	**2,82**	2,82	28,18	281,84
0,50	**3,16**	3,16	31,62	316,23
0,55	**3,55**	3,55	35,48	354,81
0,60	**3,98**	3,98	39,81	398,11
0,65	**4,47**	4,47	44,67	446,68
0,70	**5,01**	5,01	50,12	501,19
0,75	**5,62**	5,62	56,23	562,34
0,80	**6,31**	6,31	63,10	630,96
0,85	**7,08**	7,08	70,79	707,95
0,90	**7,94**	7,94	79,43	794,33
0,95	**8,91**	8,91	89,13	891,25
1,00	**10**	10,00	100,00	1000,00

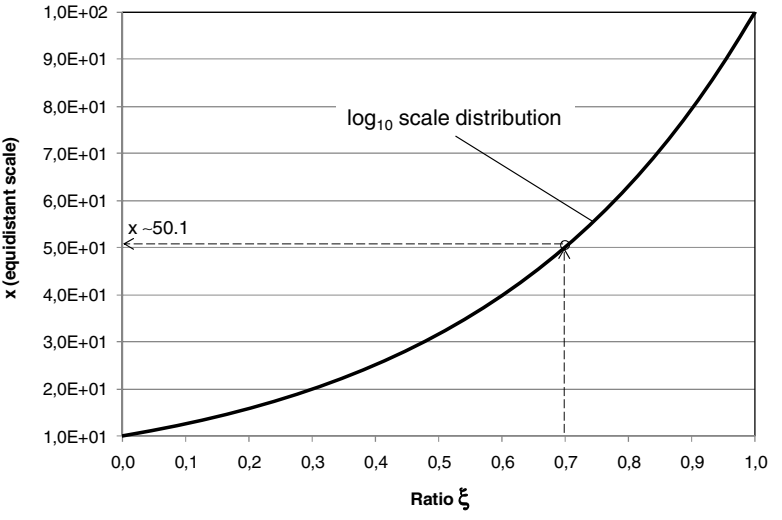

Fig. 2 Calculating x from the $\log_{10}$ scale distribution X at $\xi = 0.7$

Appendix C

Solutions of the Characteristic Equation with Complex Coefficients

The characteristic quadratic equation with complex coefficients is derived from the vibration equation [2].

$$D(s) \equiv s^2 + (a + jb)s + (c + jd) = 0 \tag{1}$$

where

$s = (\alpha \pm j\omega_n) \in \mathbf{C}$ are the complex eigenvalues;

$a, b, c, d \in \mathbf{R}$ are the real numbers.

By solving the characteristic equation $D(s) = 0$, the eigenvalue s results in

$$\begin{aligned} s_{1,2} &= -\left(\frac{a+jb}{2}\right) \pm \sqrt{\left(\frac{a+jb}{2}\right)^2 - (c+jd)} \\ &= \alpha \pm j\omega_n \end{aligned} \tag{2}$$

After calculating and rearranging the real and imaginary terms of the eigenvalue, one obtains

$$\alpha = -\frac{a}{2} \pm \frac{1}{\sqrt{2}} \left[\left(\frac{a^2 - b^2}{4} - c \right) + \sqrt{\left(\frac{a^2 - b^2}{4} - c \right)^2 + \left(\frac{ab}{2} - d \right)^2} \right]^{\frac{1}{2}} \tag{3}$$

and

$$\omega_n = \frac{b}{2} \pm \frac{1}{\sqrt{2}} \left[-\left(\frac{a^2 - b^2}{4} - c \right) + \sqrt{\left(\frac{a^2 - b^2}{4} - c \right)^2 + \left(\frac{ab}{2} - d \right)^2} \right]^{\frac{1}{2}} \tag{4}$$

Necessary condition for the rotordynamic stability is that the real term α of the eigenvalue must be negative (s. Section 4.2).

$$\alpha = -\frac{a}{2} \pm \frac{1}{\sqrt{2}} \left[-E + \sqrt{E^2 + F^2} \right]^{\frac{1}{2}} < 0 \tag{5}$$

within

$$E \equiv -\left(\frac{a^2 - b^2}{4} - c \right) = -\left(\frac{a^2 - b^2}{4} \right) + c \tag{6a}$$

$$F \equiv \left(\frac{ab}{2} - d \right) \tag{6b}$$

Thus, the stability condition ($\alpha < 0$) for the rotor given in eq. (5) becomes after a few calculation steps.

$$a(ac + bd) - d^2 \geq 0 \tag{7}$$

Appendix D

Normal Distribution Density Function and Probability Function

The normal distribution density function is defined with parameters $\bar{z}$ and σ as follows:

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z - \bar{z}}{\sigma} \right)^2 \right] \tag{1}$$

where
$\bar{z}$ is the mean value of the sampling values;
σ is the standard deviation resulted from the sampling values.

$$\sigma = \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} (z_i - \bar{z})^2} \tag{2}$$

within

$$\bar{z} = \frac{1}{N} \sum_{i=1}^{N} z_i \tag{3}$$

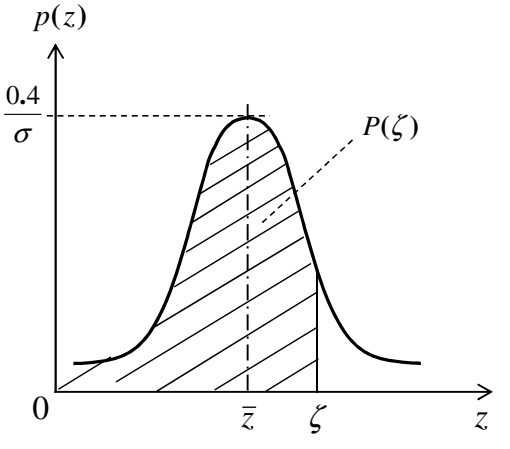

Fig. 1 Distribution density function p(z) and its probability function

The normal distribution function is called the *Gaussian density function* with a bell shape; its probability distribution function is calculated by integrating p(z) from -∞ to ζ. This integrated value *P(ζ)* is the area under the bell curve, as shown in Fig. 1.

$$P(\zeta) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\zeta} \exp\left[-\frac{1}{2}\left(\frac{z - \bar{z}}{\sigma} \right)^2 \right] dz \tag{4}$$

By substituting z by the new dimensionless variable,

$$c \equiv \frac{z - \bar{z}}{\sigma} \tag{5}$$

the normal distribution function p(z) is written in

$$p(c) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{c^2}{2} \right] \tag{6}$$

The new probability function is written in the new variable c.

$$P(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} \exp\left[-\frac{1}{2} c^2 \right] dc \tag{7}$$

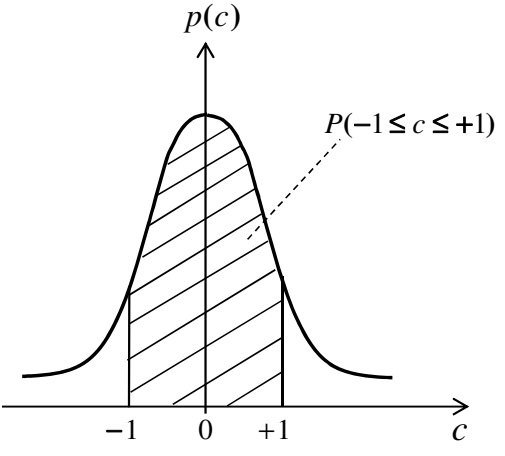

Fig. 2 Distribution density function p(c) and its probability function P(c)

Thus,

$$P(-1 \le c \le +1) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \exp\left[-\frac{1}{2} c^2 \right] dc \tag{8}$$

The probability distribution values for various parameters are calculated from Eq. (8).

$$\begin{cases} P(-1 \le c \le +1) = 68.3\% \\ P(-2 \le c \le +2) = 95.4\% \\ P(-3 \le c \le +3) = 99.7\% \\ P(-4 \le c \le +4) = 99.99\% \end{cases} \tag{9}$$

This result indicates that a production with $\pm 3\sigma$ would deliver 99.7% of the products that fulfill the given lowest and highest tolerances.

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Further Readings

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