Appendix A
Summary and Addenda

There is no new material in the first four sections of Appendix A. In Sects. A.1, A.2, definitions of the four kinds of solutions to the homogeneous Dirichlet problem which are discussed in this course are placed in one section for easy comparison. In Sects. A.3 and A.4, the main results reported in the text concerning the well-posedness of the Dirichlet problem for elliptic–hyperbolic equations of Keldysh type are collected. The last two sections of Appendix A contain new material. In A.5, a comparison is made between what is known about the two canonical classes of equations. Section A.6 contains a brief discussion of a very new method for attacking elliptic–hyperbolic boundary value problems.

A.1 Various Notions of Solution

1. *Distribution solution*: If u is a distribution solution to the equation $Lu = f$, then $u \in L^2(\Omega) \ni \forall \xi \in H_0^1(\Omega; \mathcal{K})$ for which $L^*\xi \in L^2(\Omega)$,

$$\left(u, L^*\xi\right) = \langle f, \xi \rangle,$$

   where (,) is the L^2 inner product and ⟨ , ⟩ is the Lax duality bracket.
2. *Weak solution* (first alternative): If u is a weak solution to the equation $Lu = f$, then $u \in H_0^1(\Omega, \mathcal{K}) \ni \forall \xi \in H_0^1(\Omega; \mathcal{K})$,

$$\langle Lu, \xi \rangle \equiv -\int\int_\Omega \left(\mathcal{K} u_x \xi_x + u_y \xi_y\right) dxdy = \langle f, \xi \rangle \tag{A.1}$$

   where

$$Lu = [\mathcal{K} u_x]_x + u_{yy} = f \tag{A.2}$$

   with u vanishing identically on $\partial\Omega$. Again, ⟨ , ⟩ is the duality pairing between $H_0^1(\Omega; \mathcal{K})$ and $H^{-1}(\Omega; \mathcal{K})$.

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The equivalence on the extreme left-hand side of (A.1) can be understood by integrating Lu and ξ by parts, using the vanishing of u on the boundary.

If the weight function is not taken to be the type-change function, then it may still be possible to show that a distribution solution lies in $H_0^1(\Omega; k)$ for some $k(x, y)$. This was done in Theorem 5 of [18] for the case $K = x - y^2$ and $k = y^2$. While it may be acceptable to call a solution that lies in L^2 a "distribution solution," calling a solution which lies even in a weighted Sobolev space "distributional" is a bit of a stretch. So perhaps such solutions ought to be called *generalized*. It is not clear that such generalized solutions, even if they lie in weighted H^1, are unique unless $k = K$.

3. *Weak solution* (second alternative): If u is a weak solution to the equation $Lu = f$, then there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$||u_n - u||_{H_0^1(\Omega;\mathscr{K})} \to 0 \text{ and } ||Lu_n - f||_{H^{-1}(\Omega;\mathscr{K})} \to 0 \tag{A.3}$$

as n tends to infinity.

Definitions 2 and 3 can be shown to be equivalent for L having the special form given by (A.2). Taking $\xi = u$ in (A.1), we obtain

$$-\int\int_\Omega \left(Ku_x^2 + u_y^2\right) dxdy = (f, u).$$

Because C_0^∞ is dense in $H_0^1(\Omega; \mathscr{K})$, the equivalence of the two definitions follows.

4. *Strong solution*: If $\mathbf{u} \in L^2$ is a strong solution to the equation $L\mathbf{u} = \mathbf{f}$, where $\mathbf{f}$ is a given L^2 vector, with given boundary conditions, then there exists a sequence $\mathbf{u}^\nu$ of continuously differentiable vectors, satisfying the boundary conditions, for which $\mathbf{u}^\nu$ converges to $\mathbf{u}$ in L^2 and $L\mathbf{u}^\nu$ converges to $\mathbf{f}$ in L^2.

Definition 4 is often applied to first-order systems in which $\mathbf{u} = (u_1, u_2)$ and we can take $u_1 = \varphi_x$ and $u_2 = \varphi_y$. In that case, $\mathbf{u} \in L^2$ is equivalent to $\varphi \in H^{1,2}$.

Any of the above definitions can be extended to mixed Dirichlet–Neumann problems – which are discussed in Sects. 3.3 and 3.5 – by imposing Dirichlet and Neumann conditions on disjoint proper subsets of the boundary.

A.2 Comparison of Methods

Suppose that we want to establish a fundamental inequality of the general form (3.34); that is, we want to show that

$$\|u\|_U \le C \|L^* u\|_V$$

for a suitable choice of function spaces U and V, and all $u \in C_0^\infty(\Omega)$.

If we try the abc-method, then we will be seeking functions a, b, and c such that, for some $\delta > 0$,

$$\int\int_{\Omega} \left(au + bu_x + cu_y\right) L^* u \, dxdy \geq \delta \int\int_{\Omega} |\mathscr{K}| u_x^2 + u_y^2 dxdy.$$

If we find such a, b, and c, then we reason that

$$\int\int_{\Omega} \left(au + bu_x + cu_y\right) L^* u \, dxdy \leq \|au + bu_x + cu_y\|_{L^2(\Omega)} \|L^* u\|_{L^2(\Omega)}$$
$$\leq C'_{\mathscr{K}} \|u\|_{H_0^1(\Omega;\mathscr{K})} \|L^* u\|_{L^2(\Omega)},$$

where $C'_{\mathscr{K}}$ depends on $\sup\left(|\mathscr{K}/\mathscr{K}'|\right)$. Thus we are forced to choose $U = H_0^1(\Omega; \mathscr{K})$ and $V = L^2(\Omega)$ in the fundamental inequality (3.34). Now we define, as in Sect. 3.4,

$$J_f\left(L^*\xi\right) \equiv \langle f, \xi\rangle \leq \|f\|_{H^{-1}(\Omega;\mathscr{K})} \|\xi\|_{H_0^1(\Omega;\mathscr{K})} \leq C \|f\|_{H^{-1}(\Omega;\mathscr{K})} \|L^*\xi\|_{L^2(\Omega)}. \tag{A.4}$$

We find that if $f \in H^{-1}(\Omega; \mathscr{K})$, then J_f is a bounded function on the subspace of L^2 consisting of elements $\xi \in C_0^\infty(\Omega)$ for which $L^*\xi$ is bounded in L^2. Hahn–Banach arguments extend inequality (A.4) to the L^2-closure and we apply the Riesz Representation Theorem in the inner product space L^2. This allows us to show the existence of a *distribution solution* in L^2.

Now suppose that we prove (3.34) using the integral variant of the *abc* method, under the hypothesis that Ω is star-shaped with respect to the vector field $-(b, c)$. Defining H as in (4.44) and (4.45) we find, as in (4.57) and (4.58), that

$$\delta \int\int_{\Omega} \left(|\mathscr{K}| v_x^2 + v_y^2\right) dxdy \leq \left(v, L^* H v\right) = \left(v, L^* u\right) \leq \|v\|_{H_0^1(\Omega;\mathscr{K})} \|L^* u\|_{H^{-1}(\Omega;\mathscr{K})}.$$

After dividing through by the $H_0^1(\Omega; \mathscr{K})$-norm of v, we find that we can bound the left-hand side of the preceding inequality below by the $L^2(\Omega; |\mathscr{K}|)$-norm of u, using (4.44) and (4.45). Replacing (A.4) by (4.60), we find that

$$J_f\left(L^*\xi\right) \leq C ||f||_{L^2\left(\Omega; |\mathscr{K}|^{-1}\right)} ||L^*\xi||_{H^{-1}(\Omega;\mathscr{K})}.$$

Now applying Hahn–Banach arguments as in the preceding case, we find that we are led by duality to apply the Riesz Representation Theorem in the inner product space $H_0^1(\Omega; \mathscr{K})$ rather than the inner product space L^2. That is, we find that there is a $u \in H_0^1(\Omega; \mathscr{K})$ such that

$$\langle u, L^*\xi\rangle = (f, \xi)_{L^2(\Omega)} \tag{A.5}$$

$\forall \xi \in H_0^1(\Omega; \mathscr{K})$ where $L^* : H_0^1(\Omega; \mathscr{K}) \to H^{-1}(\Omega; \mathscr{K})$ is a unique, continuous extension of the original operator (c.f. (2.8) of [6]). Note that the space weighted-H^{-1} for $L^*\xi$ is appropriate for obtaining u in weighted-H^1, as indicated by the duality bracket on the left-hand side of (A.5). However, due to the form of (A.1), we cannot actually apply this method in an obvious way unless $L = L^*$.

Instead of the L^2 solution which resulted from applying the Riesz Representation Theorem to the *abc*-method in the previous case, in the present case the Riesz Representation Theorem has given us a solution in weighted-H^1. For that reason, and because definition (A.3) can be used to derive uniqueness, it is appropriate to call the solution *weak*.

So the advantage of the integral variant of the *abc* method over the conventional *abc*-method is that the former involves estimates that are one derivative higher than those of the latter method, leading to the application of the Riesz Representation theorem in a higher (although weighted) Sobolev space. Because when applying the Riesz Representation Theorem, $L^*\xi$ is dual to the solution u in the Lax duality bracket, $L^*\xi \in L^2(\Omega)$ implies $u \in L^2(\Omega)$, whereas $L^*\xi \in H^{-1}(\Omega; \mathscr{K})$ implies $u \in H_0^1(\Omega; \mathscr{K})$.

A.3 The Existence of Solutions

In the following, results for inhomogeneous equations having homogeneous boundary conditions possess analogues for homogeneous equations having inhomogeneous boundary conditions via the arguments of Sect. 2.6.

A.3.1 Distribution Solutions

Theorem A.1 (c.f. Theorem 3.3). *Let Ω be a bounded, connected domain having piecewise C^1 boundary. The Dirichlet problem for the equation*

$$\mathscr{K}(x)u_{xx} + u_{yy} + \mathscr{K}'(x)u_x = f(x, y) \tag{A.6}$$

with boundary condition

$$u(x, y) = 0 \,\forall (x, y) \in \partial\Omega$$

possesses a distribution solution $u \in L^2(\Omega)$ for every $f \in H^{-1}(\Omega; \mathscr{K})$, provided $\mathscr{K} = x$.

Remark. See also [18] for the cold-plasma case, [15] for a case related to the Monge–Ampère equation, and [6] for the original method introduced in the context of Tricomi-type equations.

A.3.2 Weak Solutions

Theorem A.2 (c.f. Theorem 4.2). *There exists a unique, weak solution to the Dirichlet problem of Theorem A.1 provided the type-change function* $\mathcal{K}(x)$ *is replaced by the type-change function* $\mathcal{K}(x,y) = x - y^2$*, and in addition the following hypotheses are satisfied:* x *is non-negative on* Ω *and the origin of coordinates lies on* $\partial\Omega$*;* Ω *is star-shaped with respect to the flow of the vector field* $V = -(b,c)$ *for* $b = mx$ *and* $c = \mu y$*, where* m *and* μ *are positive constants and* m *exceeds* 3μ*.*

Theorem A.3 (c.f. Theorem 4.3). *Consider an equation having the form*

$$Lu \equiv x^{2k+1}u_{xx} + u_{yy} + c_1 x^{2k} u_x + c_2 u = 0, \tag{A.7}$$

where $k \in \mathbf{Z}^+$ *and the constants* c_1 *and* c_2 *satisfy* $c_1 < k+1$ *and* $c_2 < 0$ *with* $|c_2|$ *sufficiently large. Let a portion of the line* $x = 0$ *lie in* Ω *and let the point* $(0,0)$ *lie on* $\partial\Omega$*. Assume that* Ω *is star-shaped with respect to the vector field* $V = -(b,c)$*, where* $b = mx$*, and* $c = \mu y$*. Let* μ *be a positive constant and let*

$$m = \begin{cases} -a/\ell + \mu/2\ell - \delta/\ell \text{ in } \Omega^+ \\ -a/\ell + \mu/2\ell + \delta/\ell \text{ in } \Omega^- \end{cases}$$

for a positive constant δ*, where* $\ell = k + 1 - c_1$*. Let* a *be a negative constant of sufficiently large magnitude. In particular, let* a *have sufficiently large magnitude that* m *is positive. Then for every* $f \in L^2\left(\Omega; |\mathcal{K}|^{-1}\right)$ *there is a distribution solution* u *to (A.7), with*

$$u(x,y) = 0 \; \forall (x,y) \in \partial\Omega,$$

lying in $H_0^1(\Omega;\mathcal{K})$ *for* $\mathcal{K} = x^{2k+1}$*.*

Remark. For the original Tricomi case of the preceding two results, see [6].

Theorem A.4 (Magnanini–Talenti [14] (c.f. Theorem 5.1)). *Let* D_R*,* $R > 1$*, be a disc of radius* R *centered at the origin of coordinates in* $\mathbf{R}^2$ *and let* Ω *be a subset of* D_R *that has positive distance from both the boundary and the center of* D_R*. Suppose that* $f \in L^2[-\pi,\pi]$ *is a given function. Choose polar coordinates* (r,θ)*, where* $0 < r < \infty$*,* $-\pi < \theta \le \pi$*. Then there is a unique function* V *lying in* $H^{1,2}(\Omega)$*, which satisfies*

$$\left(r^2 - 1\right) V_{rr} + rV_r + V_{\theta\theta} = 0$$

weakly in D_R *and smoothly in the punctured disc* D_r*, where* $0 < r < R$*. Moreover,* $V = f$ *on* ∂D *in the sense that*

$$\lim_{r \uparrow R} \int_{-\pi}^{\pi} \left|V\left(r \cdot e^{i\theta}\right) - f(\theta)\right|^2 d\theta = 0.$$

Finally, V has an explicit representation in the form of a series

$$V\left(re^{i\theta}\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{T_k(r)}{T_k(R)} \cdot \{a_k \cos(k\theta) + b_k \sin(k\theta)\},$$

where

$$a_k = \pi^{-1} \int_{-\pi}^{\pi} f(\theta) \cos(k\theta)\, d\theta;$$

$$b_k = \pi^{-1} \int_{-\pi}^{\pi} f(\theta) \sin(k\theta)\, d\theta;$$

$$T_k(r) = \begin{cases} \cos(k \arccos r) & \text{if } r \le 1, \\ \frac{1}{2}\left[\left(r + \sqrt{r^2-1}\right)^k + \left(r - \sqrt{r^2-1}\right)^k\right] & \text{if } r > 1. \end{cases}$$

Theorem A.5 (c.f. Theorem 6.3). *Denote by Ω a region of the plane for which part of the boundary $\partial\Omega$ consists of a family $\mathcal{G}$ of curves composed of points satisfying*

$$\left(1 - y^2\right) dx^2 + 2xy\,dx\,dy + (1 - x^2)dy^2 = 0$$

and the remainder $C = \partial\Omega \backslash \mathcal{G}$ of the boundary consists of points (x, y) which do not satisfy that equation. In addition to the requirements that Ω contain an arc of the unit circle and that y^2 is bounded above away from unity, we require that any points satisfying the equation

$$xy + 1 - y^2 = 0$$

lie in the complement of Ω. Let $\partial\Omega$ be oriented in the counter-clockwise direction and have piecewise continuous tangent. Let the parameter θ lie in the interval $[0, \pi/4]$ and denote by Ω the region of the first and fourth quadrants bounded by the characteristic line

$$\mathcal{G}_1 : x\cos\theta + y\sin\theta = 1,$$

the characteristic line

$$\mathcal{G}_2 : x\cos\theta - y\sin\theta = 1,$$

and a smooth curve C. Let C intersect the lines $\mathcal{G}_1, \mathcal{G}_2$ at two distinct points c_1, c_2, respectively. Assume that $dy \le 0$ on C if γ_1 and γ_2 are negative definite and that $dy \ge 0$ if γ_1 and γ_2 are positive definite. Suppose that the bounded functions γ_1 and γ_2 are definite with the same sign. Then there exists a weak solution of the boundary value problem

$$Lu = \mathbf{f} \ \ in\ \Omega;$$
$$(L\mathbf{u})_1 = \left[\left(1-x^2\right)u_1\right]_x - 2xyu_{1y} + \left[\left(1-y^2\right)u_2\right]_y + \gamma_2 u_1,$$
$$(L\mathbf{u})_2 = \left(1-y^2\right)\left(u_{1y}-u_{2x}\right) + \gamma_1 u_2,$$

where

$$\gamma_i = \left(xy+1-y^2\right)\Gamma_i,\ \ i = 1,2;$$
$$u_1\frac{dx}{ds} + u_2\frac{dy}{ds} = 0 \ on\ \partial\Omega$$

for every $\mathbf{f}$ *in the space* H *consisting of pairs of measurable functions* $\mathbf{h} = (h_1, h_2)$ *for which the norm*

$$\|\mathbf{h}\|^* = \left[\int\int_\Omega \left(|\gamma_1(x,y)|^{-1} h_1^2 + |\gamma_2(x,y)|^{-1} h_2^2\right) dxdy\right]^{1/2}$$

is finite.

A.3.3 Strong Solutions

Theorem A.6 (c.f. Theorem 3.4). *Let* Ω *be a bounded, connected domain of* $\mathbf{R}^2$ *having* C^2 *boundary* $\partial\Omega$*, oriented in a counterclockwise direction. Let* $\partial\Omega_1^+$ *be a (possibly empty and not necessarily proper) subset of* $\partial\Omega^+$*. Let the inequality*

$$bn_1 + cn_2 \geq 0$$

be satisfied on $\partial\Omega^+\backslash\partial\Omega_1^+$*. On* $\partial\Omega_1^+$ *let*

$$bn_1 + cn_2 \leq 0$$

and on $\partial\Omega\backslash\partial\Omega^+$*, let*

$$-bn_1 + cn_2 \geq 0.$$

Let $b(x,y)$ *and* $c(x,y)$ *satisfy*

$$b^2 + c^2\mathscr{K} \neq 0$$

on Ω*, with neither* b *nor* c *vanishing on* Ω^+*, and let*

$$\mathscr{K}\left(bn_1 - cn_2\right)^2 + \left(c\mathscr{K}n_1 + bn_2\right)^2 \leq 0 \ on\ \partial\Omega\backslash\partial\Omega^+.$$

Let L be given by the system

$$Lu = \mathbf{f}, \tag{A.8}$$

$$Lu = \begin{pmatrix} \mathscr{K}(x,y) & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_y + \text{zeroth-order terms} \tag{A.9}$$

and let EL be symmetric positive, where

$$E = \begin{pmatrix} b & -c\mathscr{K} \\ c & b \end{pmatrix}.$$

Let the Dirichlet condition

$$-u_1 n_2 + u_2 n_1 = 0$$

be satisfied on $\partial\Omega^+ \backslash \partial\Omega_1^+$ and let the Neumann condition

$$\mathscr{K} u_1 n_1 + u_2 n_2 = 0$$

be satisfied on $\partial\Omega_1^+$. Then the equations (A.8), (A.9) possess a strong solution on Ω for every $\mathbf{f} \in L^2(\Omega)$. If in particular, the operator L is given by

$$(L\mathbf{u})_1 = x u_{1x} + u_{2y} + \kappa_1 u_1 + \kappa_2 u_2,$$
$$(L\mathbf{u})_2 = u_{1y} - u_{2x},$$

where κ_1 and κ_2 are constants, then sufficient conditions for the system to be symmetric positive are

$$2b\kappa_1 - b_x\mathscr{K} - b + c_y\mathscr{K} > 0 \text{ in } \Omega$$

and

$$\left(2b\kappa_1 - b_x\mathscr{K} - b + c_y\mathscr{K}\right)\left(2c\kappa_2 + b_x - c_y\right) - \left(b\kappa_2 + c\kappa_1 - c_x\mathscr{K} - c - b_y\right)^2 > 0 \text{ in } \Omega.$$

Theorem A.7 (c.f. Theorem 5.2). *Consider a system having the form*

$$Lw = A^1 w_r + A^2 w_\theta + Bw = F,$$

where L is a first-order operator, $w = (w_1(r,\theta), w_2(r,\theta))$, $F = (f, 0)$,

$$A^1 = \begin{pmatrix} \mathscr{K}(r) & 0 \\ 0 & -1 \end{pmatrix}, \; A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} \mathcal{K}'(r) & k \\ 0 & 0 \end{pmatrix}.$$

where $\mathcal{K}'(r) > 0$; k *is a nonzero constant;* $\mathcal{K}(r) < 0$ *for* $0 \leq r < r_{crit}$ *and* $\mathcal{K}(r) > 0$ *for* $r_{crit} < r \leq R$. *Suppose that there is a positive constant* ν_0 *such that* $\mathcal{K}'(r) \geq \nu_0$. *Let there be continuous functions* $\sigma(\theta)$ *and* $\tau(\theta)$ *such that the boundary condition*

$$\sigma(\theta)w_1 + \tau(\theta)w_2 = 0$$

is satisfied on the boundary $r = R$, *where the product* $\sigma(\theta)\tau(\theta)$ *is either strictly positive or strictly negative and has sign opposite to the sign of* k. *Then the resulting boundary value problem possesses a strong solution on the closed disc* $\{(r,\theta)\,|0 \leq r \leq R\}$ *provided* $|\mathcal{K}(0)|$ *is sufficiently small.*

A.3.4 Classical Solutions

Theorem A.8 (Cinquini-Cibrario [1] (c.f. Sect. 3.6.2)). *Given the domain bounded in the half-plane* $x \geq 0$ *by the curve*

$$C = \left\{(x,y)\,|4x + y^2 = 1\right\}$$

and in the half-plane $x < 0$ *by the intersecting characteristic lines*

$$\Gamma^{\pm} = \left\{(x,y)\,|y = \pm 2\left(\sqrt{-x} - 2\right)\right\},$$

and the equation

$$xu_{yy} + u_{yy} = 0,$$

then a solution to this equation exists which is equal on C *to a function of the form* $x\varphi(y)$, *where* $\varphi(y)$ *is finite and continuous on* C, *including the endpoints, and vanishes on the* y*-axis. The solution is analytic in the interior of the domain.*

Theorem A.9 (Gu [2,3] (c.f. Theorem 6.4)). *For* $s > 1$, *let* $f \in H^{1,s}(\Omega)$ *and* $a > -(n/2) + s$. *Then there exists a unique solution* $\varphi \in H^{1,s+1}(\Omega)$ *to the equation*

$$L\varphi \equiv \left(\delta^{ij} - x^i x^j\right)\partial_{ij}\varphi + 2ax^i\partial_i\varphi - a\,(a+1)\,\varphi = f, \tag{A.10}$$

which satisfies the boundary conditions

$$\varphi_{|\partial\Omega} = 0, \;\; \frac{\partial\varphi}{\partial n}_{|\partial\Omega} = 0; \tag{A.11}$$

here $\partial/\partial n$ indicates differentiation in the direction of the outward-pointing normal. In the special case $s \geq (n/2) + 2$, a classical solution of (A.10), (A.11) exists. In addition, under the conditions of the theorem the solution is unique whenever $a < -n/2$.

A.4 The Nonexistence of Solutions

Theorem A.10 (c.f. Theorem 3.1). *Consider the equation*

$$\mathscr{K}(x)u_{xx} + u_{yy} + \frac{\mathscr{K}'(x)}{2}u_x = 0,$$

where $\mathscr{K}$ satisfies $\mathscr{K}(0) = 0$ and $x\mathscr{K}(x) > 0$ for $x \neq 0$. Assume that $\mathscr{K}$ is $\mathbf{C}^1$, and monotonic on $\mathscr{D}^-$. Define constants a, b, d, and m, where $m < a \leq 0 < d$ and $b > 0$. Consider the domain $\mathscr{D}$ formed by the line segments

$$\mathscr{L}_1 = \{(x, y)\,|a \leq x \leq d, y = -b\};$$

$$\mathscr{L}_2 = \{(x, y)\,|x = d, -b \leq y \leq b\};$$

$$\mathscr{L}_3 = \{(x, y)\,|a \leq x \leq d, y = b\};$$

the characteristic line Γ_1 joining the points $(m, 0)$ and $(a, -b)$; and the characteristic line Γ_2 joining the points $(m, 0)$ and (a, b). Let the solution u be sufficiently smooth so that the integral

$$I = \int_0^{(x,y)} \left[\mathscr{K}(x)u_x^2 - u_y^2\right] dy - 2u_x u_y dx$$

is continuous on $\mathscr{D} \cup \partial\mathscr{D}$. If u vanishes identically on the non-characteristic boundary, then $u \equiv 0$ on all of $\mathscr{D}$. Consequently, the closed Dirichlet problem on $\mathscr{D}$ is over-determined.

Remark. For the original Tricomi case, see [16], Theorem 2. The conclusions of the theorem extend to certain equations having type-change functions which do not precisely satisfy the conditions satisfied by $\mathscr{K}$, on domains which are not identical to $\mathscr{D}$. These include equations for the Laplace–Beltrami operator on extended $\mathbf{P}^2$ (Theorem 6.1) and on a relativistically rotating disc (Problem 11, Appendix B); an equation arising in non-geometrical optics (Theorem 6.2), and an equation characterizing the behavior of electromagnetic waves in zero-temperature plasma – this last result is due to Morawetz, Stevens, and Weitzner [17]. It is also possible to extend this theorem to certain mixed Dirichlet–Neumann problems (Theorem 3.2).

A.5 Rough Comparison of Tricomi and Keldysh Classes

Recall that there are at least three fundamental differences between the analytic properties of equations of Tricomi type and those of Keldysh type:

1. Characteristic lines associated with equations of Keldysh type degenerate at the parabolic transition, in that they intersect the sonic curve tangentially.
2. Differential operators of Tricomi type tend to be of real principal type, whereas we do not expect operators of Keldysh type to possess this property.
3. Equations of Tricomi type are formally self-adjoint in their second-order terms, whereas equations of Keldysh type require the addition of a suitable first-order term in order to become formally self-adjoint.

As a result of these three differences, there is a significant difference between what we know about the two classes. Here we list five typical differences. The list is by no means exhaustive, and reflects the interests of the text. It is, in that sense, a motivation for Appendix B on suggested directions for future research.

1. Lupo, Morawetz and Payne [6] have established the existence of a unique H^1_{loc} weak solution to the homogeneous closed Dirichlet problem for an inhomogeneous Tricomi equation of the form (1.1). Moreover, these authors showed that if the inhomogeneous term f satisfies $f_x \in L^2\left(\Omega, |y|^{-1}\right)$, then the solution lies in $H^2_{loc}(\Omega)$. In that case, u is continuous by the Sobolev Theorem. It is known that the corresponding weak solution for the formally self-adjoint form of the cold plasma model (Chap. 4) – which is an equation of Keldysh type having somewhat similar structure – is not $H^{1,2}$ in any neighborhood of the origin [17]. See item *vii*) of Appendix B, Problem 6.
2. A maximum principle has been proven by Lupo and Payne for generalized solutions to the *Tricomi problem* (solutions prescribed on the elliptic part of the boundary and a characteristic) [11]. One would expect some kind of extension of this result to the Keldysh case, but no such extension appears to exist in the literature. See Problem 9 of Appendix B.
3. Lupo and Payne have proven a series of results on open boundary value problems for semilinear Tricomi operators [7], [9–13]. These authors have, in particular, investigated a variational approach to semilinear equations of Tricomi type [9]. They have also studied, with occasional collaborators, the spectral properties of linear and semilinear Tricomi operators [5, 8, 12]. I know of no such results for elliptic–hyperbolic operators of Keldysh type. See Problems 15, 12, and 18 of Appendix B.
4. As was indicated earlier, there is a microlocal theory for equations of Tricomi type which has no obvious analogue for equations of Keldysh type. This results in technical information about the solvability of boundary value problems and the propagation of singularities which is lacking for equations of Keldysh type; see [19] and references therein.
5. An existence theorem for the closed Neumann problem has been proven for the Lavrent'ev–Bitsadze equation (2.5) by Pilant [20]. No other result for conormal

conditions on the entire boundary is known for elliptic–hyperbolic equations of either type. See Problem 13 of Appendix B. Some results have been proven by Lupo, Morawetz, and Payne [6] for the mixed Dirichlet–Neumann problem for systems corresponding to equations having the Tricomi form (3.6), (3.7), (3.8); these extend easily to certain equations having the corresponding Keldysh form (3.7), (3.10) [18].

A.6 Weak Solutions in Anisotropic Sobolev Spaces (*After M. Khuri*)

We outline a very recent method introduced by M. Khuri [4], which has not yet been published. For details, see arXiv:1106.4000v1.

Define the Sobolev space $H^{(m,\ell)}(\Omega)$ to consist of functions for which derivatives up to the m^{th} partial derivative in x, and the ℓ^{th} partial derivative in y, are square-integrable. We have the norms

$$||u||^2_{(m,\ell)} \equiv \int_\Omega \sum_{0\le s\le m, 0\le t\le \ell} \left(\partial_x^s \partial_y^t u\right)^2$$

and

$$||v||_{(-m,-\ell)} = \sup_{u\in H^{(m,\ell)}(\Omega)} \frac{|\,(u,v)\,|}{||u||_{(m,\ell)}}.$$

We obtain the rigged triple

$$H^{(m,\ell)}(\Omega) \subset L^2(\Omega) \subset H^{(-m,-\ell)}(\Omega).$$

Consider the boundary value problem

$$Lu \equiv Ku_{xx} + u_{yy} + Au_x + Bu_y = f \text{ in } \Omega \tag{A.12}$$

$$Bu \equiv \alpha u_x + \beta u_y + \gamma u = 0 \text{ on } \partial\Omega, \tag{A.13}$$

and the adjoint problem

$$L^* v = g \text{ in } \Omega, \tag{A.14}$$

$$B^* v = 0 \text{ on } \partial\Omega, \tag{A.15}$$

where as usual the superscripted asterisk denotes formal adjoint. The coefficients K, A, B, α, β, and γ are assumed to be sufficiently smooth on $\overline{\Omega}$. We say that u is a *weak solution* of the boundary value problem (A.12), (A.13) if

$$\left(u, L^* v\right) = (f, v) \;\; \forall v \in C^\infty_{B^*}(\overline{\Omega}),$$

where $C^{\infty}_{B^*}(\overline{\Omega})$ is the space of smooth functions up to the boundary on Ω which satisfy the boundary conditions (A.15).

Theorem A.11 (Khuri [4]). *Let $m, \ell, s, t \in \mathbf{Z}^+ \cup \{0\}$. There exists a weak solution $u \in H^{(m,\ell)}(\Omega)$ to the boundary value problem (A.12), (A.13) if and only if there exists a constant C such that*

$$||v||_{(-s,-t)} \leq C\,||L^*v||_{(-m,-\ell)} \;\; \forall v \in C^{\infty}_{B^*}(\overline{\Omega}). \tag{A.16}$$

For a proof we refer the reader to the appendix to Khuri's paper.

In order to apply this result, one proceeds in a roughly analogous manner to the case of weighted Sobolev spaces discussed in Chap. 4. Initially, consider the auxiliary boundary value problem

$$Mu = v \text{ in } \Omega,$$
$$\tilde{B}u = 0 \text{ on } \partial\Omega,$$

where the differential operator M and the boundary operator $\tilde{B}$ must be chosen, based on the conditions of the original problem. Apply integration by parts to obtain

$$\left(L^*v, u\right) - (v, Lu) = \int_{\partial\Omega} I_1\,(u, v)$$

and

$$(Mu, Lu) = \int_{\Omega} I_2\,(u, u) + \int_{\partial\Omega} I_3\,(u, u)$$

for quadratic forms I_1, I_2, and I_3. The method requires that M, $\tilde{B}$, and B^* be chosen so that the following three inequalities are satisfied:

$$\int_{\Omega} I_2\,(u, u) \geq C^{-1}||u||^2_{(m,\ell)},$$
$$||v||_{(-s,-t)} \leq C\,||u||_{(m,\ell)}, \tag{A.17}$$

and

$$\int_{\partial\Omega} \left[I_1\,(u, Mu) + I_3\,(u, u)\right] \geq 0. \tag{A.18}$$

If these three inequalities can be established, then one reasons as follows:

$$||u||_{(m,\ell)}||L^*v||_{(-m,-\ell)} \geq \left(L^*v, u\right)$$
$$= (v, Lu) + \int_{\partial\Omega} I_1\,(u, v)$$
$$= (Mu, Lu) + \int_{\partial\Omega} I_1\,(u, v)$$

$$= \int_{\Omega} I_2(u,u) + \int_{\partial\Omega} [I_1(u, Mu) + I_3(u,u)]$$
$$\geq C^{-1} ||u||^2_{(m,\ell)}.$$

Dividing through by the $H^{(m,\ell)}$-norm of u and then applying (A.17), one obtains (A.16). Then the weak existence of solutions to the original problem (A.12), (A.13) will follow from Theorem A.11.

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Appendix B
Directions for Future Research

In this intentionally speculative appendix, 21 possible lines of investigation for equations of the general form studied in this review are discussed. These include both research into properties of the equations themselves and possible areas of application.

1. With very few exceptions (e.g., [35,130]), virtually nothing about the existence of solutions to the equations considered in these notes is known for dimensions exceeding two. The situation is somewhat better for equations of Tricomi type, at least since the famous paper by Protter in 1954 [101]. See [122] for a recent application and, e.g., [43] for an earlier higher-dimensional paper. In addition, there are higher-dimensional results using microlocal methods (see, e.g., [89, 90]), and series of technical papers by Karatoprakliev [44–48] and Sorokina [111–116]; see also [51]. Friedrichs' theory of symmetric positive operators is an n-dimensional argument, and although it is often applied in dimension 2, it has been employed in its full n-dimensional generality in many cases involving equations of Tricomi type; see, e.g., [44, 46, 79, 100, 111–114, 124]. One expects that some of these arguments are extendable to equations of Keldysh type.

One of the obstructions to higher-dimensional results for elliptic–hyperbolic equations is the linearization technique – most of the equations originate in non-linear models. The requirements for the hodograph method, for example, are very restrictive. Another obstruction appears to result from the nature of the multiplier methods, which convert the equation to a form in which an energy inequality such as (4.55) can be applied. These methods (that is, the Friedrichs *abc* method) are easiest to apply when the matrix is 2×2; the *abc* method can be quite difficult to apply even in the 2×2 case. Finally, underlying many of the current methods for equations of mixed type is the theory of complex variables; see, e.g., the application of the *abc* method by Morawetz in [76].

Due to the importance of ideas from complex variables in classical elliptic–hyperbolic analysis, it is possible that the three-dimensional, and even four-dimensional cases could be attacked using the quaternion representation. Wu uses

T.H. Otway, *The Dirichlet Problem for Elliptic-Hyperbolic Equations of Keldysh Type*, Lecture Notes in Mathematics 2043, DOI 10.1007/978-3-642-24415-5,

quaternions to represent the three-dimensional case of the (hyperbolic) water wave equations in [129]. She does this in order to be able to use a Cauchy integral formula for the 3-D problem, analogously to the use of the Poisson integral formula for the 2-D problem. It might be natural, in a model of rotational compressible flow, to represent the velocity field as a quaternion-valued 1-form. It is even possible that the 2-form representing the vorticity of the flow can be given an interpretation as the curvature associated to the quaternion-valued connection 1-form. If so, the invariance of the vorticity under galilean transformations of the velocity would inherit a geometrical interpretation. Notice that, for an irrotational flow, the vorticity is zero and we are in the flat, geometrically trivial case. See, e.g., [42] and Sect. 1.5 of [7] for examples of this kind of thinking in other contexts. The continuity equations for such a model might turn out to be bundle-valued elliptic–hyperbolic equations having an immediate extension to $\mathbf{R}^n$; c.f. (4.3) of [72].

There are of course many four-dimensional applications of elliptic–hyperbolic equations (Sect. 6.4.5), which have been pursued in physical contexts.

2. Similar remarks can be made about higher-order equations, although in that case it must be said that compelling physical examples seem to be lacking, which is certainly not the case for higher-dimensional second-order problems. Boundary value problems for higher-order elliptic–hyperbolic equations are treated in [87,88]; see the references in [87] for a few other examples, but the literature on this topic seems to be very small.

The few results that exist demonstrate the richness of higher-order equations. The equation considered in [87] (see also [86]) is n-dimensional and depends on several parameters. Depending on the choice of those parameters, the equation is either of mixed hyperbolic–parabolic or mixed elliptic–parabolic type. A concrete example is provided by the hyperbolic–parabolic equation

$$\begin{aligned}&[\sin(\pi t) - 1]\,\partial_t^6 u(x,y,t) + A\partial_t^5 u + \partial_x^6 u(x,y,t)\\ &\quad + \partial_y^6 u(x,y,t) + [\cos(\pi t) - C]\,u(x,y,t) = f(x,y,t),\end{aligned} \tag{B.1}$$

were A is a constant; C is a positive constant; $t \in (0,1)$; $(x,y) \in \mathscr{D}$; where

$$\mathscr{D} = \left\{x, y | x^2 + y^2 < R\right\};$$

$R =$ const. > 0; f is a prescribed L^2-function on $\mathscr{D} \times (0,1)$. The boundary conditions are

$$\frac{\partial^{|\alpha|}}{\partial_x^{\alpha_1}\partial_y^{\alpha_2}} u(t,x,y)_{|\Gamma} = 0 \tag{B.2}$$

for $|\alpha| = \alpha_1 + \alpha_2 \le 2$, where $\Gamma = \partial\mathscr{D} \times (0,1)$;

$$\partial_t^i u(1,x,y) = \frac{1}{2}\partial_t^i u(0,x,y), \tag{B.3}$$

where $i = 0,1,2,3,4,5$, for all $(x,y) \in \overline{\mathscr{D}}$.

It can be shown that if the constants A and C are sufficiently large, then a unique solution to this boundary value problem exists in the generalized sense that

$$\left(u, L^* v\right) = (f, v) \ \ \forall v \in \tilde{C}_*^\infty(\overline{G}),$$

where the parentheses denote L^2-inner product; L^* is the formal adjoint of the differential operator of (B.1); $C_*^\infty(\overline{G})$ is the space of infinitely smooth functions satisfying boundary conditions adjoint to (B.2), (B.3); $\overline{G}$ is the closure of the domain $G \equiv \mathscr{D} \times (0, T)$ for $T > 0$. The solution u is an element of the anisotropic function space $H^{5,3}_{(t,x,y)}(G)$. The space $H^{p,q}_{(t,x,y)}(G)$ is defined as the closure of the space of infinitely smooth functions on $\overline{G}$ which satisfy the boundary conditions (B.2), (B.3) with respect to the norm

$$||u||^2_{p,q} = \int_G \sum_{qi+p|\alpha| \le pq} \left[\partial_t^i \frac{\partial^{|\alpha|}}{\partial_x^{\alpha_1} \partial_y^{\alpha_2}} u\,(t,x,y) \right]^2 dt dx dy.$$

The proof uses the ideas of Berezanskii and his school, as outlined in Chaps. 3 and 4, in the context of anisotropic function spaces. See also item *ii*) of Problem 4, and Sect. A.6. Does this example suggest a higher-order analogue for the method described in Sect. A.6?

3. With a few exceptions, there do not seem to be results in the literature for equations of Keldysh type having multiple parabolic transitions, although such results have been known for Tricomi-type equations for a long time (see, e.g., [106, 113] for early examples; [102] for a later example; and [55, 103] for recent examples).

Multiple parabolic transitions are, in fact, a feature of one of the earliest treatments of elliptic–hyperbolic equations. In 1929, Bateman [12] considered the equation

$$\left(1 - x^2\right) u_{xx} - 2x u_x - \left(1 - y^2\right) u_{yy} + 2y u_y = 0. \tag{B.4}$$

This equation, which can be seen to be of Keldysh type by expressing it in polar coordinates, is hyperbolic when $1 - x^2$ and $1 - y^2$ have the same sign and elliptic when they have different signs. Bateman found some solutions in terms of special functions and made some observations about the apparent lack of uniqueness in certain boundary value problems; but as far as I know there are no formal results for the corresponding boundary value problem in the literature.

Much later, but nonetheless more than 30 years ago, Gu [35] considered the class of n-dimensional equations

$$\left[e(r^2)\delta^{ij} - x^i x^j\right] u_{x^i x^j} + 2a x^i u_{x^j} - a\,(a+1)\,u = f,$$

where $r^2 = \left(x^1\right)^2 + \cdots (x^n)^2$ and $e\left(r^2\right)$ is a smooth function such that $e\left(r^2\right) - r^2$ is:

positive if $0 \leq r^2 < c_1, c_2 < r^2 < c_3$;

negative if $c_1 < r^2 < c_2, r^2 > 5$;

zero if $c_4 < r^2 < c_5$;

here $c_1, \ldots, c_5$ are constants. Taking the domain to be a closed and bounded region containing the sphere $r^2 = c_5$ and assuming that the tangent planes to the boundary of the domain do not meet this sphere, there are two elliptic regions ($0 \leq r^2 < c_1$, $c_2 < r^2 < c_3$), two hyperbolic regions ($c_2 < r^2 < c_1$, $r^2 > 5$), and one parabolic region ($c_4 < r^2 < c_5$). Gu proves that if a is sufficiently large, then the equation admits a C^2 solution satisfying the conditions

$$u = 0, \ \partial u/\partial n = 0$$

on the domain boundary, where $\partial/\partial n$ indicates differentiation in the direction of the outward-pointing normal. Moreover, if $-a$ is sufficiently large, then there is only one classical or weak solution to this boundary value problem.

Gu's arguments are based on the theory that he developed for the equations considered in Sects. 6.4.2 and 6.4.3. As in the case of (6.36), in some sense (B.4) is a generalization of an equation treated in this review – in this case, (5.53). But again, Gu's method relies on hypotheses which explicitly exclude that case. Nevertheless, one could modify a number of equations considered in this text in a similar way, and presumably obtain similar results.

See also the extensive work by Popivanov [96–98], concerning equations having the form

$$K(y)u_{xx} + M(x)u_{yy} + \text{ lower-order } = f \tag{B.5}$$

where $u = u(x, y)$, $f = f(x, y)$, $K(0) = M(0) = 0$, $yK(y) > 0$, $xM(x) > 0$, $K'(y) > 0$, and $M'(x) > 0$. The equation is studied from the point of view of symmetric positive differential equations. A multidimensional generalization has also been studied [99].

4. The following two problems concern the auxiliary problem in the integral variant of the *abc* method.

i) What can be said in general about the solution of the Dirichlet problem for first-order hyperbolic systems generalizing (4.44), (4.45)? In addition to its intrinsic interest, this question must be answered if the integral variant of the *abc* method as applied in Sect. 4.3 is to be extended to a large class of equations. The current literature on the Dirichlet problem for hyperbolic equations is inadequate in many respects: First, it tends to be restricted to scalar second-order equations having the form of the wave equation; second, the coefficients must satisfy delicate number-theoretic conditions, whereas the conditions on the coefficients in the system (4.44), (4.45) have to do with the geometry of the domain with respect to a vector field. For a review of the literature on the Dirichlet problem for hyperbolic equations, see [126].

ii) Very recently, M. Khuri introduced an extension of the integral variant of the *abc* method in which the weighted spaces employed in this text, and in the recent literature on this method, are replaced by anisotropic function spaces [50]; Appendix A.6 contains a very brief review of this method. An open-ended direction for future research would be to investigate systematically the effects of this method on the problems reviewed in this text; c.f. the remark at the end of Problem 2.

Anisotropic Sobolev spaces are appropriate for solutions which have greater regularity in one direction than in another. This is typical of elliptic–hyperbolic equations of the form (1.3). Despite its promise, it is unlikely that Khuri's method for exploiting these spaces will entirely replace methods based on the weighted function spaces discussed in earlier chapters. The latter spaces are well-adapted to solutions which have essential singularities on a subset of their domain. Such solutions arise frequently in equations of Keldysh type. For example, solutions to the cold plasma model and to the Laplace–Beltrami equation on mixed Riemannian–Lorentzian metrics are typically singular at one or more points on the sonic curve. Essential singularities also characterize fundamental solutions of Cinquini-Cibrario's equation for certain coefficients of the first-order term.

5. What, in general, is the relation between the vector field $V = -(b, c)$ and the differential equation that permits the integral variant of the *abc* method to work under an assumption that the domain is star-shaped with respect to V? Obviously, this problem is related to Problem 4, but is more fundamental. For example, the choice of multiplier in open boundary value problems for symmetric positive differential equations seems to have a natural interpretation in terms of starlike boundaries; consider, for example, the hypotheses on the boundary in Theorem 3.4.

In [64], invariance of the differential equation under dilations is crucial; see also [91]. But this invariance is not required in applying the method to the closed Dirichlet problem in [58], and is also not exploited in these notes, except in the context of special solutions (Sects. 3.7 and 4.4).

In [58, 85], only vector fields of the form $(b, c) = (mx, \mu y)$ are considered, where m and μ are constants. Can one prove weak existence for more general kinds of elliptic–hyperbolic Dirichlet problems by considering more general classes of vector fields?

The integral variant of the *abc* method was used in [58] to solve the closed Dirichlet problem for a wide class of elliptic–hyperbolic equations of Tricomi type. It was extended in [85] to a particular equation of Keldysh type. An obvious program would be to find the largest class of equations of mixed elliptic–hyperbolic type for which one could show the existence of a weak solution to a closed Dirichlet problem via the integral variant of the *abc* method. Presumably, answering the questions of the preceding paragraphs would contribute to that program. In particular, is there a notion of weak solution that would extend the method discussed in Sect. 4.3 to operators which are not formally self-adjoint?

6. In addition to Problems 4 and 5, the following problems are suggested by the material in Chap. 4:

i) The nature of the singularity at the origin of the cold plasma model equations is poorly understood. The only results appear to be the analytic results of [95] and the numerical experiments of [77]. Numerical experiments show the presence of singularities propagating away from the origin along characteristic lines. Neither the physical nor the analytic meaning of these singularities is clear at present.

ii) More generally, questions of the dependence of a solution on boundary data are poorly understood for equations of elliptic–hyperbolic type. One expects singularities to be reflected in the boundary conditions that one is able to impose, so an answer to item *i*) would contribute to understanding how a solution to an elliptic–hyperbolic equation depends on the smoothness of boundary data.

iii) What is the most general type-change function for which Theorem 4.2 remains true for some vector field V? (In particular, see items *v*) and *vi*), below.)

iv) All these questions are also interesting for the variant of the model equation having the form

$$\left(x - y^2\right) u_{xx} - u_{yy} + \text{ lower-order terms} = 0, \tag{B.6}$$

a variant which we generally ignored in this text. The methods of Sect. 4.3 appear to fail utterly in this case.

v) The heart of the proof of Theorem 4.2 is the fundamental estimate of Lemma 4.1, the proof of which would fail without the term y^2 in the type-change function $\mathscr{K}(x, y) = x - y^2$. Moreover, the singular structure of the solution, which is its most physically and mathematically interesting feature, is apparently derived from the geometry of the parabola $y = x^{1/2}$ at the origin of coordinates. This raises the question of whether there is a larger, mathematically natural class of type-change functions for which these kind of arguments work. For example, what can one say about equations that change type along a conic section? How are the various geometric features of this class of curves reflected in the analysis of the problem? Because the type-change functions of Chap. 5 and 6 are circles, this program would unify the discussions of Chaps. 4–6. Note, however, that different analytic methods are used in Chaps. 4–6.

Distribution solutions for a closed Dirichlet problem having a type-change function in the form of a hyperbola are studied in the very recent paper [75].

vi) The questions posed in item *v*) may also be asked about toric sections. That is, can one say something about type-change functions having the general form

$$\mathscr{K}(x, y) = \left(x^2 + y^2\right)^2 + ax^2 + by^2 + cx + dy + e,$$

where a, b, c, d, and e are constants, and if so, how does the geometry of these curves affect the analysis? Of course, such type-change functions are also natural generalizations of those discussed in Chaps. 5 and 6.

vii) Lupo, Morawetz and Payne [58] have established the existence of a unique H^1_{loc} weak solution to the homogeneous closed Dirichlet problem for an inhomogeneous Tricomi equation of the form (1.1). It is known that the corresponding weak solution for the formally self-adjoint form of the cold plasma model (Chap. 4) is not $H^{1,2}$ in any neighborhood of the origin [77]; c.f. item 1 of Sect. A.5. It is natural to ask whether H^1_{loc} solutions to closed boundary value problems exist for *any* equations of Keldysh type having the structure of (1.3).

7. Concerning the discussion of of the fundamental solution for Cinquini-Cibrario's equation in Sect. 3.7:

i) Are there fundamental solutions to the cold plasma model equations? Some of the individual arguments in [9–11, 20, 131] extend to the cold plasma model (Sect. 4.4); but apparent problems arise due to the absence of a closed form for the characteristic lines of the cold plasma model equations.

ii) Fundamental solutions are used to associate an integral equation with a boundary value problem. This integral equation determines a Green's-function solution to the boundary value problem. Thus the existence of a fundamental solution for Cinquini-Cibrario's equation should open the way for the solution of open boundary value problems by the method of Green's functions. (I am grateful to an anonymous referee for making this point.) The boundary condition would be imposed on the elliptic boundary, and appropriate conditions on the domain boundary would be required. The regularity of the fundamental solution described in Sect. 3.7, for various lower-order terms, must be taken into account. The self-adjoint case is particularly interesting, for reasons that are clear from the examples in this text. This problem is, as far as I know, also open for the Tricomi equation, although in that case a more regular fundamental solution exists [9–11], and the Green's function method has already been applied to the degenerately elliptic case [104]. In particular, the authors of [9] suggest that unnatural restrictions on the boundary that accompany the classic papers on the existence of solutions to open boundary value problems for Tricomi's equation could be removed by the use of the Barros-Neto–Gelfand solution; a similar hope exists for Green's functions created by integrating the Chen solution to Cinquini-Cibrario's equation over a suitable domain.

8. Recall that we use the term *elliptic boundary* to refer to that part of the domain boundary on which the type-change function is positive. Similarly, by the *hyperbolic boundary* we mean the collection of boundary arcs for which the type-change function is negative. We define a *fully elliptic–hyperbolic* boundary value problem to be a boundary value problem for which the hyperbolic boundary is non-empty and the subset of the hyperbolic boundary on which data have been prescribed is

also non-empty. Note that this definition is independent of whether the boundary value problem is open or closed.

The fact that the data in Theorems 5.1 and 5.2 are prescribed only on the elliptic boundary is an important restriction, as it is the hyperbolic boundary on which the Dirichlet problem typically becomes over-determined. So there is obvious interest in extending Theorems 5.1 and 5.2 to fully elliptic–hyperbolic boundary value problems. For example, does a solution to the equations of Theorems 5.1 and 5.2 exist in an annulus about the sonic curve, as was shown for the Tricomi-type equation in [121]? One would also like a result for harmonic fields on the extended projective disc which is analogous to Theorem 5.1. In that case the boundary would be fully hyperbolic, which is much harder, but again one could look for a solution in an annulus.

Also regarding harmonic fields on the extended projective disc, there should be a proof of the existence of weak solutions to an open Dirichlet problem which does not require a rearrangement of the lower-order terms between the two equations in the system (c.f. Sect. 6.3.1).

9. There is no obvious reason why one should not have maximum principles for equations of Keldysh type which are analogous to those derived in [3, 62] for equations of Tricomi type; but there do not seem to be any such results in the literature. (I am indebted to Yuxi Zheng for this observation.) Similarly, eigenvalue problems for the equations of Sect. 6 are only known for very special cases [130] and appear to be completely unknown for the equations of Chaps. 2–4; c.f. Problem 12.

10. The existence and nature of water caustics have been the subject of both theoretical and experimental research. In Sect. 3.3.1 of [73], trapped water waves on a ridge and in a submarine trough are analyzed. It is shown that the geometrical optics approximation is valid and that the rays form an envelope, producing a caustic; see also [8, 105]. These analyses have been supported by experimental studies of ocean caustics [18, 93]. To what extent do the methods of Chap. 5 extend to water caustics?

There are certain apparent obstructions. For example, the refractive index is highly variable in realistic applications to water waves (see below). Also, water caustics often occur in the context of shoaling waves, in which turbulent effects tend to work against optical analogies. Nevertheless, an apparently reasonable model based on a Ludwig–Kravtsov-like system has been formulated by Chao for water waves near a caustic [17] – see also [18]. Chao's model proceeds from the fact that the equations of motion for an inviscid and incompressible liquid in simple harmonic motion, bounded below by an impervious and rigid bottom, can be written in the linearized, dimensionless form [74]

$$\lambda^2\varphi_{zz} + \Delta\varphi = 0, \; -h\,(x,y) \le z \le 0. \tag{B.7}$$

At $z = -h\,(x,y)$,

$$\lambda^2\varphi_z + \nabla h \cdot \nabla\varphi = 0, \tag{B.8}$$

and at $z = 0$,

$$\varphi_z = \varphi \tag{B.9}$$

and

$$\tilde{\eta} = \mathrm{Re}\left(-i\varphi e^{-i\tau}\right). \tag{B.10}$$

Here $\lambda = \mathscr{L}\omega^2/g$, where ω is the angular frequency of the wave, g is the acceleration due to gravity, and $\mathscr{L}$ is the horizontal scale length of the bottom contours; $\varphi = \nu\Phi$, where $\Phi = \Phi(x, y, z)$ is the velocity potential of the wave in cartesian coordinates (x, y, z), z is the vertical axis positive upward from the equilibrium water level, and $\nu = \omega^3/g^2$; h is the product of the water depth and the scaling factor $\lambda/\mathscr{L}$; $\tilde{\eta}$ is the product of the surface fluctuation and the same scaling factor; $i^2 = -1$; and $\tau = \omega t$, where t is time.

Expressing φ in the form

$$\varphi = w(x, y)\cosh\left[k(h + z)\right], \tag{B.11}$$

where k is a re-scaling of the wave number by the quantity $\lambda\mathscr{L}$, we notice that $\varphi_{zz} = k^2\varphi$ and (B.7)–(B.10) imply (5.12) in the form

$$\left(\Delta + k^2\lambda^2\right)\varphi = 0.$$

In this case the geometrical optics approximation is equivalent to letting λ tend to infinity. By the definition of λ, this would require the horizontal scale length of the bottom to be large relative to the deep-water wavelength $2\pi g/\omega^2$. Moreover, the arguments of [49], Sect. 3, can be used to obtain the identity

$$\lambda = (kh/S_b)\tanh kh,$$

where S_b denotes the bottom slope. This implies that, for fixed kh, λ will be large when the bottom is close to horizontal. Based on these considerations, Chao [17] estimates that that the geometrical optics approximation is reasonable for most areas of shoaling water except those in the neighborhood of the shoreline, in which regions the linearization itself fails as a result of surf effects. This form of the Helmholtz equation is not only mathematically equivalent to the equation satisfied by standing waves in wave optics; the physical interpretations of the coefficients k, λ are analogous as well, in the sense that the caustic associated to a shoaling wave is a refractive effect.

Condition (B.11) fails at a caustic, motivating the Ludwig–Kravtsov ansatz. In this context the Kravtsov–Ludwig ansatz consists in expressing the leading term of the velocity potential φ, for large λ, in the form

$$\varphi(x, y) = e^{i\lambda\theta(x,y)}\cosh\left[k(h + z)\right] \times \left\{\gamma_0(x, y)A\left[\lambda^{2/3}\rho(x, y)\right] + \frac{i\gamma_1(x, y)}{\lambda^{1/3}}A'\left[\lambda^{2/3}\rho(x, y)\right]\right\},$$

where the notation ρ, θ, γ_0, γ_1, and A is as in Sect. 5.1. We obtain from the Helmholtz equation an expression of the form

$$(\nabla\theta)^2 - \rho\,(\nabla\rho)^2 - k^2 = \nabla\theta \cdot \nabla\rho = 0. \tag{B.12}$$

Because we assume that the propagation speed is approximately equal to a nonzero constant c over a sufficiently short interval of time, we can replace the wave number k by a constant inversely proportional to c. Under such an assumption it is convenient to re-scale θ and ρ by defining new variables $\tilde{\theta} = k^{-1}\theta$ and $\tilde{\rho} = k^{-2/3}\rho$. In that case we can divide (B.12) by k^2 and replace the term k by the number 1. This results in an equation identical to (5.42).

Thus it is natural to wonder about the extent to which the arguments of [24, 25, 66–70] are applicable to Chao's model. A related question is the extent, if any, to which the method of evanescent wave tracking and complex ray tracing (see, e.g., [31, 32]) is applicable to water waves.

A potential difficulty in the water wave case is the inversion of the solution in the hodograph plane. It is likely that there will be some nonlinear terms in the physical plane which must be neglected if the system is to be put into the homogeneous form required for the hodograph transformation. In the case of optics, the refractive index can be taken to be constant, which forces the inhomogeneous terms in the nonlinear equation to vanish. That simplification does not seem to be realistic in the case of water waves.

We note that water caustics are not only caused by the propagation of deep-water waves into shoaling water. For example, circular water caustics may arise in the contexts of a marine explosion or the impact of a high-velocity body; c.f. [13, 127].

11. The distance element associated to the metric for a stationary, rotating, axisymmetric field has the general form [56]

$$ds^2 = -A(r)dt^2 + 2B(r)d\varphi dt + C(r)d\varphi^2 + D(r)\left(dr^2 + dz^2\right). \tag{B.13}$$

This metric is Lorentzian provided the matrix determinant

$$g = -\left(AC + B^2\right)D^2$$

is negative.

Examples include the *van Stockum metric* [118], for which $A(r) = 1$, $B(r) = \omega r^2$, $C(r) = r^2\left(1 - \omega^2 r^2\right)$, and $D(r) = \exp[-\omega^2 r^2]$. This metric arises in a model of a rotating infinite cylinder of dust, held stationary within a vacuum by the balance of centrifugal and gravitational forces on the dust. Because the variable φ in (B.13) is the angular coordinate, any azimuthal curve for which the variables t, r, and z are held constant will have invariant length

$$ds^2 = 4\pi^2 C(r).$$

The resulting integral curve will be a closed, time-like curve provided $C(r) < 0$. The role of such curves in van Stockum's model was apparently first realized by Tipler [119], long after the model was originally introduced. Closed, time-like curves are associated with certain violations of causality and for that reason have attracted interest.

The z-coordinate, representing the axis of rotation, is not very important in the analysis of the metric (B.13). If we ignore this variable, then in the stationary case we obtain, for $D(r) = 1$, the distance formula for the extended projected disc:

$$ds^2 = r^2\left(1 - \omega^2 r^2\right) d\varphi^2 + dr^2.$$

In this interpretation the angular velocity in van Stockum's model has become the hyperbolic curvature $K = -\omega^2$.

Writing (6.16) in its second-order form, we obtain the wave equation on the curved metric g (*i.e.*, the Laplace–Beltrami equation)

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{|g|}\frac{\partial u}{\partial x^j}\right) = 0. \tag{B.14}$$

This equation is often reduced by imposing an invariance condition which amounts to a consideration of stationary waves; see, e.g., Sect. 2 of [121]. Applying this operator to the metric associated to (B.13), we obtain the differential operator of (1.4) with $\mathscr{K}(r) = C/D$ and $k = 1/2$.

Thus in particular, we know from Theorem 3.1 not to expect classical solutions to a closed Dirichlet problem on a domain such as the one constructed in Sect. 3.3. It is therefore natural to ask whether strong solutions exist, by applying the approach of Theorem 5.2.

Applying the first-order form of the equation, (6.16), with $u_1 = u_x$, $u_2 = u_y$, and taking $A(t) = B(t) = 0$ in (B.13) we obtain, ignoring the z-coordinate, the system

$$\left[\frac{(|C|D)_r - 2|C|D_r}{2|C|D^2}\right]u_1 + \frac{u_{1r}}{D} + \frac{u_{2\varphi}}{C} = f \tag{B.15}$$

$$u_{1\varphi} - u_{2r} = 0. \tag{B.16}$$

In order to apply the methods of Theorem 5.2, we might consider this system in the disc

$$\Omega = \{(r,\varphi)\,|0 \le r \le R, 0 < \varphi \le 2\pi\},$$

and assume that $D(r)$ exceeds zero on Ω and that $C(r)$ changes from positive to negative sign on a circle $r = r_{crit}$ in the interior of Ω. The given function f on the right-hand side of (B.15) has been inserted for mathematical generality; see also Sect. 2.6. Multiplying (B.15) by C and carrying out the indicated operations, we obtain the simpler equation

$$Mu_1 + \frac{C}{D}u_{1r} + u_{2\varphi} = \tilde{f}, \tag{B.17}$$

where

$$M = \frac{C_r D - CD_r}{2D^2}$$

and $\tilde{f} = Cf$. It is easy to check that this system is not symmetric positive and that the methods used to prove Theorem 5.2 do not apply in any obvious way.

Similarly, the methods used in Chap. 4 to obtain weak solutions do not have an obvious application. Thus it remains an interesting open problem to determine conditions under which weak or strong solutions exist for the Laplace–Beltrami equations on a relativistically rotating disc on which closed, time-like lines are permitted.

12. Lupo, Payne, and occasional collaborators have studied the spectral properties of linear and semilinear Tricomi operators [57,59,63]. I know of no such results for elliptic–hyperbolic operators of Keldysh type.

13. An existence theorem for the closed Neumann problem has been proven for the Lavrent'ev–Bitsadze equation (2.5) by Pilant [94]. No other result for conormal conditions on the entire boundary is known for elliptic–hyperbolic equations of either type.

B.1 Nonlinear Equations

The problem of extending these methods to nonlinear equations deserves a discussion of its own.

14. *A nonlinear extension of problem 10:* The focusing of wave action in a caustic region has been advanced as an explanation for giant *rogue waves* which have been observed, for example, in the Agulhas current off the southeast coast of Africa; see [36, 53, 92, 108, 128]. In this case as well, related experimental studies have been conducted; see, e.g., [38] and, for deep-water examples, the statistical studies cited in Sect. 1 of [27].

In studying the interaction of a wave with an opposing current, a ray approximation is indicated by comparison of the length of even large wind-induced swells, such as the ones correlated with accidents in the Agulhas, with the scale of horizontal variations of the current. (This point is made in [108], p. 417.) Models of rogue waves based on the theory of caustics have been criticized on different grounds: that the incoming ocean wave would have to enter the zone of variable currents with a single direction. Otherwise, the wave would be too diffuse to focus into a caustic; c.f. [27], Sect. 3.

An analogy can be drawn to optical caustics at the bottom of a swimming pool on a *sunny* day (unidirectional incoming wave) versus their absence on a *cloudy* day (diffuse incoming wave). In a swimming-pool caustic, the light waves scatter by refraction through the water. In the model of rogue waves, the incoming wave is refracted by collision with the fast-moving Agulhas current. A sunny day would correspond to a unidirectional current, which would presumably be very rare (maybe too rare). A cloudy day would correspond to the usual case of a current composed of individual waves moving in multiple directions. This analysis suggests that focusing *can* produce extreme waves; what fraction of the observed examples actually *is* produced by them is not so clear. As a reaction to this kind of criticism, recent papers have concentrated on a statistical analysis of the probability of a caustic-producing current [37]; see also the recent experimental work reported in [120].

However, this problem may also be attacked through a better understanding of the relevant partial differential equations, which remain poorly understood. There exist deep-water models for effects exerted on water waves by reflection of rays at a caustic, which take into account diffractive effects arising from nonlinear terms in the governing equations. Those models are based on the observation that the governing equations for the wave amplitude can be put into the approximate form of a nonlinear Schrödinger equation [132]. Using this kind of approach, it has been estimated that a three-fold amplification of wave amplitudes could occur near a caustic in the collision of a wave with the Agulhas Current [108]. Moreover, rogue waves in this model possess an asymmetry that accounts for reports of a deep trough that precedes the steep forward face observed in rogue waves. The presence of such a trough, and the asymmetrical steepness of the forward face of the wave, explain some of the destructive effects of such waves on oil tankers. (No change of equation type is associated with such models.)

But extreme waves also occur in shallower waters. So it is possible that a better understanding of shallow-water models such as the one described in item *x*) of Sect. 2.7.2 might also be welcome. These have been used in studying, for example, the focusing of tidal waters by a narrowing at the mouth of channel, inducing a change in their velocity as the tide enters the still water of the channel [123]. A similar effect occurs as a wave progresses up a sloping beach. In each case the change in the velocity profile corresponds to the focusing effects in optics which arise from changes in the refractive index of the medium. A geometrical optics approximation is reasonable in this case as well, as the relevant dimension for applying the geometrical optics approximation is the ratio of the width of the channel entrance to the horizontal scale of the incoming tide. In cases for which this ratio is small, ray effects will dominate over diffractive effects at the mouth of the channel. Some technical issues regarding the geometrical optics approach in shallow water theory are addressed in Sect. 1 of [49].

While there is already a large interdisciplinary literature on various aspects of extreme waves, the literature on elliptic–hyperbolic transitions in hydrodynamics is considerably smaller. A possible resource is the large literature on elliptic–hyperbolic transitions and shock waves in gas dynamics, which are analogous in some respects, but not analogous in others.

15. Existence/nonexistence theorems for semilinear forms of Cinquini-Cibrario's equation, even in the degenerate elliptic form studied by Keldysh, would presumably have applications to the study of magnetically dominated plasmas. (See, e.g., the discussion following (16), (17) in [125]; there is interest in various contexts for $\overline{Q}$ having a polynomial dependence on $\overline{P}$ in (16) of that reference. A similar remark applies to the discussion preceding (8) in [40].) An approach to the nonexistence question might be to try to extend the methods of [64, 65] to certain equations of Keldysh type. Tricomi problems for semilinear equations of Tricomi type are investigated in [60]; those results may have extensions to equations of Keldysh type.

16. A quasilinear theory of symmetric positive operators is outlined in [34], the Appendix to [35], and [124]; but the quasilinear case has been much less intensively studied than the linear case, both in terms of its mathematical properties and its applications. Quasilinear elliptic–hyperbolic problems in general, other than those associated with gas dynamics, have not been studied very much except in linearized forms. Recent exceptions include [22, 23]. Applications of methods originating in gas dynamics to various problems in general relativity have been pioneered by Smoller in papers with various collaborators; see, e.g., [110]. The hypotheses in those papers are, appropriately, intimately connected with the physical model, and it is not clear how they would apply to a general mathematical theory for equations of Keldysh type. Of course the distinction between Keldysh type and Tricomi type applies to linear and, by an obvious extension, semilinear equations. However, certain quasilinear equations become equations of Keldysh type in a natural linearization. An example (in addition to those in Chaps. 5 and 6) is

$$\left((u+x)\,u_x - \frac{u}{2}\right)_x + u_{yy} = 0$$

for $u = u\,(x, y)$, which is studied in [15, 16].

We note that recently, Friedrichs' criteria for admissible boundary conditions have been related to three apparently different sets of intrinsic geometric conditions in graph spaces [30]. Subsequently, those intrinsic boundary conditions were shown to be equivalent when reinterpreted in the context of Kreĭn spaces [5]. These papers are examples of a resurgence of interest in symmetric positive operators among theoretical numerical analysts; see also [14,29,41]. It is possible that these reformulations of the theory in more abstract contexts will facilitate quasilinear extensions.

17. The technique reviewed in Sect. 2.6, for deducing the existence of solutions to an inhomogeneous Dirichlet problem for a homogeneous equation by solving a homogeneous Dirichlet problem for the corresponding inhomogeneous equation, does not extend to nonlinear equations. Thus the entire issue of Dirichlet problems having inhomogeneous data becomes important for nonlinear equations to a degree that is not encountered in the linear case. As noted in Sect. 5.3, inhomogeneous boundary conditions become more complicated under linearization of the associated equation by the hodograph map; so difficulties in boundary conditions for nonlinear equations are only rarely improved by a hodograph linearization, even under the restrictive conditions in which such linearizations are possible.

18. Because the ellipticity condition for Euler–Lagrange equations can double as a convexity condition for many of the standard energy functionals, variational approaches tend to be associated with equations of elliptic type. But, for example, General Relativity is a variational theory having strictly hyperbolic variational equations; and, as illustrated in Sect. 2.7, a large class of quasilinear elliptic–hyperbolic equations can be derived as Euler–Lagrange equations of an energy functional. We will consider two candidates for further research – one involving semilinear equations and the other involving quasilinear equations:

i) Variational methods for open boundary value problems involving semilinear equations of Tricomi type have been pioneered by Lupo and Payne ([60]; see also [57, 61]). The papers employ a *dual variational method*, in which a semilinear partial differential equation of the form $Lu = F(u)$, having variational structure, is solved by first inverting the corresponding linear operator and then treating the resulting equation

$$u = L^{-1}F(u) \tag{B.18}$$

by variational means.

Consider an open Tricomi boundary value problem of the form considered in (3.1)–(3.3), but with a nonlinear term $Rf(u)$ added to (3.1). Here R is the reflection operator on $L^2(\Omega)$ induced by composition with the map taking points (x, y) of $\mathbf{R}^2$ into points $(-x, y)$, where Ω has the general form of the domain in Fig. 3.1. In particular, Ω is symmetric about the y-axis and is bounded smoothly by a Jordan curve in the elliptic region of the equation and piecewise smoothly by two characteristic lines Γ_1 and Γ_2.

For a fixed arc $\Gamma \in \partial\Omega$, the associated Sobolev space is the closure of H^1_Γ with respect to the H^1-norm of the set $C_\Gamma\left(\overline{\Omega}\right)$ consisting of smooth functions on $\overline{\Omega}$ vanishing identically on Γ. This space has a dual, H^{-1}_Γ, formed in the expected way (c.f. [61], (1.1)).

The fundamental difficulty with applying variational methods in this context is that the Tricomi operator for an open boundary value problem does not map H^1_Γ onto its dual, but rather onto the dual of the adjoint problem. If, for example, the solution is constrained to vanish on the elliptic boundary and a characteristic curve Γ_1, then the dual of the adjoint problem has data vanishing on the elliptic boundary and the characteristic curve Γ_2. In the dual variational method, the Tricomi operator is composed with the operator R, which induces an isometric isomorphism between the adjoint boundary spaces. In this way, a variational structure can be associated with the boundary value problem. The underlying idea is to then take advantage of the compactness of the inverse of the linear Tricomi operator T by solving (B.18) for $L = RT$ and $F = Rf$.

One question is whether this method can be extended to treat equations associated with different or more general elliptic–hyperbolic operators. But there are also interesting avenues of research for the dual variational method in its current application to the Tricomi operator. For example, because it is necessary to obtain the continuity of the Nemitskii operator associated to the nonlinearity, this method requires the nonlinearities to have at most asymptotically linear growth. In order to treat nonlinearities having superlinear growth, one would have to develop a suitable L^p theory for the linear Tricomi operator; see [61], Sect. 1.

ii) The nonlinear Hodge–Frobenius equations (5.76), (5.77) for a differential form of arbitrary degree can be derived by applying variational arguments to a nonlinear Hodge energy functional as in Sect. 2.7.1. We obtain ([72], Sect. 5.1.1)

$$d * [\rho(Q)\omega] = -d\eta \wedge * [\rho(Q)\omega] , \tag{B.19}$$

where now ω is a gradient-recursive k-form and η is a 0-form. The other notation is as in Sect. 5.6.1. This is a quasilinear equation of roughly similar form to the class considered by Tso [124]. It is natural to wonder whether this quasilinear variational equation, which for appropriate ρ may be of elliptic–hyperbolic type, can be treated by the quasilinear extensions of the theory of symmetric positive operators introduced by Gu and Tso.

A condition broadly analogous to the Frobenius condition (5.77) arises if ω is taken to be a Lie-algebra-valued 2-form F_A, where A is a Lie-algebra-valued 1-form. In that case, the second Bianchi identity

$$dF_A = -[A, F_A], \tag{B.20}$$

where [,] denotes the Lie bracket, has a form analogous to (5.77).

Precisely, let X be a vector bundle over a smooth, finite, oriented, n-dimensional Riemannian manifold M. Suppose that X has compact structure group $G \subset SO(m)$. Let $A \in \Gamma\,(M, ad\ X \otimes T^*M)$ be a connection 1-form on X having curvature 2-form

$$F_A = dA + \frac{1}{2}[A,\ A] = dA + A \wedge A,$$

where [,] is the bracket of the Lie algebra $\Im$, the fiber of the adjoint bundle $ad\ X$. Sections of the automorphism bundle $Aut\ X$ are *gauge transformations,* acting tensorially on F_A but affinely on A; see, *e.g.,* [71].

One can form energy functionals analogous to the nonlinear Hodge energy, in which $Q = |F_A|^2 = \langle F_A, F_A\rangle$ is an inner product on the fibers of the bundle $ad\ X \otimes \Lambda^2\,(T^*M)$. The inner product on $ad\ X$ is induced by the normalized trace inner product on $SO(m)$ and that on $\Lambda^2\,(T^*M)$, by the exterior product $*\,(F_A \wedge *F_A)$.

A nonabelian variational problem analogous to (5.77), (5.76) is described briefly in Sect. 5.1 of [84]. One is led to consider smooth variations having the form

$$var\,(E) = \int_M \rho(Q) var(Q) dM = \int_M \rho(Q) \frac{d}{dt}_{|t=0} |F_{A+t\psi}|^2 dM$$
$$= \int_M \rho(Q) \frac{d}{dt}_{|t=0} \left| F_A + t D_A \psi + t^2 \psi \wedge \psi \right|^2 dM,$$

where $D_A = d + [A, \,]$ is the exterior covariant derivative in the bundle. The Euler–Lagrange equations are

$$\delta \left(\rho(Q) F_A \right) = - * [A, *\rho(Q) F_A] \,. \tag{B.21}$$

In addition, we have the Bianchi identity (B.20).

If we write (5.77) in components

$$d\omega^a = \Gamma^a_b \wedge \omega^a,$$

then if $-\Gamma$ is interpreted as a connection 1-form, (5.77) can be interpreted as the vanishing of an exterior covariant derivative, which is the analytic content of (B.20). Moreover, we recover the well known algebraic requirement that Γ must satisfy

$$\left(d\Gamma^a_b - \Gamma^a_c \wedge \Gamma^c_b \right) \wedge \omega^b = 0$$

(c.f. (4-2.3) of [28]) as a zero-curvature condition:

$$[F_\Gamma, \omega] = 0.$$

If a suitable modification of the methods of [124] can be applied to (B.19), (5.77), then the question arises whether they can be applied to the nonabelian extension (B.20), (B.21). It may be necessary to introduce a suitably large lower-order perturbation in order for there to be any hope of using these methods.

On the one hand it seems absurd to suggest the study of the elliptic–hyperbolic form of systems such as (B.20), (B.21) in a text which devotes an entire chapter to an equation as simple as the cold plasma model equation (4.35). But on the other hand, the equations of motion for a plasma in their full generality are not noticeably simpler than the system (B.20), (B.21). A major feature of the study of such a system is the search for physically or geometrically reasonable special cases in which the equations simplify to analytically tractable forms.

19. The prescription of asymptotic boundary conditions at infinity is the global analogue of the Dirichlet problem. Variational conditions which lead to analytic results under growth hypotheses at infinity are notable for being at least superficially independent of type, especially if the variations are taken by reparametrization of the underlying domain – so-called *r-variations* [4] – rather than in the infinitesimal deformation space of the solution.

For example, consider an energy integral having the very general form

$$E = \int_M w\left(|du|^p\right) dv_g, \;\; p > 0,$$

where M is a Riemannian manifold having local metric g and u is a map from M into another Riemannian manifold N. This functional was introduced in [80] in the context of a Liouville theorem for a stationary point under r-variations, with prescribed growth conditions on the energy. The function w is assumed to satisfy the conditions

$$0 \leq \dot{w}(t) \leq K_1, \tag{B.22}$$

$$K_2 t \leq w(t) \tag{B.23}$$

for constants $K_1 \geq 0$ and $K_2 > 0$. The map u is an example of what would later be called an *F-harmonic map* [6]. Condition (B.23) and the boundedness of the derivative in condition (B.22) are not a feature of F-harmonic maps in general; they are imposed in order to derive the Liouville theorem, which is not satisfied for an r-stationary F-harmonic map without some extra hypotheses. The type of the variational equations of this object is not strictly specified without some further condition on the function w.

In the special case in which

$$w\left(|du|^p\right) = \int_0^Q \rho(s) ds,$$

where $Q = |du|^2$, the functional E reduces to the nonlinear Hodge energy of Sect. 2.7.1 (modulo the multiplicative constant 1/2). However, the variations satisfied by r-stationary points of E are quite different from those which produce the nonlinear Hodge equations (5.71), (5.72), as the latter are taken in the infinitesimal deformation space of the map rather than in the reparametrization space of the underlying domain.

Asymptotic conditions are imposed on both the manifold and the energy in [80] under which r-stationary points are forced to be globally trivial.

One purpose of Liouville theorems is to decide how singular a geometric object must be in order to avoid global triviality. For this reason, the domain manifold in [80] is allowed to have a singular set of prescribed Hausdorff dimension. In addition to mappings, Liouville theorems have been established for variational points that live on a vector bundle, and which are associated, by a particular choice of mass density, to energies of generalized Yang-Mills or Yang-Mills–Born-Infeld type. See, e.g., Sect. 5 of [84] and the references therein – also Sect. 1 of [83] and Sect. 4 of [107]. The very recent paper [26] introduces an associated triviality condition for the Dirichlet problem and a collection of vanishing theorems for differential forms. See also Sect. 2 of [82], Sect. 1 of [39], and Sect. 4.2 of [72].

It is natural to ask what geometric and analytic hypotheses would be necessary and/or sufficient in order to extend these results from the case of Riemannian manifolds to a class of semi-Riemannian manifolds. Another obvious application of the results would be to apply them to specific densities of physical or geometric interest, such as those cited in Sect. 2.7.

20. I am indebted to Yisong Yang for drawing my attention to the following problem, which arises in quantum field theory [1, 2].

By a series of fortunate approximations, the partition function for quantum electrodynamics can be reduced to a relativistic model in which the quarks are coupled to a pair of classical gauge fields. The latter are represented by a vector-valued gauge potential $\mathbf{A}$ and a scalar-valued field ϕ. (Note that a "scalar" in this context is what mathematicians might call a *weighted relative tensor* – see, e.g., the discussion on pp. 105–107 of [81].) Additional choices reduce the analysis to a problem in nonlinear electrostatics with the governing equations

$$\nabla \cdot (\sigma \nabla \Phi) = 0, \tag{B.24}$$

where Φ is a – suitably interpreted – flux, and σ a prescribed function of the cylindrical radial coordinate ρ and the scalar $|\nabla \Phi|$. This equation is superficially similar to (2.35) of Sect. 2.7; but the function

$$\sigma = \sigma\left(\rho, |\nabla \Phi|\right)$$

is *not* assumed to be a quadratic function of $\nabla \Phi$; see (20b) and (21) of [2].

Defining the inward-pointing unit normal and its normal derivative in terms of Φ via

$$\hat{\mathbf{n}} = \frac{\nabla \Phi}{|\nabla \Phi|}, \;\; \partial_n = \hat{\mathbf{n}} \cdot \nabla,$$

it can be shown that (B.24) reduces to

$$\left[\partial_\rho^2 + \partial_z^2 + (\alpha - 1)\, \partial_n^2\right] \Phi - \alpha \rho^{-1} \partial_\rho \Phi = 0, \tag{B.25}$$

where

$$\alpha = 1 + \frac{\partial \log \sigma}{\partial \log |\nabla \Phi|}.$$

Defining $\hat{\ell}$ to be the unit tangent vector to the surface of constant Φ and ∂_ℓ to be the corresponding tangential derivative, the differential operator of (B.25) can be put into the form

$$L\Phi = \left[\partial_\ell^2 + \alpha \partial_n^2\right] \Phi.$$

That is, L is elliptic, hyperbolic, or parabolic, depending on whether α is positive, negative, or zero. The nonlinearities of this operator are considerably wilder than those of the operators studied in this text, and it is not clear how L can be effectively linearized. However, the model underlying this analysis is very rich, and opportunities may be found for further simplification.

For example, in the *leading logarithm model* considered in [1, 2], still more choices reduce (B.25) to a degenerately elliptic equation. It is not clear whether (B.25), which applies to any effective action density depending quadratically on the electric field, is accessible without imposing severe physical restrictions on the generality of the model.

21. The final problem is, in a sense, the most difficult. It is the problem that the linear theory of elliptic–hyperbolic equations yields relatively little qualitative insight into the nonlinear theory. Physical models of classical fields are generally nonlinear, which means that they are in most cases beyond the reach of linear analysis.

This is similar to the case of hyperbolic differential equations. But it is in sharp distinction to the theory of elliptic (and parabolic) differential equations, in which the qualitative behavior of the potential equation (and heat equation) is a rough guide to the corresponding nonlinear theory; c.f. [33, 52, 78].

Consider for example the nonlinear equations which arise in the two-dimensional, compressible Euler equations for an ideal fluid. We briefly review a few aspects of this vast and complex topic, following [133, 135]. The governing equations in this case are:

i) The conservation of mass:

$$\rho_t + (\rho u)_x + (\rho v)_y = 0;$$

ii) The conservation of linear momentum in u :

$$(\rho u)_t + \left(\rho u^2 + p\right)_x + (\rho uv)_y = 0;$$

iii) The conservation of linear momentum in v :

$$(\rho v)_t + (\rho uv)_x + \left(\rho v^2 + p\right)_y = 0;$$

iv) The conservation of energy:

$$(\rho E)_t + (\rho uE + up)_x + (\rho vE + vp)_y = 0,$$

where

$$E = \frac{u^2 + v^2}{2} + e,$$

for $(x, y) \in \mathbf{R}^+$, $t \in \mathbf{R}$. Here e denotes the internal energy of the system. For a polytropic gas,

$$e = \frac{p}{(\gamma - 1)\,\rho},$$

where $\gamma > 1$ is the adiabatic constant of the gas.

A simpler model, in which we take $\rho = 1$ and ignore inertial contributions, is the *pressure-gradient system*:

$$u_t + p_x = 0,$$
$$v_t + p_y = 0,$$
$$E_t + (pu)_x + (pv)_y = 0.$$

Assuming that solutions are sufficiently smooth and applying the coordinate transformation

$$p = (\gamma - 1)\, P,\ t = \frac{T}{\gamma - 1},$$

this system reduces to the single equation

$$\left(\frac{P_T}{P}\right)_T - \left(P_{xx} + P_{yy}\right) = 0. \tag{B.26}$$

The Cauchy problem for even this relatively simple form of the equations is an open problem.

There is also a hydrodynamic interpretation of this model, representing flow in a shallow channel with a bump [21]; c.f. Problem 14.

Some progress has been made in the analysis of self-similar solutions, which occur naturally in many contexts. For example, it is often useful to seek solutions depending on the new variables

$$\xi = \frac{x}{T},\ \eta = \frac{y}{T}$$

in which case (B.26) assumes the form

$$\left(P - \xi^2\right) P_{\xi\xi} - 2\xi\eta P_{\xi\eta} + \left(P - \eta^2\right) P_{\eta\eta} + \frac{\left(\xi P_\xi + \eta P_\eta\right)^2}{P} - 2\left(\xi P_\xi + \eta P_\eta\right) = 0. \tag{B.27}$$

Near the origin this equation is elliptic, but far from the origin, it is hyperbolic. The well-posedness of boundary value problems outside of the elliptic regime is a rich source of open problems. Another source of open problems is the reflection of the wave off of obstacles having various geometries; see, e.g., [19, 134] and references therein.

Consideration of such problems, even on a superficial level, would take us very far afield, and for that reason it is natural to end our course here.

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