Appendix A
Spherical Coordinates

The formalism for Landau level quantization on the sphere developed in Sect. 2.1.5 requires vector analysis in spherical coordinates. In this appendix, we will briefly review the conventions. Vectors and vector fields are given by

$$\mathbf{r} = r \mathbf{e}_r,$$

$$\mathbf{v}(r) = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi,$$  \hspace{1cm} (A.1)

with

$$\mathbf{e}_r = \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} \cos \phi \cos \theta \\ \sin \phi \cos \theta \\ -\sin \theta \end{pmatrix}, \quad \mathbf{e}_\phi = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}.$$  \hspace{1cm} (A.3)

where $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$. This implies

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi, \quad \mathbf{e}_\theta \times \mathbf{e}_\phi = \mathbf{e}_r, \quad \mathbf{e}_\phi \times \mathbf{e}_r = \mathbf{e}_\theta,$$  \hspace{1cm} (A.4)

and

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_\phi}{\partial \theta} = 0,$$

$$\frac{\partial \mathbf{e}_r}{\partial \phi} = \sin \theta \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos \theta \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin \theta \mathbf{e}_r - \cos \theta \mathbf{e}_\theta.$$  \hspace{1cm} (A.5)

With

$$\mathbf{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$  \hspace{1cm} (A.6)

we obtain

$$\nabla \mathbf{v} = \frac{1}{r^2} \frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta v_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}.$$  \hspace{1cm} (A.7)
\[ \nabla \times \mathbf{v} = e_r \frac{1}{r \sin \theta} \left( \frac{\partial (\sin \theta v_\phi)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi} \right) + e_\theta \left( \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{1}{r} \frac{\partial (rv_\phi)}{\partial r} \right) + e_\phi \left( \frac{1}{r} \frac{\partial (rv_\theta)}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right), \]  
(A.8)

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \]  
(A.9)
Appendix B
Fourier Sums for One-Dimensional Lattices

In this appendix we collect and proof some useful formulas for the explicit calculations of the Haldane–Shastry model. In particular, we provide the Fourier sums required for the evaluation of the coefficients $A_l$ in (2.2.40) using two different methods, first by contour integration loosely following Laughlin et al. [1], and second by Feynmanesque algebra.

For $\eta_\alpha = e^{i\frac{2\pi}{N} \alpha}$ with $\alpha = 1, \ldots, N$ the following hold:

(a) $\eta_\alpha^N = 1$. \hfill (B.1)

(b) $\sum_{\alpha=1}^{N} \eta_\alpha^m = N \delta_{m,0} \mod N$. \hfill (B.2)

(c) $\prod_{\alpha=1}^{N} (\eta - \eta_\alpha) = \eta^N - 1$. \hfill (B.3)

**Proof** The $\eta_\alpha$ are by definition roots of 1. \hfill \square

(d) $\sum_{\alpha=1}^{N} \frac{1}{\eta - \eta_\alpha} = \frac{N\eta^{N-1}}{\eta^N - 1}$. \hfill (B.4)

**Proof** Take $\frac{\partial}{\partial \eta}$ of (B.3) and divide both sides by $\eta^N - 1$. \hfill \square
Proof Substitute $\eta_\alpha \rightarrow \frac{1}{\eta_\alpha}, \eta \rightarrow \frac{1}{\eta}$ in (B.4) and divide by $(-\eta)$.

\begin{equation}
\sum_{\alpha=1}^{N} \frac{\eta_\alpha}{\eta - \eta_\alpha} = \frac{N}{\eta^N - 1}.
\end{equation}

(f)

\begin{equation}
\sum_{\alpha,\beta,\gamma=1}^{N} \frac{\eta_\gamma^2}{(\eta_\alpha - \eta_\gamma)(\eta_\beta - \eta_\gamma)} = \frac{N(N-1)(N-2)}{3}.
\end{equation}

Proof Use the algebraic identity

\begin{equation}
\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} = 1.
\end{equation}

(g)

\begin{equation}
\sum_{\alpha,\beta=1}^{N-1} \frac{1}{(\eta_\alpha - 1)(\eta_\beta - 1)} = \frac{(N-1)(N-2)}{3}.
\end{equation}

Proof Substitute $\eta_\alpha \rightarrow \eta_\alpha \eta_\gamma, \eta_\beta \rightarrow \eta_\beta \eta_\gamma$ in (B.6)

(h)

\begin{equation}
\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{\eta_\alpha - 1} = \frac{N+1}{2} - m, \quad 1 \leq m \leq N
\end{equation}

Proof by contour integration Use Cauchy’s theorem \[2\] for the function

\[ f(z) = \frac{z^{m-1}}{z-1}, \quad N \geq 2, \]

with the contours shown in Fig. B.1 yields

\begin{align*}
\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{\eta_\alpha - 1} &= \frac{1}{2\pi i} \sum_{\alpha=1}^{N-1} \oint_{C} \frac{z^{m-1}}{z - 1 - \eta_\alpha} \, dz \\
&= \frac{N}{2\pi i} \oint_{C} \frac{z^{m-1}}{(z-1)(z^N - 1)} \, dz \\
&= \frac{N}{2\pi i} \oint_{C'} \frac{z^{m-1}}{(z-1)(z^N - 1)} \, dz = f(z)
\end{align*}
where we have first used (B.5) and then deformed the contour $C$ such that the radius of circle goes to infinity, used that the circle at infinity does not contribute to the integral as the integrand falls off as at least $1/z^2$ for $m \leq N$, and finally reversed the direction of integration to replace $C$ by $C'$.

Since $f(z)$ has a pole of second order at $z = 1$, the residue is given by

$$c_{-1} = \lim_{z \to 1} z \frac{d}{dz} (z - 1)^2 f(z) = -\lim_{z \to 1} \frac{g'(z)}{g^2(z)}$$

With

$$g(z) = \frac{1}{z^{m-1}} \frac{z^N - 1}{z - 1} = \sum_{k=1}^{N} z^{k-m} \overset{z \to 1}{\longrightarrow} N,$$  \hspace{1cm} (B.10)$$

$$g'(z) = \sum_{k=1}^{N} (k-m)z^{k-m-1} \overset{z \to 1}{\longrightarrow} \frac{N(N+1)}{2} - mN,$$  \hspace{1cm} (B.11)$$

we obtain

$$\sum_{\alpha=1}^{N-1} \eta_{\alpha}^m = -Nc_{-1} = \frac{(N + 1)}{2} - m.$$ 

**Proof by algebra** With the definition

$$S_m \equiv \sum_{\alpha=1}^{N-1} \frac{\eta_{\alpha}^m}{\eta_{\alpha} - 1},$$

we find

$$S_{m+1} - S_m = \sum_{\alpha=1}^{N-1} \eta_{\alpha}^m = \begin{cases} -1, & 1 \leq m \leq N - 1, \\ N - 1, & m = 0. \end{cases}$$

and

$$S_0 = \sum_{\alpha=1}^{N-1} \frac{1}{\eta_{\alpha} - 1} = \sum_{\alpha=1}^{N-1} \frac{\eta_{\alpha}}{\eta_{\alpha} - 1} = -S_1,$$
where we substituted $\eta_\alpha \to \frac{1}{\eta_\alpha}$. This directly implies

$$S_1 = -S_0 = \frac{N - 1}{2} \quad \text{and} \quad S_m = \frac{(N + 1)}{2} - m, \quad 1 \leq m \leq N.$$

(i)

$$\sum_{\alpha=1}^{N-1} \frac{1}{\eta_\alpha - 1} = -\frac{N - 1}{2}. \quad (B.12)$$

Proof Use (B.9) with $m = N$.

(j)

$$\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{(\eta_\alpha - 1)^2} = -\frac{N^2 - 1}{12} + \frac{(m - 1)(N - m + 1)}{2}, \quad 1 \leq m \leq N. \quad (B.13)$$

Proof by contour integration In analogy to the proof of (B.9) we write

$$\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{(\eta_\alpha - 1)^2} = \frac{1}{2\pi i} \sum_{\alpha=1}^{N-1} \int_C \frac{z^{m-1}}{(z - 1)^2} \frac{\eta_\alpha}{z - \eta_\alpha} \, dz$$

$$= \frac{N}{2\pi i} \int_C \frac{z^{m-1}}{(z - 1)^2(z^N - 1)} \, dz$$

$$= -\frac{N}{2\pi i} \int_{-h(z)} \left( \frac{z^{m-1}}{(z - 1)^2(z^N - 1)} \right) \, dz$$

where we have again used (B.5) and replaced the contour $C$ by $C'$. As $h(z)$ has a now pole of third order at $z = 1$, the residue is given by

$$c_{-1} = \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} \left( (z - 1)^3 h(z) \right) = \lim_{z \to 1} \left( -\frac{g''(z)}{2g(z)} + \frac{(g'(z))^2}{g^3(z)} w \right).$$

With $g(1)$ and $g'(1)$ as given by (B.10) and (B.11) and

$$g''(z) = \sum_{k=1}^{N} (k - m)(k - m - 1)z^{k-m-2}$$

$$\xrightarrow{z \to 1} \frac{N(N+1)(2N+1)}{6} - (2m + 1) \frac{N(N+1)}{2} + m(m + 1),$$

we find after some algebra that $-Nc_{-1}$ equals the expression on the right of (B.13).

Proof by algebra With the definition

$$R_m \equiv \sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{(\eta_\alpha - 1)^2},$$

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we find

\[ R_{m+1} - R_m = \sum_{\alpha=1}^{N-1} \frac{\eta^m_{\alpha}}{(\eta_{\alpha} - 1)} = S_m \]

and

\[
R_0 = \sum_{\alpha=1}^{N-1} \frac{1}{(\eta_{\alpha} - 1)^2} \\
= \sum_{\alpha=1}^{N-1} \sum_{\beta=1}^{N-1} \frac{1}{(\eta_{\alpha} - 1)(\eta_{\beta} - 1)} - \sum_{\alpha, \beta=1}^{N-1} \frac{1}{(\eta_{\alpha} - 1)(\eta_{\beta} - 1)} \\
= -\frac{(N-1)(N-5)}{12},
\]

where we have used (B.12) and (B.8). This implies

\[
R_{m+1} = R_0 + S_0 + \sum_{n=1}^{m} S_n \\
= -\frac{N^2 - 1}{12} + \sum_{n=1}^{m} \left( \frac{(N+1) - n}{2} \right) \\
= -\frac{N^2 - 1}{12} + \frac{m(N-m)}{2},
\]

for \(1 \leq m \leq N\).

(k)

\[
\sum_{\alpha=1}^{N-1} \frac{1}{(\eta_{\alpha} - 1)^2} = -\frac{(N-1)(N-5)}{12}.
\]

**Proof** Use (B.13) with \(m = N\).

(l)

\[
\sum_{\alpha=1}^{N-1} \frac{\eta^m_{\alpha}}{|\eta_{\alpha} - 1|^2} = \frac{N^2 - 1}{12} - \frac{m(N-m)}{2}, \quad 0 \leq m \leq N.
\]

**Proof** Use (B.13) with \(m \to m + 1\).

(m)

\[
\sum_{\alpha=1}^{N} \frac{\eta_{\alpha} + \eta_{\beta}}{\eta_{\alpha} - \eta_{\beta}} = 0.
\]
\textit{Proof} Substitute $\eta_{\alpha} \rightarrow \frac{1}{\eta_{\alpha}}$, $\eta_{\beta} \rightarrow \frac{1}{\eta_{\beta}}$ in one of the terms or use (B.9) and (B.12).

\[ \sum_{\alpha=1}^{N} \frac{1}{\eta_{\alpha} - \eta_{\beta}} \cdot \frac{1}{\eta_{\alpha} - \eta_{\gamma}} = - \frac{\eta_{\beta}}{\eta_{\beta} - \eta_{\gamma}} \left( \frac{\eta_{\gamma}}{\eta_{\alpha} - \eta_{\beta}} + \frac{\eta_{\beta}}{\eta_{\beta} - \eta_{\gamma}} \right) \]

which follows directly from (B.9), we write

\[ \sum_{\alpha=1}^{N} \frac{1}{\eta_{\alpha} - \eta_{\beta}} \cdot \frac{1}{\eta_{\alpha} - \eta_{\gamma}} = - \sum_{\alpha=1}^{N} \frac{\eta_{\beta}}{\eta_{\beta} - \eta_{\gamma}} \left( \frac{\eta_{\alpha}}{\eta_{\alpha} - \eta_{\beta}} + \frac{\eta_{\beta}}{\eta_{\beta} - \eta_{\gamma}} \right) \]

\[ = \frac{\eta_{\beta}}{\eta_{\beta} - \eta_{\gamma}} \left( \frac{\eta_{\gamma}}{\eta_{\beta} - \eta_{\gamma}} + \frac{\eta_{\beta}}{\eta_{\beta} - \eta_{\gamma}} \right) \]

\[ = \frac{\eta_{\beta}}{\eta_{\beta} - \eta_{\gamma}} \left( 1 + \frac{2\eta_{\gamma}}{\eta_{\beta} - \eta_{\gamma}} \right) \]

\[ = - \frac{\eta_{\beta}}{\eta_{\beta} - \eta_{\gamma}} + \frac{2}{|\eta_{\beta} - \eta_{\gamma}|^2}. \]

\[ \sum_{\alpha \neq \beta, \gamma}^{N} \frac{1}{(\eta_{\alpha} - \eta_{\beta})(\eta_{\alpha} - \eta_{\gamma})} = \sum_{\alpha \neq \beta, \gamma}^{N} \frac{2A_{\gamma\beta}}{|\eta_{\beta} - \eta_{\gamma}|^2} - \frac{1}{2} \sum_{\alpha \neq \beta, \gamma}^{N} A_{\gamma\beta}. \]

\textit{Proof} Use (B.17).
In this appendix, we review a few very well known relations for angular momentum operators [3, 4]. The components of the angular momentum operator $\mathbf{J}$ obey the SU(2) Lie algebra

$$[J^a, J^b] = i \epsilon^{abc} J^c \quad \text{for} \quad a, b, c = x, y, z.$$  \hspace{1cm} (C.1)

Since $[J^2, J^z] = 0$, we can choose a basis of simultaneous eigenstates of $J^2$ and $J^z$,

$$J^2 |j, m\rangle = j(j + 1) |j, m\rangle,$$
$$J^z |j, m\rangle = m |j, m\rangle,$$  \hspace{1cm} (C.2)

where $m = -j, \ldots, j$. With $J^\pm \equiv J^x \pm i J^y$, we have

$$[J^z, J^\pm] = \pm J^\pm.$$  \hspace{1cm} (C.3)

We further have

$$J^+ J^- = (J^x)^2 + (J^y)^2 - i[J^x, J^y] = J^2 - (J^x)^2 + J^z,$$
$$J^- J^+ = J^2 - (J^z)^2 - J^z,$$  \hspace{1cm} (C.4)

and therefore

$$[J^+, J^-] = 2J^z.$$  \hspace{1cm} (C.5)

Equations (C.3) and (C.4) further imply

$$J^\pm |j, m\rangle = \sqrt{j(j + 1) - m(m \pm 1)} |j, m \pm 1\rangle,$$  \hspace{1cm} (C.6)

where we have chosen the phases between $J^- |j, m\rangle$ and $|j, m - 1\rangle$ real.
Appendix D
Tensor Decompositions of Spin Operators

In this appendix, we will write out the tensor components \([3, 4]\) of all the tensors of different order we can form from one, two, or three spins operators.

D.1 One Spin Operator

A single spin \(S\) transforms as a vector under rotations, which we normalize such that the \(m = 0\) component equals \(S^z\) (see (3.5.10) in Sect. 3.5). The components of \(V^m\) are

\[
V^1 = -\frac{1}{\sqrt{2}} S^+, \\
V^0 = \frac{1}{\sqrt{2}} [S^-, V^1] = S^z, \\
V^{-1} = \frac{1}{\sqrt{2}} [S^-, V^0] = \frac{1}{\sqrt{2}} S^-.
\]  

(D.1.1)

D.2 Two Spin Operators

Since each spin operator transforms as a vector, and the representation content of four vectors is given by

\[
1 \otimes 1 = 0 \oplus 1 \oplus 2,
\]

we can form one scalar, one vector, and one tensor of second order from two spin operators \(S_1\) and \(S_2\). The scalar is given by

\[
U_{12} = S_1 S_2 = \frac{1}{2} (S_1^+ S_2^- + S_1^- S_2^+) + S_1^z S_2^z
\]  

(D.2.1)
and the vector by \(-i(S_1 \times S_2)\). Written out in components, we obtain

\[
V_{12}^1 = \frac{i}{\sqrt{2}} (S_1 \times S_2)^+ = \frac{1}{\sqrt{2}} \left( S_1^+ S_2^z - S_1^z S_2^+ \right),
\]

\[
V_{12}^0 = -i(S_1 \times S_2)^z = \frac{1}{2} \left( S_1^+ S_2^- - S_1^- S_2^+ \right),
\]

\[
V_{12}^{-1} = -\frac{i}{\sqrt{2}} (S_1 \times S_2)^- = \frac{1}{\sqrt{2}} \left( S_1^- S_2^z - S_1^z S_2^- \right).
\]

With regard to the 2nd order tensor, note that \(S_1^+ S_2^+\) is the only operator we can construct with two spin operators which raises the \(S_z^{\text{tot}}\) quantum number by two. It must hence be proportional to the \(m = 2\) component of the 2nd order tensor. As there is no particularly propitious way to normalize this tensor, we simply set the \(m = 2\) component equal to \(S_1^+ S_2^+\), and then obtain the other components using (3.5.9). This yields

\[
T_{12}^2 = S_1^+ S_2^+,
\]

\[
T_{12}^1 = \frac{1}{2} \left[ S_1^- + S_2^-, T_{12}^2 \right] = -S_1^z S_2^- - S_1^- S_2^z,
\]

\[
T_{12}^0 = \frac{1}{\sqrt{6}} \left[ S_1^- + S_2^-, T_{12}^1 \right] = \frac{1}{\sqrt{6}} \left( 4S_1^z S_2^+ - S_1^+ S_2^z - S_1^- S_2^+ \right),
\]

\[
T_{12}^{-1} = \frac{1}{\sqrt{6}} \left[ S_1^- + S_2^-, T_{12}^0 \right] = S_1^z S_2^- + S_1^- S_2^z,
\]

\[
T_{12}^{-2} = \frac{1}{2} \left[ S_1^- + S_2^-, T_{12}^{-1} \right] = S_1^- S_2^-.
\]

Equations (D.2.1) and (D.2.3) imply

\[
\frac{1}{2} \left( S_1^+ S_2^z + S_1^- S_2^+ \right) = \frac{2}{3} S_1 S_2 - \frac{1}{\sqrt{6}} T_{12}^0,
\]

\[
S_1^z S_2^z = \frac{1}{3} S_1 S_2 + \frac{1}{\sqrt{6}} T_{12}^0.
\]

Combining (D.2.4) with (D.2.2) yields

\[
S_1^+ S_2^- = \frac{2}{3} S_1 S_2 - i(S_1 \times S_2)^z - \frac{1}{\sqrt{6}} T_{12}^0,
\]

\[
S_1^- S_2^+ = \frac{2}{3} S_1 S_2 + i(S_1 \times S_2)^z - \frac{1}{\sqrt{6}} T_{12}^0.
\]

---

1 We denote general tensors of order \(j\) with \(T^{(j)}\) and 2nd order tensors with \(T\).
For $S_1 = S_2$, (D.2.6) reduces with $S_1 \times S_1 = iS_1$ to

$$
S_1^+ S_1^- = \frac{2}{3} S_1^2 + S_1^z - \frac{1}{\sqrt{6}} T_{11}^0,
$$
$$
S_1^- S_1^+ = \frac{2}{3} S_1^2 - S_1^z - \frac{1}{\sqrt{6}} T_{11}^0.
$$

(D.2.7)

### D.3 Three Spin Operators

Since

$$
1 \otimes 1 \otimes 1 = 0 \oplus 3 \cdot 1 \oplus 2 \cdot 2 \oplus 3,
$$

we can form one scalar, three vectors, two tensors of second order, and one tensor of third order, from three spin operators $S_1$, $S_2$, and $S_3$.

The scalar is given by

$$
U_{123} = -iS_1(S_2 \times S_3)
= \frac{1}{2} S_1^z (S_2^+ S_3^- - S_2^- S_3^+) + 2 \text{ cyclic permutations}
= \frac{1}{2} \left( S_1^z S_2^+ S_3^- + S_1^z S_2^- S_3^+ + S_1^- S_2^z S_3^+ 
- S_1^+ S_2^- S_3^+ - S_1^- S_2^z S_3^- - S_1^+ S_2^z S_3^- \right).
$$

(D.3.1)

The three vectors are given by

$$
S_1(S_2 S_3), \quad S_1(S_2)S_3, \quad \text{and} \quad (S_1 S_2)S_3,
$$

(D.3.2)

where the scalar product in the second expression is understood to contract $S_1$ and $S_3$. The components for each $m$ are according to the conventions specified in (D.1.1). For later purposes, we write for the $m = 0$ components,

$$
V_{a,123}^0 = S_1^z (S_2 S_3) = \frac{1}{2} (S_1^z S_2^+ S_3^- + S_1^z S_2^- S_3^+) + S_1^z S_2^z S_3^z,
$$
$$
V_{b,123}^0 = S_1(S_2^z)S_3 = \frac{1}{2} (S_1^- S_2^z S_3^+ + S_1^+ S_2^z S_3^-) + S_1^z S_2^2 S_3^z,
$$
$$
V_{c,123}^0 = (S_1 S_2)S_3^z = \frac{1}{2} (S_1^+ S_2^- S_3^+ + S_1^- S_2^+ S_3^-) + S_1^z S_2^2 S_3^z.
$$

(D.3.3)
To obtain a tensor operator of second order, or more precisely the $m = 2$ component of it, all we need to do is to form the product of the $m = 1$ components of two vector operators constructed out of the three spins, like $S_1$ and $-i(S_2 \times S_3)$ or $-i(S_1 \times S_2)$ and $S_3$. In this way, we construct the tensor operators of second order

\[
T^2_{a,123} = -iS_1^+ (S_2 \times S_3)^+ ,
\]

\[
T^2_{b,123} = -i(S_1 \times S_2)^+ S_3^+ .
\]

The other components are obtained as in (D.2.3). As we are primarily interested in the $m = 0$ component, we may use (D.2.3) directly to write

\[
T^0_{a,123} = -\frac{i}{\sqrt{6}} \left[ 4S_1^+ (S_2 \times S_3)^z - S_1^+ (S_2 \times S_3^-) - S_1^- (S_2 \times S_3)^+ \right]
\]

\[
= \frac{1}{\sqrt{6}} \left[ 2S_1^+ (S_2^+ S_3^- - S_2^- S_3^+) - S_1^+ S_2^- S_3^+ + S_1^+ S_2^+ S_3^- 
+ S_1^- S_2^+ S_3^- - S_1^- S_2^- S_3^+ \right],
\]

and similarly for $T^0_{b,123}$, which can be obtained from $T^0_{b,123}$ by a cyclical permutation of the superscripts $+, -, z$. Note that there is no third tensor of this kind, as the sum of the three tensors obtained from (D.3.5) by cyclic permutations of the superscripts equals zero.

We obtain the tensor of third order with the method we used to obtain the second order tensor (D.2.3) formed by two spins:

\[
W^3_{123} = -S_1^+ S_2^+ S_3^+ ,
\]

\[
W^2_{123} = \frac{1}{\sqrt{6}} \left[ S_1^- + S_2^- + S_3^- , W^3_{123} \right]
\]

\[
= -\frac{1}{\sqrt{6}} \left[ S_1^- , S_1^+ \right] S_2^+ S_3^+ + 2 \text{ cycl. permutations}
\]

\[
= \frac{\sqrt{2}}{3} S_1^+ S_2^+ S_3^+ + 2 \text{ cycl. permutations},
\]

\[
W^1_{123} = \frac{1}{\sqrt{10}} \left[ S_1^- + S_2^- + S_3^- , W^2_{123} \right]
\]

\[
= \frac{1}{\sqrt{15}} \left( [S_1^- , S_1^+] S_2^+ S_3^+ + S_1^+ [S_2^- + S_3^- , S_2^+ S_3^+] \right) + 2 \text{ cycl. permutations},
\]

\[
= \frac{1}{\sqrt{15}} \left( S_1^- S_2^+ S_3^+ - 4S_1^+ S_2^+ S_3^+ \right) + 2 \text{ cycl. permutations},
\]

\[\text{(D.3.5)}\]
The permutations here always refer to permutations of the superscripts $+, -, z$, as otherwise we would have to assume again that none of the three spin operators are identical. In particular, writing out the $m = 0$ yields

\[
W_{123}^0 = -\frac{1}{\sqrt{5}} \left( S_1^- S_2^+ S_3^z + S_1^+ S_2^z S_3^- + S_1^z S_2^- S_3^+ \right) + \frac{4}{5} S_1^z S_2^z S_3^z.
\]  

Combining (D.3.3) and (D.3.7), we obtain

\[
S_1^z S_2^z S_3^z = \frac{1}{5} \left( V_{a,123}^0 + V_{b,123}^0 + V_{c,123}^0 \right) + \frac{1}{2\sqrt{5}} W_{123}^0,
\]  

and hence

\[
\frac{1}{2} S_1^z \left( S_2^+ S_3^- + S_2^- S_3^+ \right) = V_{a,123}^0 - S_1^z S_2^z S_3^z
\]

\[= \frac{4}{5} V_{a,123}^0 - \frac{1}{5} V_{b,123}^0 - \frac{1}{5} V_{c,123}^0 - \frac{1}{2\sqrt{5}} W_{123}^0.
\]  

From (D.3.1) and (D.3.5) we obtain

\[
W_{123}^0 = \frac{1}{\sqrt{12}} [S_1^- + S_2^- + S_3^-, W_{123}^1]
\]

\[
+ \frac{1}{6\sqrt{5}} \left( S_1^- [S_2^- + S_3^-, S_2^+ S_3^+] - 4[S_1^- + S_2^- S_3^], S_1^z S_2^z S_3^z \right) - 4S_1^z S_2^z [S_3^-, S_3^+] + 2 \text{ cycl. permutations}
\]

\[
= -\frac{1}{\sqrt{5}} (S_1^- S_2^+ S_3^z - 4S_1^- S_2^z S_3^z) + 2 \text{ cycl. permutations},
\]

\[
W_{123}^{-1} = \frac{1}{\sqrt{12}} [S_1^- + S_2^- + S_3^-, W_{123}^0]
\]

\[
= -\frac{1}{\sqrt{15}} (S_1^- S_2^+ S_3^z - 4S_1^- S_2^z S_3^z) + 2 \text{ cycl. permutations},
\]

\[
W_{123}^{-2} = \frac{1}{\sqrt{10}} [S_1^- + S_2^- + S_3^-, W_{123}^{-1}]
\]

\[
= \frac{\sqrt{2}}{3} S_1^- S_2^+ S_3^z + 2 \text{ cycl. permutations},
\]

\[
W_{123}^{-3} = \frac{1}{\sqrt{6}} [S_1^- + S_2^- + S_3^-, W_{123}^{-2}]
\]

\[
= S_1^- S_2^- S_3^-.
\]
\[ \frac{1}{2} S_1^z (S_2^+ S_3^- - S_2^- S_3^+) = \frac{1}{3} U_{123} + \frac{1}{\sqrt{6}} T^0_{a,123}. \] (D.3.10)

Combining (D.3.9) and (D.3.10) we finally obtain

\[
S_1^z S_2^+ S_3^- = \frac{1}{3} U_{123} + \frac{1}{5} \left( 4 V^0_{a,123} - V^0_{b,123} - V^0_{c,123} \right) + \frac{1}{\sqrt{6}} T^0_{a,123} - \frac{1}{2\sqrt{5}} W^0_{123}
\]

\[ = \frac{1}{3} S_1 (S_2 \times S_3) + \frac{1}{5} \left[ 4 S_1^z (S_2 S_3) - S_1 (S_2^z S_3) - (S_1 S_2) S_3^z \right] + \frac{1}{\sqrt{6}} T^0_{a,123} - \frac{1}{2\sqrt{5}} W^0_{123}, \]

(D.3.11)

\[
S_1^z S_2^- S_3^+ = -\frac{1}{3} U_{123} + \frac{1}{5} \left( 4 V^0_{a,123} - V^0_{b,123} - V^0_{c,123} \right) - \frac{1}{\sqrt{6}} T^0_{a,123} - \frac{1}{2\sqrt{5}} W^0_{123}
\]

\[ = -\frac{1}{3} S_1 (S_2 \times S_3) + \frac{1}{5} \left[ 4 S_1^z (S_2 S_3) - S_1 (S_2^z S_3) - (S_1 S_2) S_3^z \right] - \frac{1}{\sqrt{6}} T^0_{a,123} - \frac{1}{2\sqrt{5}} W^0_{123}, \]

(D.3.12)

References

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