Appendix A
Lorentz Invariant Distributions

Theorem. There are precisely two, linearly indendent, Lorentz invariant distributions $\Delta_i(z;\ m)$ which obey the Klein–Gordon equation for mass m. These are

$$\Delta_0(z;m) := -\frac{\mathrm{i}}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2\omega_k}(\mathrm{e}^{-\mathrm{i}kz} - \mathrm{e}^{\mathrm{i}kz}), \tag{A.1}$$

$$\Delta_1(z;m) := \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2\omega_k}(\mathrm{e}^{-\mathrm{i}kz} + \mathrm{e}^{\mathrm{i}kz}), \tag{A.2}$$

with $\omega_k \equiv k^0 = \sqrt{\boldsymbol{k}^2 + m^2}$. These distributions have the following properties:

(i) $$\{\Box + m^2\}\Delta_i(z;m) = 0,\ i = 0, 1, \tag{A.3}$$

(ii) $$\Delta_0(z^0 = 0, \boldsymbol{z}; m) = 0, \tag{A.4a}$$

(iii) $$\left.\frac{\partial}{\partial z^0}\Delta_0(z;m)\right|_{z^0=0} = -\delta(\boldsymbol{z}), \tag{A.4b}$$

(iv) $$\Delta_0(-z;m) = -\Delta_0(z;m), \qquad \Delta_1(-z;m) = \Delta_1(z;m), \tag{A.5}$$

(v) $$\Delta_0(z;m) = 0 \text{ for } z^2 < 0. \tag{A.6}$$

Note, however, that $\Delta_1\ (z;\ m)$ does not vanish for spacelike argument, $z^2 < 0$.

Proof. Part (i) is obvious because Δ_i are linear superpositions of exponentials which satisfy the Klein–Gordon equation. Part (ii) is also obvious by noting that, with $z^0 = 0$, the integrand is an *odd* function. Part (iii) gives minus the sum of the integrals $\int \mathrm{d}^3k \exp\{\pm i\boldsymbol{k}\cdot\boldsymbol{z}\}$ divided by $2(2\pi)^3$. This is a representation of the three-dimensional delta distribution. Part (iv) is again obvious by noting that the integrand in (A.1) is antisymmetric while that in (A.2) is symmetric when z is replaced by $-z$.

To prove (v) we first note that Δ_0, (A.1), can also be written as

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$$\Delta_0(z;m) = -\frac{\mathrm{i}}{(2\pi)^3}\int \mathrm{d}^4 q e^{-\mathrm{i}qz}\delta(q^2-m^2)\varepsilon(q^0), \tag{A.7}$$

where $\varepsilon(q^0) = q^0/|q^0|$ is the sign of the time component of the four-vector q. Note that unlike the variable k in (A.1) and (A.2), the four-vector q is unconstrained, i.e. has four independent components. That (A.7) and (A.1) represent the same distribution is seen by using the equality

$$\begin{aligned}\delta(q^2-m^2) &= \delta((q^0)^2-(\boldsymbol{q}^2+m^2)) \\ &= \frac{1}{2q^0}\left\{\delta(q^0-\sqrt{\boldsymbol{q}^2+m^2})+\delta(q^0+\sqrt{\boldsymbol{q}^0+m^2})\right\}\end{aligned}$$

and by doing the integral over the variable q^0. This gives (A.1) with $\boldsymbol{k}=\boldsymbol{q}$, $k^0 \equiv \omega_k = \sqrt{\boldsymbol{q}^2+m^2}$. Now, for *time-like* z, i.e. for $z^2>0$, both z^2 and the sign of z^0, sign z^0 are Lorentz invariants. Thus, for time-like arguments the distribution Δ_0, being itself Lorentz invariant, must be a funtion of z^2 and of sign z^0. This is in accord with the antisymmetry noted in the first equation (A.5). For *space-like* arguments, however, the distinction between positive and negative values of z^0 is not Lorentz invariant. Therefore, for space-like z the distribution Δ_0 is a function of the invariant z^2 only. From this observation one concludes that

$$\Delta(-z;m) = +\Delta(z;m).$$

As this is in contradiction with the antisymmetry (A.5), we conclude that part (v) of the theorem is true. Regarding the distribution Δ_1 we note that this argument does not hold: Indeed, Δ_1 is symmetric under the transformation $z \to -z$, and, hence, does not vanish for space-like z.

Appendix B
S-Matrix, Cross Sections, Decay Probabilities

Write the scattering matrix (somewhat symbolically) as

$$S_{\text{fi}} = \delta_{\text{fi}} + R_{\text{fi}}, \tag{B.1}$$

where f and i are asymptotic free states. δ_{fi} means "no scattering" and R_{fi} is the reaction matrix proper. R_{fi} necessarily contains a δ-distribution expressing conservation of total energy and momentum. Besides this distribution it is convenient to take out a factor $\mathrm{i}(2\pi)^4$ and to define the T-matrix by

$$R_{\text{fi}} = \mathrm{i}(2\pi)^4 \delta(P_{\text{f}} - P_{\text{i}}) T_{\text{fi}}. \tag{B.2}$$

The differential cross section for the reaction

$$\text{a} + \text{b} \to 1 + 2 + \cdots + N$$

(all particles being described asymptotically by plane waves), is given by the general expression

$$\mathrm{d}\sigma_{\text{fi}}(\text{a} + \text{b} \to 1 + \cdots + N) = \frac{(2\pi)^{10}\delta(P_{\text{f}} - P_{\text{i}})}{2E_{\text{a}} 2E_{\text{b}} |\boldsymbol{v}_{\text{ab}}|} |T_{\text{fi}}|^2 \prod_{n=1}^{N} \frac{\mathrm{d}^3 p^{(n)}}{2E_N}. \tag{B.3}$$

In this expression

$$E_{\text{a}} = \sqrt{\boldsymbol{p}^{(\text{a})2} + m_{\text{a}}^2}, \qquad E_{\text{b}} = \sqrt{\boldsymbol{p}^{(\text{b})2} + m_{\text{b}}^2}, \qquad E_{\text{n}} = \sqrt{\boldsymbol{p}^{(\text{n})2} + m_{\text{n}}^2};$$

$$P_{\text{i}} = p^{(\text{a})} + p^{(\text{b})}, \qquad P_{\text{f}} = \sum_{n=1}^{N} p^{(n)},$$

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and $|\boldsymbol{v}_{\rm ab}|$ is the relative velocity of the incoming particles, $\mathrm{d}^3 p^{(n)}/2E_n$ is the Lorentz invariant volume element in the phase space of particle number n. This term as well as the incoming flux factor in the first denominator is in accord with our covariant normalization $\langle \boldsymbol{p}'|\boldsymbol{p}\rangle = 2E_p \delta(\boldsymbol{p}'-\boldsymbol{p})$. Formula (B.3) holds in all systems of reference where the momenta $\boldsymbol{p}^{(\rm a)}$ and $\boldsymbol{p}^{(\rm b)}$ are collinear (e.g. the laboratory and centre-of-mass systems). It can be extended to any system by replacing the flux factor (so-called Møller factor) with the invariant on the r.h.s. of the following equation:

$$E_{\rm a} E_{\rm b} |\boldsymbol{v}_{\rm ab}| = \sqrt{(p^{(\rm a)} \cdot p^{(\rm b)})^2 - p^{(\rm a)2} p^{(\rm b)2}}. \tag{B.4}$$

(One should verify that this equation does indeed hold if the 3-momenta of particles a and b are collinear.)

The observable cross sections are obtained from (B.3) by integration over those momentum variables in the final state which are not observed. Similarly, depending on whether or not particles a and b have nonvanishing spin and are polarized, the appropriate average over spin projections must be taken. If the spin orientations in the final state are not discriminated, (B.3) must be summed over them.

In a similar fashion the differential decay rate of a particle a, with mass $m_{\rm a}$ and momentum q, into a final state with N particles,

$$\mathrm{a} \to 1 + 2 + \cdots + N,$$

is given by

$$\mathrm{d}\Gamma_{\rm fi} = (2\pi)^4 \delta(p^{(1)} + p^{(2)} + \cdots + p^{(N)} - q) \frac{(2\pi)^3}{2E_q} |T_{\rm fi}|^2 \prod_{n=1}^{N} \frac{\mathrm{d}^3 p^{(n)}}{2E_n}. \tag{B.5}$$

Again, depending on what shall be observed, integration over some of the momentum variables and, possibly, sums over spin projections in the final state must be performed. If the decaying particle has nonvanishing spin and if the spin orientation is not known, the formula must be averaged over all spin projections.

From (B.5) one sees that the squared decay amplitude has the dimension $[|T_{\rm fi}|^2] = E^{2(3-N)}$. Thus, in a two-body decay the dimension is (energy)^2, whereas in a three-body decay T is dimensionless. This can be useful in checking calculations.

Let us consider a few examples:

(i) *Differential cross section for $a + b \to 1 + 2$*. The example of elastic scattering of two massive particles (masses m and M, respectively) is worked out in detail in Sect. 2.4.2. Examples of neutrino reactions are treated in Sects. 3.2.4 and 4.1.2e.

(ii) *Two-body decay*. In the rest system of the decaying particle the two particles in the final state have the momenta

$$p^{(1)} = \{E_1, \boldsymbol{\kappa}\},\ p^{(2)} = \{E_2, -\boldsymbol{\kappa}\},$$

with $E_1 + E_2 = m_\mathrm{a}$, $E_i = \left(m_i^2 + \kappa^2\right)^{1/2}$, and $\kappa := |\boldsymbol{\kappa}|$. Integrating over the 3-momentum of particle 2 one obtains from (B.5)

$$\mathrm{d}^3\Gamma = \frac{(2\pi)^7}{8m_\mathrm{a}E_1E_2}|T(\mathrm{a}\to 1+2)|^2\delta(E_1+E_2-m_\mathrm{a})\mathrm{d}^3p^{(1)},$$

where $\mathrm{d}^3p^{(1)}$ can be expressed in polar coordinates, $\mathrm{d}^3p^{(1)} = \kappa^2\mathrm{d}\kappa\mathrm{d}\Omega$. Noting than $\kappa\mathrm{d}\kappa = E_1\mathrm{d}E_1$ one can convert the integration over κ into an integration over E_1. For this we need the derivative of the argument of the δ-distribution with respect to E_1, viz.

$$\frac{\mathrm{d}}{\mathrm{d}E_1}\{E_1+E_2-m_\mathrm{a}\} = 1 + \frac{\mathrm{d}E_2}{\mathrm{d}\kappa}\frac{\mathrm{d}\kappa}{\mathrm{d}E_1} = \frac{E_1+E_2}{E_2} = \frac{m_a}{E_2},$$

one obtains

$$\mathrm{d}^2\Gamma = \frac{(2\pi)^7\kappa}{8m_\mathrm{a}^2}|T(\mathrm{a}\to 1+2)|^2\mathrm{d}\Omega.$$

The decay probability is independent of the azimuth φ. Integrating over this angle we have

$$\mathrm{d}\Gamma = \frac{(2\pi)^8\kappa}{8m_\mathrm{a}^2}|T(\mathrm{a}\to 1+2)|^2\mathrm{d}(\cos\theta), \tag{B.6}$$

where θ is the opening angle between the spin expectation value of the decaying particle a and the momentum of particle 1. Equation (4.96) for the decay of a polarized τ into a pion and a neutrino provides an example for this case. If the decaying particle is spinless, or if it has spin but is unpolarized, $|T|^2$ is isotropic. Integrating over $\mathrm{d}(\cos\theta)$ one obtains the total decay rate

$$\Gamma = \frac{(2\pi)^8\kappa}{4m_\mathrm{a}^2}|T(\mathrm{a}\to 1+2)|^2. \tag{B.7}$$

(iii) *Three-body decays*. Here we distinguish several situations: If two of the particles in the final state are not observed (cf. the example of $\mu \to \mathrm{e}\nu\bar{\nu}$) one proceeds as described in Sect. 4.1.2a and obtains a differential decay rate

$$\mathrm{d}^2\Gamma/\mathrm{d}E\mathrm{d}(\cos\theta),$$

where θ is the opening angle between the spin of the decaying particle and the momentum of the observed particle in the final state. In other situations one may proceed as follows. Integrate first (B.5) over $\mathrm{d}^3p^{(3)}$ to obtain

$$d^6\Gamma = \frac{(2\pi)^7}{16m_a}\frac{\kappa_1\kappa_2}{E_3}|T(a \to 123)|^2\delta(E_1 + E_2 + E_3 - m_a)$$
$$\times\; dE_1\, dE_2\, d\,\Omega_1\, d\,\Omega_2,$$

where $E_3 = \left[m_3^2 + (\boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)})^2\right]^{1/2}$ and $\kappa_i := |\boldsymbol{p}^{(i)}|$. Then integrate over $d\Omega_1$ for particle 1 which is emitted isotropically, take $\boldsymbol{p}^{(1)}$ as the 3-axis and make use of the axial symmetry around this direction, viz.

$$d^3\Gamma = \frac{(2\pi)^9}{8m_a}\frac{\kappa_1\kappa_2}{E_3}|T(a \to 123)|^2\delta(E_1 + E_2 + E_3 - m_a)dE_1\, dE_2 d(\cos\theta), \tag{B.8}$$

where now
$$E_3 = (m_3^2 + \kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2\cos\theta)^{1/2}.$$

This formula may be transformed to the variables E_1, E_2, E_3 by means of the Jacobian

$$\frac{\partial(E_1, E_2, \cos\theta)}{\partial(E_1, E_2, E_3)} = \frac{E_3}{\kappa_1\kappa_2}$$

and finally to the variables $s := E_2 + E_3$ and $t := E_2 - E_3$, giving

$$d^2\Gamma = \frac{(2\pi)^9}{16m_a}|T(a \to 123)|^2 dE_1 dt, \tag{B.9}$$

from which the total rate is obtained by integration over the kinematic range of E_1 and of t.

Note that T contains a factor $(2\pi)^{-3/2}$ for each external particle. So $|T(a \to 1 + 2)|^2$ produces a factor $(2\pi)^{-9}$, $|T(a \to 123)|^2$ produces a factor $(2\pi)^{-12}$.

Appendix C1
Some Feynman Rules for Quantum Electrodynamics of Spin-1/2 Particles $f^{\pm}$

The rules hold for the matrix R, as defined by (B.1). The T-matrix is obtained upon comparison with the defining equation (B.2).

(i) *Diagrams*. One draws all connected diagrams of the process under consideration, at the order n in the coupling constant that one wishes to calculate. External and internal fermion lines are provided with arrows which point in the direction of the flow of *negative* charge. The momenta of internal lines are chosen such as to follow the arrow. All factors prescribed by the following rules must be written down from *right* to *left* following the direction of the arrows.

(ii) *External lines*. For each external, incoming f^- write a spinor in momentum space $u_f(p)$, for each incoming f^+ write $\overline{v_f(p)}$. Similarly, for an outgoing f^- write $\overline{u_f(p)}$, for an outgoing f^+ write a $v_f(p)$. For an incoming or outgoing photon write a polarization vector $\varepsilon_\alpha(\boldsymbol{k},\ \lambda)$ with the index α to be contracted with γ^α at the fermion vertex to which it couples. In addition, each external particle obtains a factor $(2\pi)^{-3/2}$.

(iii) *Vertices*. Each vertex (ffγ) has a factor $e\gamma^\alpha$ and a δ-distribution for energy–momentum conservation at that vertex.

(iv) *Internal fermion lines*. An internal fermion line is represented by a propagator

$$\frac{\not{p} + m_f}{p^2 - m_f^2 + i\varepsilon},$$

where the direction of p is chosen in accordance with rule (i).

(v) *Internal photon lines*. An internal photon line with momentum k connects two vertices (ffγ) characterized by the Lorentz indices α and β [cf. rule (iii)] and yields a factor

$$-\frac{g^{\alpha\beta}}{k^2 + i\varepsilon}.$$

(vi) *Integrations*. All internal momenta must be integrated over. In all cases this yields a δ-distribution $\delta(P_i - P_f)$ for conservation of total energy–momentum.

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In orders of e which are higher than the lowest nontrivial order this rule also gives rise to some nontrivial integrations over internal momenta. Such integrals can turn out to be divergent and must then be analyzed in the framework of regularization and renormalization.

(vii) *Factors*. R_{fi} has a factor $(-)^P$ where P is the permutation of the fermions in the final state, as well as a factor $(-)^L$ if L is the number of closed fermion loops. In addition, R_{fi} obtains the following factors:

$$\mathrm{i}^{n+f_{\mathrm{i}}+b_{\mathrm{i}}}(2\pi)^{4(n-f_{\mathrm{i}}-b_{\mathrm{i}})},$$

where n is the order of perturbation theory, f_{i} the number of internal fermion lines, b_{i} the number of internal photon lines.

(viii) *Closed fermion loops*. Closed loops which couple to an *odd* number of photon lines give a vanishing amplitude. This is a consequence of C-invariance of QED.

(ix) *External potentials*. An external potential is an approximation for the interaction with a very heavy particle which therefore can absorb or provide an arbitrary amount of 3-momentum. Therefore, for an external potential one has to write a δ-distribution only for energy conservation, whilst the vertex factor $e\gamma^{\alpha}$ must be replaced by

$$\delta^{\alpha 0}\frac{Ze}{(2\pi)^3}\frac{\upsilon(\boldsymbol{k})}{k^2},$$

where Ze is the total charge that creates the potential, $\upsilon(\boldsymbol{k})$ is the form factor of the corresponding charge distribution,

$$\upsilon(\boldsymbol{k}) = \int \mathrm{d}^3x\mathrm{e}^{-ik\cdot x}\rho(\boldsymbol{x}),$$

with $\int \rho(\boldsymbol{x})\mathrm{d}^3x = 1$.

Appendix C2
Traces

The following formulae are all derived from the equations (1.73–1.75):

$$\mathrm{tr}\,\mathbb{1} = 4, \qquad \mathrm{tr}\gamma^\alpha = \mathrm{tr}\gamma_5 = 0. \tag{C.1}$$

The trace of a product with an *odd* number of factors vanishes. For products with an even number of γ-matrices the following relations are useful:

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\} = 4g^{\alpha\beta}, \tag{C.2}$$

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma_5\} = 0, \tag{C.3}$$

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\tau\} = 4\{g^{\alpha\beta}g^{\sigma\tau} - g^{\alpha\sigma}g^{\beta\tau} + g^{\alpha\tau}g^{\beta\sigma}\}, \tag{C.4}$$

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\tau\gamma_5\} = 4i\varepsilon^{\alpha\beta\sigma\tau}, \tag{C.5}$$

$$\begin{aligned}\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau\} &= g^{\alpha\beta}\mathrm{tr}\{\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau\} - g^{\alpha\mu}\mathrm{tr}\{\gamma^\beta\gamma^\nu\gamma^\sigma\gamma^\tau\} \\ &\quad + g^{\alpha\nu}\mathrm{tr}\{\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\tau\} - g^{\alpha\sigma}\mathrm{tr}\{\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\tau\} \\ &\quad + g^{\alpha\tau}\mathrm{tr}\{\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma\}.\end{aligned} \tag{C.6}$$

In many cases the Lorentz indices of some of the γ-matrices in a product have to be contracted. The following formulae are then useful:

$$\gamma_\alpha\gamma^\alpha = 4, \qquad \gamma_\alpha\not{a}\gamma^\alpha = -2\not{a}, \tag{C.7}$$

$$\gamma_\alpha\not{a}\not{b}\gamma^\alpha = 4ab, \quad \gamma_\alpha\not{a}\not{b}\not{c}\gamma^\alpha = -2\not{c}\not{b}\not{a}, \tag{C.8}$$

$$\gamma_\alpha\not{a}\not{b}\not{c}\not{d}\gamma^\alpha = 2(\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}). \tag{C.9}$$

Note that in our conventions $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, $\varepsilon_{0123} = +1$. Note also the relation

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\sigma\tau} = -2\{\delta^\mu_\sigma\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\sigma\}. \tag{C.10}$$

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Appendix D
The Group SU(3)

The group SU(3) is defined as the set of all complex 3×3 matrices which are unitary and have determinant 1,

$$\mathrm{SU}(3) = \{U \in M_3(\mathbb{C}) | U^\dagger U = \mathbb{1},\ \det U = 1\}. \tag{D.1}$$

A matrix U in n complex dimensions depends on $2n^2$ real parameters. The unitarity condition $U^\dagger U = \mathbb{1}$ gives n^2 real constraints (n conditions for the diagonal, $2n(n-1)/2$ conditions for the nondiagonal matrix elements), the determinant gives one more real constraint. Therefore $U \in \mathrm{SU}(n)$ depends on $n^2 - 1$ real parameters. In particular, the elements of SU(3) depend on 8 real parameters. This is also seen from their representation (3.102) as exponential series in terms of hermitean 3×3 matrices with vanishing trace,

$$U = \exp\{\mathrm{i}H\} \quad \text{with} \quad \mathrm{tr} H = 0.$$

The condition on the trace of H is seen most easily after diagonalization of H, hence of U, whereby U becomes $U = \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\lambda}{}_1, \ldots, \mathrm{e}^{\mathrm{i}\lambda}{}_3\right)$. The constraint det $U = 1$ yields $\Sigma\lambda_i = 0$, hence tr $H = 0$. Now, any hermitean 3×3 matrix H can be written as a linear combination of 8 linearly independent matrices as follows,

$$H = \sum_{k=1}^{8} \Lambda_k \left(\frac{\lambda_k}{2}\right).$$

The matrices λ_k have a standard form, called Gell-Mann matrices after M. Gell-Mann who constructed them in analogy to the Pauli matrices (1.24), viz.

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$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\lambda_2=\begin{pmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&0\\0&0&0\end{pmatrix}\lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$
$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\lambda_5=\begin{pmatrix}0&0&-\mathrm{i}\\0&0&0\\\mathrm{i}&0&0\end{pmatrix}\lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\qquad\text{(D.2)}$$
$$\lambda_7=\begin{pmatrix}0&0&0\\0&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix}\lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

Apart from a factor 2, the matrices (D.2) are the generators in the three-dimensional representation $\underline{3}$, called the fundamental representation,

$$U(T_k)=\left(\frac{\lambda_k}{2}\right)\quad\text{with normalization tr}\left\{\left(\frac{\lambda_k}{2}\right)\left(\frac{\lambda_j}{2}\right)\right\}=\frac{1}{2}\delta_{kj}\,.\qquad\text{(D.3)}$$

SU(3) has rank 2. This is seen from eqs. (D.2) which show that two generators, T_3 and T_8, are simultaneously diagonal. Furthermore, from the explicit representation (D.2) it is obvious that the generators (T_1, T_2, T_3) form an SU(2) subgroup of SU(3). In the "eightfold way" where baryons and mesons made up of u, d and s quarks are classified according to the flavour group $\mathrm{SU_f}(3)$, this SU(2) subgroup is the (strong) isospin group. In a similar fashion one verifies that the sets

$$\left(U_1=T_6, U_2=T_7, U_3=-\frac{1}{2}T_3+\frac{\sqrt{3}}{2}T_3\right)\text{ and}$$
$$\left(V_1=T_4, V_2=T_5, V_3=\frac{1}{2}T_3+\frac{\sqrt{3}}{2}T_8\right)$$

also generate SU(2) subgroups of SU(3). In analogy to the isospin they are called, respectively, U-spin and V-spin.

In SU(3), i.e. when $\mathrm{SU_f}(3)$ is interpreted as the flavour group classifying light mesons and baryons, the electric charge operator is

$$Q_{\text{e.m.}}=T_3+\frac{1}{2}Y,\qquad\text{with}\quad Y:=\frac{2}{\sqrt{3}}T_8,\qquad\text{(D.4)}$$

Y denoting the operator of hypercharge with respect to strong interactions. One verifies easily that Q commutes with the generators U_i. This means that U-spin connects particles which carry the same electric charge.

Invariant tensors in SU(3) are δ_i^k, ε_{ijk} and ε^{ijk}. They are used in constructing irreducible, unitary representations of SU(3) from the fundamental representation $\underline{3}$, where

$$U(T_3) = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, 0\right), \qquad U(Y) = \text{diag}\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right),$$
$$U(Q) = \text{diag}\left(+\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right),$$

and from its conjugate. The representation conjugate to $\underline{3}$ is the antitriplet $\underline{\bar{3}}$ which cannot be identified with the triplet $\underline{3}$–unlike SU(2) where the doublet and its conjugate are equivalent to each other.

Important Clebsch–Gordan decompositions of SU(3) are

$$\underline{3} \times \underline{\bar{3}} = \underline{1} + \underline{8}, \; \underline{3} \times \underline{3} = \underline{\bar{3}}_{\mathrm{a}} + \underline{6}_{\mathrm{s}}, \; \underline{3} \times \underline{3} \times \underline{3} = \underline{1}_{\mathrm{a}} + \underline{8} + \underline{8} + \underline{10}_{\mathrm{s}}, \tag{D.5}$$

where the subscripts ‘a’ and ‘s’ indicate that these states are antisymmetric or symmetric, respectively, under exchange of any two of their constituents. In $\mathrm{SU_f}(3)$ the triplet $\underline{3}$ is used to classify quarks (u,d,s), its conjugate $\underline{\bar{3}}$ then describes their antiparticles $(\bar{\mathrm{u}}, \bar{\mathrm{d}}, \bar{\mathrm{s}})$. The singlets $\underline{1}$, the octets $\underline{8}$ and the decuplets $\underline{10}$ serve to classify physical hadrons, mesons made up of quarks and antiquarks and baryons made up of three quarks.

Appendix E
Dirac Equation with Central Fields

The Hamiltonian form (1.82a) of the Dirac equation is well adapted for a discussion of interactions with external fields. For the case of an external, spherically symmetric potential $V(r)$ and for stationary states ($\propto e^{-iEt}$) it reads

$$E\Psi(r) = \{-i\boldsymbol{\alpha} \cdot \nabla + V(r)\mathbb{1} + m\beta\}\Psi(\boldsymbol{r}), \tag{E.1}$$

with α and β as given by (1.81). Using the vector identities

$$\begin{aligned} \nabla &= \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \nabla) - \hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \nabla) \\ &= \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \nabla) - \frac{i}{r}\hat{\boldsymbol{r}} \times \boldsymbol{l} \end{aligned}$$

one has

$$\boldsymbol{\alpha} \cdot \nabla = \boldsymbol{\alpha} \cdot \hat{\boldsymbol{r}}\frac{\partial}{\partial r} - \frac{i}{r}\boldsymbol{\alpha} \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{l}) = \gamma_5 \boldsymbol{S} \cdot \hat{\boldsymbol{r}} \left\{ \frac{\partial}{\partial r} - \frac{1}{r}\boldsymbol{S} \cdot \boldsymbol{l} \right\},$$

where γ_5 is given by (1.78), whilst the matrix $\boldsymbol{S}$ stands for

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

Finally, upon introduction of Dirac's angular momentum operator

$$K := \beta(\boldsymbol{S} \cdot \boldsymbol{l} + \mathbb{1}) \equiv \begin{pmatrix} K^{(0)} & 0 \\ 0 & -K^{(0)} \end{pmatrix} \tag{E.2}$$

with $K^{(0)} = \boldsymbol{\sigma} \cdot \boldsymbol{l} + \mathbb{1}$, equation (E.1) takes the form

$$\left\{ -i\gamma_5 \boldsymbol{S} \cdot \hat{\boldsymbol{r}} \left(\frac{\partial}{\partial r} + \frac{1}{r} - \beta\frac{K}{r} \right) + V(r)\mathbb{1} + \beta m \right\} \Psi = E\Psi =: H\Psi. \tag{E.3}$$

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One verifies by explicit calculation that K commutes with H, $[H, K] = 0$, but that H neither commutes with the orbital angular momentum nor with the spin. The operator K contains the entire dependence on angular momenta so that (E.3) lends itself to separation into radial and angular coordinates. To see this, we note first that $K^{(0)}$ can be written as

$$K^{(0)} = \boldsymbol{\sigma} \cdot \boldsymbol{l} + \mathbb{1} = 2\boldsymbol{s} \cdot \boldsymbol{l} + \mathbb{1} = \boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2 + \mathbb{1}.$$

Its eigenfunctions are the coupled states

$$\varphi_{jlm} = \sum_{m_l m_s} \left(l m_l, \frac{1}{2} m_s | j m \right) Y_{l m_l} \chi_{m_s}.$$

Denote the eigenvalues of $K^{(0)}$ by $-\kappa$, i.e.

$$K^{(0)} \varphi_{jlm} = -\kappa \varphi_{jlm} \qquad \text{with} \quad \kappa = -j(j+1) + l(l+1) - \frac{1}{4}.$$

Note also that $(K^{(0)})^2 = \mathbb{1} + \boldsymbol{\sigma} \cdot \boldsymbol{l} + \boldsymbol{l}^2 = \boldsymbol{j}^2 - \boldsymbol{s}^2 + \mathbb{1}$, from which one deduces $\kappa^2 = \left(j + \frac{1}{2}\right)^2$. From these formulae one sees that

$$\begin{aligned} &\text{for } \kappa > 0: \quad && l = \kappa, \\ &\text{for } \kappa < 0: && l = -\kappa - 1, \\ &\text{in all cases} && j = |\kappa| - \frac{1}{2}. \end{aligned} \tag{E.4}$$

Therefore, the eigenfunctions of total angular momentum $\boldsymbol{j}$ can be written in the compact notation $\varphi_{jlm} \equiv \varphi_{\kappa m}$, the modulus of κ giving the value of j, the sign giving the value of $l = j \pm \frac{1}{2}$, according to the rules (E.4). As $K^{(0)} \varphi_{\kappa m} = -\kappa \varphi_{\kappa m}$, the eigenvalues and eigenfunctions of K, (E.2), are

$$K \begin{pmatrix} \varphi_{\kappa m} \\ \varphi_{-\kappa m} \end{pmatrix} = -\kappa \begin{pmatrix} \varphi_{\kappa m} \\ \varphi_{-\kappa m} \end{pmatrix}.$$

For the eigenfunctions $\boldsymbol{\Psi}$ of (E.1) or (E.3) one makes the ansatz

$$\boldsymbol{\Psi}_{\kappa m}(r, \hat{\boldsymbol{r}}) = \begin{pmatrix} g_\kappa(r) \varphi_{\kappa m}(\hat{\boldsymbol{r}}) \\ \mathrm{i} f_\kappa(r) \varphi_{-\kappa m}(\hat{\boldsymbol{r}}) \end{pmatrix}, \tag{E.5}$$

the factor i being introduced for convenience so that the resulting differential equations for the radial functions f and g become real. As a last step one verifies by explicit calculation that $(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}) \varphi_{\kappa m} = -\varphi_{-\kappa m}$. With these tools at hand one deduces from (E.3) the following system of differential equations:

$$
\begin{aligned}
f_\kappa' &= \frac{\kappa - 1}{r} f_\kappa - (E - V(r) - m) g_\kappa, \\
g_\kappa' &= -\frac{\kappa + 1}{r} g_\kappa + (E - V(r) + m) f_\kappa .
\end{aligned}
\tag{E.6}
$$

Clearly, this result does not depend on the specific representation (1.82a) of the Dirac equation we started from. For example in the representation (1.74) we would obtain

$$
\boldsymbol{\Psi}_{\kappa m}(\boldsymbol{r}) = \frac{1}{\sqrt{2}} \begin{pmatrix} g_\kappa \varphi_{\kappa m} + \mathrm{i} f_\kappa \varphi_{-\kappa m} \\ g_\kappa \varphi_{\kappa m} - \mathrm{i} f_\kappa \varphi_{-\kappa m} \end{pmatrix}
$$

[cf. (2.107)].

Equation (E.5) shows very clearly that the central field solutions are not eigenfunctions of orbital angular momentum: For example for $\kappa = -1$, the upper component has $l = 0$, the lower has $\bar{l} = 1$. Thus, the relativistic analogue of an s-state has a component proportional to a p-state, cf. the discussing of the M1-transition 2s $\rightarrow$ 1 s in Sect. 4.1.3.

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Exercises: Further Hints and Selected Solutions

Chapter 1: Fermion Fields and their Properties

1.1. The Lagrangian density for a Majorana field is given by (1.108). One takes the partial derivatives with respect to, say, ϕ^*_B and with respect to $\partial_\mu \phi^*_B$,

$$\frac{\partial \mathscr{L}}{\partial \phi^*_B} = \frac{\mathrm{i}}{2}(\hat{\sigma}^\mu \partial_\mu)^{Bb}\phi_b + m\varepsilon^{BC}\phi^*_C \tag{1}$$

$$\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \phi^*_B\right)} = -\frac{\mathrm{i}}{2}(\hat{\sigma}^\mu)^{Bb}\phi_b. \tag{2}$$

The Euler–Lagrange equation which reads (1) – $\partial_\mu(2) = 0$ yields

$$\mathrm{i}(\hat{\sigma}^\mu \partial_\mu)^{Bb}\phi_b + m\varepsilon^{BC}\phi^*_C = 0.$$

This is (1.107b). Similarly, (1.107a) is obtained by taking the derivatives with respect to ϕ_b and to $\partial_\mu \phi_b$.

1.3. With $J_2 = \sigma^{(2)}/2$ and noting that all even powers of $\sigma^{(i)}$ are equal to the unit matrix, while all odd powers are equal to $\sigma^{(i)}$,

$$(\sigma^{(i)})^{2n} = (\sigma^{(i)2})^n = \mathbb{1}, \quad (\sigma^{(i)})^{2n+1} = \sigma^{(i)},$$

one finds

$$\mathrm{e}^{\mathrm{i}\theta \mathrm{J}_2} = \mathrm{e}^{\mathrm{i}\theta/2\sigma^{(2)}} = \mathbb{1}\cos\frac{\theta}{2} + i\sigma^{(2)}\sin\frac{\theta}{2}.$$

With $\mu = (1/2, -1/2)$ counting the rows, $m = (1/2, -1/2)$ counting the columns, one has

$$(\mathrm{e}^{\mathrm{i}\pi J_2})_{\mu m} = \mathrm{i}(\sigma^{(2)})_{\mu m} - (-)^{1/2-\mu}\delta_{\mu,-m} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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Let U denote the transformation that effects the transition to contragedience. Regarding rotations, this simply means that U must be such that

$$UD(R)U^{-1} = (D(R)^{-1})^{\mathrm{T}} = D^*(R), \tag{3}$$

for any rotation matrix $D(R)$. When expressed in terms of the generators, this gives the condition

$$U\mathrm{e}^{\mathrm{i}\alpha_k J_k}U^{-1} = \mathrm{e}^{\mathrm{i}\alpha_k(U J_k U^{-1})} = \mathrm{e}^{-\mathrm{i}\alpha_k J_k^*}, \tag{4}$$

and, hence

$$UJ_kU^{-1} = -J_k^*, \quad \text{or} \quad UJ_k + J_k^*U = 0. \tag{5}$$

The phase convention which is standard in the theory of angular momentum (and which goes back to Condon and Shortley) yields J_1 real and positive, J_2 pure imaginary, and, of course, J_3 real and diagonal. Therefore the condition (5) is met if we choose

$$U = D(0, \pi, 0) = \mathrm{e}^{\mathrm{i}\pi J_2},$$

because a rotation by π about the 2-axis leaves invariant J_2 but transforms J_1 and J_3 into $-J_1$ and $-J_3$, respectively. Applying U twice leads back to the original representation.

1.4. For vanishing mass $m = 0$, and omitting the spinor indices, (1.69b) reads

$$\mathrm{i}\hat{\sigma}^\mu \partial_\mu(\mathrm{e}^{\pm\mathrm{i}px}\tilde{\phi}(p)) = 0.$$

The wave function in momentum space obeys the equation

$$(\sigma^0 p^0 - \boldsymbol{\sigma} \cdot \boldsymbol{p})\tilde{\phi}(p) = 0,$$

with $p^0 = |\boldsymbol{p}|$. Obviously, its solutions describe *positive* helicity, $\boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}| = +1$. In a similar way (1.69a) with $m = 0$ yields plane wave solutions with *negative* helicity.

1.6. With $\mathscr{L}$ as given in (1.163) we calculate the derivatives

$$\frac{\partial \mathscr{L}}{\partial \phi_A^*} = \frac{\mathrm{i}}{2}(\hat{\sigma}^\mu \partial_\mu)^{Ab}\phi_b - m_D^*\chi^A + m_1^*\varepsilon^{AB}\phi_B^*$$

$$\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi_A^*\right)} = -\frac{\mathrm{i}}{2}(\hat{\sigma}^\mu)^{Ab}\phi_b.$$

Note that according to the rules (1.56) $\varepsilon^{AB}\phi_B^* = -\varepsilon^{ab}(\phi_b)^* = +\phi^{*A}$. Therefore, the Euler–Lagrange equation reads

$$\mathrm{i}(\hat{\sigma}^\mu \partial_\mu)^{Ab}\phi_b = m_D^*\chi^A - m_1^*\phi^{*A}.$$

Taking derivatives with respect to χ^* and to $\partial_\mu\chi^*$ one obtains in the same way

$$\mathrm{i}(\sigma^{\mu}\partial_{\mu})_{aB}\chi^{B} = m_D\phi_a + m_2^{*}\chi_a^{*}.$$

1.7. The behaviour of the two-spinors ϕ_a and χ^A under P, C, and T is given explicitly in Sect. 1.5. The transformation behaviour of the Lagrangian density (1.163) is studied in the same way as in (3.50), (3.55), and (3.56), and we recommend that the reader goes through these first. It then follows that (1.163) transforms according to the pattern

$$P : \{m_D, m_1, m_2\} \to \{m_D^{*}, m_2, m_1\},$$

$$C : \{m_D, m_1, m_2\} \to \{m_D^{*}, m_2^{*}, m_1^{*}\},$$

$$T : \{m_D, m_1, m_2\} \to \{m_D, m_1^{*}, m_2^{*}\}.$$

Thus, if one combines the three discrete operations, $\mathscr{L}$ is indeed found to be invariant

$$TCP : \{m_D, m_1, m_2\} \to \{m_D, m_1, m_2\}. \tag{6}$$

1.8. The left-hand side is the direct product of the matrices (1.26) and is easily calculated to be

$$\mathbb{1}\otimes\mathbb{1} - \sum_i \sigma^{(i)}\otimes\sigma^{(i)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

The right-hand side is also a direct product but with indices c and B interchanged. As such it reads

$$-2\varepsilon_{ac}\varepsilon_{BD} = 2\begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix} = 2\begin{pmatrix} 0\ 0 & \boxed{0\ 1} \\ \boxed{0\ 0} & -1\ 0 \\ 0\ -1 & \boxed{0\ 0} \\ \boxed{1\ 0} & 0\ 0 \end{pmatrix}.$$

Interchanging c and B means interchanging the second half of the first row with the first half of the second row, as well as interchanging the second half of the third row with the first half of the fourth row. This gives indeed the result (7).

1.9. First verify that the trace of ρ is real, viz

$$(tr\rho)^{*} = \mathrm{tr}\rho^{\dagger} = \mathrm{tr}(\gamma_0^2\rho^{\dagger}) = \mathrm{tr}(\gamma_0\rho^{\dagger}\gamma_0) = \mathrm{tr}\rho.$$

Now, let A be an observable expressed as an operator in the space of Dirac fields Ψ. One verifies that its expectation value in a pure state $|n\rangle$, say $\langle A\rangle_n = \langle n|\bar{\Psi}A\Psi|n\rangle$, is real only if $\gamma^0\, A^\dagger\, \gamma^0 = A$. Thus, for a pure or a mixed state described by the density matrix ρ one finds

$$(\mathrm{tr}(\rho A))^* = \mathrm{tr}(A^\dagger \rho^\dagger) = \mathrm{tr}(\gamma^0 A^\dagger \gamma^0 \gamma^0 \rho^\dagger \gamma^0) = \mathrm{tr}(A\rho).$$

Here, we have made use of the relation (1.154b) but we did not need ρ to be hermitean.

1.10. First we work out the explicit form of ρ. Using the standard representation (1.78) one finds

$$\rho = \frac{1}{2}\begin{pmatrix} (E+m)(\mathbb{1}+\boldsymbol{\varsigma}\cdot\boldsymbol{\sigma}) & -(\boldsymbol{p}\cdot\boldsymbol{\sigma})-(\boldsymbol{p}\cdot\boldsymbol{\varsigma})+\mathrm{i}(\boldsymbol{p}\times\boldsymbol{\varsigma})\cdot\boldsymbol{\sigma} \\ (\boldsymbol{p}\cdot\boldsymbol{\sigma})-(\boldsymbol{p}\cdot\boldsymbol{\varsigma})+\mathrm{i}(\boldsymbol{p}\times\boldsymbol{\varsigma})\cdot\boldsymbol{\sigma} & -(E-m)(\mathbb{1}-\zeta\cdot\boldsymbol{\sigma})-\frac{2}{E+m}(\boldsymbol{p}\cdot\boldsymbol{\varsigma})(\boldsymbol{p}\cdot\boldsymbol{\sigma}) \end{pmatrix}. \tag{8}$$

Then choose the 3-axis such that $\boldsymbol{p} = p\hat{e}_3$. The answer is then read off from the explicit representation (8).

1.12. The matrix $A \in SL(2,\mathrm{C})$ that represents the boost with rapidity parameter λ along the direction $\hat{w}$ is given by

$$\begin{aligned} A = \mathrm{e}^{\lambda/2\boldsymbol{\sigma}\cdot\hat{\boldsymbol{w}}} &= \mathbb{1} + \left(\frac{\lambda}{2}\right)^{2n}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{w}})^{2n} + \left(\frac{\lambda}{2}\right)^{2n+1}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{w}})^{2n+1} \\ &= \mathbb{1}\cosh(\lambda/2) + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{w}}\sinh(\lambda/2). \end{aligned} \tag{9}$$

(Cf. also the solution to exercise 1.3 above). With $\sigma^{(1)T} = \sigma^{(1)}$, $\sigma^{(2)T} = -\sigma^{(2)}$, and $\sigma^{(3)T} = \sigma^{(3)}$, it is easy to find the transpose of the expression (9) as well as its inverse. The latter is

$$(A^{-1})^{\mathrm{T}} = \mathbb{1}\cosh(\lambda/2) - (\sigma^{(1)}\hat{\boldsymbol{w}}_1 - \sigma^{(2)}\hat{\boldsymbol{w}}_2 + \sigma^{(3)}\hat{\boldsymbol{w}}_3)\sinh(\lambda/2). \tag{10}$$

(Verify that (10) is indeed the inverse of A^{T}.) From (5), on the other hand, we have

$$\begin{aligned} UAU^{-1} &= \exp\left(\frac{\lambda}{2}\hat{\boldsymbol{w}}\cdot U\boldsymbol{\sigma}U^{-1}\right) = \exp\left(-\frac{\lambda}{2}\hat{\boldsymbol{w}}\cdot\boldsymbol{\sigma}^*\right) \\ &= \exp\left(-\frac{\lambda}{2}(\hat{w}_1\sigma^{(1)} - \hat{w}_2\sigma^{(2)} + \hat{w}_3\sigma^{(3)}\right). \end{aligned}$$

The right-hand side, when expanded as in (9), is identical with (10).

1.13. The momenta conjugate to ϕ_a and to ϕ^*_B are, respectively,

$$\begin{aligned} \pi^a &= \frac{\partial\mathscr{L}}{\partial(\partial_0\phi_a)} = \frac{\mathrm{i}}{2}\phi^*_B(\hat{\sigma}^0)^{Ba}, \\ \pi^{*B} &= \frac{\partial\mathscr{L}}{\partial(\partial_0\phi^*_B)} = -\frac{\mathrm{i}}{2}(\hat{\sigma}^0)^{Ba}\phi_a. \end{aligned}$$

The Hamiltonian density is

$$\mathscr{H} = \frac{\mathrm{i}}{2}\phi_B^*(\hat{\sigma}^0)^{Ba}\partial_0\phi_a - \frac{\mathrm{i}}{2}\partial_0\phi_B^*(\hat{\sigma}^0)^{Ba}\phi_a - \mathscr{L},$$

with $\mathscr{L}$ as given in (1.108). Making use of the equations of motion (1.107) and taking the integral over 3-space, at constant time, one obtains

$$H = \int_{x^0=\text{const.}} \mathrm{d}^3x\, \mathscr{H} = \frac{\mathrm{i}}{2}\int_{x^0=\text{const.}} \mathrm{d}^3x \phi_B^* \overset{\leftrightarrow}{\partial}_0 \delta^{Ba}\phi_a.$$

It is not difficult to derive from the equations of motions that a plane wave solution must have the form

$$\phi_a(x) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}px}\varphi_1(p) + \mathrm{e}^{\mathrm{i}px}\psi_1(p) \\ \mathrm{e}^{+\mathrm{i}px}\varphi_2(p) + \mathrm{e}^{-\mathrm{i}px}\psi_2(p) \end{pmatrix}$$

and that the functions φ_i and ψ_i satisfy the equations

$$\begin{aligned}
(p_0 + p_3)\varphi_1(p) + (p_1 - \mathrm{i}p_2)\psi_2(p) &= m\varphi_2^*(p) \\
(p_0 - p_3)\varphi_2(p) + (p_1 + \mathrm{i}p_2)\psi_1(p) &= m\varphi_1^*(p) \\
(p_0 + p_3)\psi_1(p) + (p_1 - \mathrm{i}p_2)\varphi_2(p) &= -m\psi_2^*(p) \\
(p_0 + p_3)\psi_2(p) + (p_1 + \mathrm{i}p_2)\varphi_1(p) &= -m\psi_1^*(p).
\end{aligned}$$

The latter are seen to be invariant under the replacement

$$\varphi_1(p^0,\boldsymbol{p}) \to \varphi_2^*(p^0,-\boldsymbol{p}), \quad \psi_2(p^0,\boldsymbol{p}) \to -\psi_1^*(p^0,-\boldsymbol{p}).$$

Thus, $|\varphi_1(p^0,\boldsymbol{p})|^2 = |\varphi_1(p^0,\, -\boldsymbol{p})|^2$, and $|\psi_1(p^0,\, \boldsymbol{p})|^2 = |\psi_2(p^0,\, -\boldsymbol{p})|^2$. Insert now the Fourier decomposition

$$\phi_a(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3p}{2E_p} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}px}\varphi_1(p) + \mathrm{e}^{\mathrm{i}px}\psi_1(p) \\ \mathrm{e}^{+\mathrm{i}px}\varphi_2(p) + \mathrm{e}^{-\mathrm{i}px}\psi_2(p) \end{pmatrix}$$

of the field ϕ and its hermitean conjugate into the expression for $\mathscr{H}$ and perform the integral over d^3x, giving a δ distribution in the momentum variables. The result is

$$H = \int \frac{\mathrm{d}^3p}{2E_p} E_p[|\varphi_1(p)|^2 - |\varphi_2(p)|^2 - |\psi_1(p)|^2 + |\psi_2(p)|^2].$$

Indeed, H vanishes because the integrand is odd.

Chapter 2: Electromagnetic Processes and Interactions

2.1. Reflection with respect to a plane through the origin and perpendicular to the 3-direction inverts the direction of the incident momentum but leaves the spin orientation unchanged. A rotation by 180° about, say, the 2-axis, brings back the incident momentum to its initial configuration but interchanges the solutions u_+ and u_-. If the interaction is invariant the scattering amplitudes for the two spin orientations must be the same.

2.2. Repeat the calculation described in Sect. 2.4.2., inserting the matrix element

$$\langle p'|j_\beta(0)|p\rangle = \frac{Z}{(2\pi)^3} F(q^2),$$

instead of (2.46′), and use $m \approx 0$.

2.3. For an infinitesimally small translation four-vector ε with $|\varepsilon_\mu| \ll 1$, (2.39) gives

$$F'(x') \approx F(x) + \varepsilon_\mu \partial^\mu F(x) = F(x) + \mathrm{i}\varepsilon_\mu [P^\mu, F(x)].$$

On the other hand, using $[P^\mu, P^\nu] = 0$, and expanding the operator $U(\varepsilon) \approx \mathbb{1} + \mathrm{i}\varepsilon_\mu P^\mu$, we find

$$U(\varepsilon)F(x)U^{-1}(\varepsilon) \approx F(x) + \mathrm{i}\varepsilon_\mu [P^\mu, F(x)].$$

Now choose ε to be $\varepsilon_\mu = a_\mu/n$ for some large n and take the limit

$$\lim_{n\to\infty}\left(\mathbb{1} + \mathrm{i}\frac{a_\mu}{n}P^\mu\right)^n = \mathrm{e}^{\mathrm{i}a_\mu P^\mu}.$$

2.4. With $\boldsymbol{x} := \boldsymbol{r}_n - \boldsymbol{r}_e$ the integral over d^3q can be calculated using spherical coordinates in $\boldsymbol{q}$-space, i.e. $\mathrm{d}^3q = q^2\mathrm{d}q\,\mathrm{d}(\cos\theta_q)\mathrm{d}\phi_q$. The integrand being even in the modulus q of $\boldsymbol{q}$ we extend the integral over q to the interval $(-\infty, +\infty)$. With $z := \cos\theta_q$ and noting that the integral over the azimuth ϕ_q gives a factor 2π, we find

$$\int \mathrm{d}^3q \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}}{q^2} = 2\pi\int_0^\infty \mathrm{d}q \int_{-1}^{+1}\mathrm{d}z\, \mathrm{e}^{\mathrm{i}qxz} = \frac{\pi}{\mathrm{i}x}\int_{-\infty}^{+\infty}\mathrm{d}q\left(\frac{\mathrm{e}^{\mathrm{i}qx}}{q} - \frac{\mathrm{e}^{-\mathrm{i}qx}}{q}\right) = \frac{2\pi^2}{x}.$$

In the second step we extended the integral over q to the interval $(-\infty, \infty)$, in the last step we made use of Cauchy's theorem.

2.7. Due to time dilation the effective lifetime that is recorded in the laboratory frame is $\tau_{\text{lab}} = \gamma\tau$, with $\gamma = E/m$. The average length over which a particle of energy E can be transported is then estimated from τ_{lab} and its velocity $v = p/(m\gamma)$ in the laboratory frame.

2.8. For negative κ we have $\ell = -\kappa - 1$ and (2.105) read, dropping the index κ,

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{-\ell - 2}{r} f - (E - V - m) g \tag{11}$$

$$\frac{\mathrm{d}g}{\mathrm{d}r} = \frac{\ell}{r} g + (E - V + m) f. \tag{12}$$

By (12) f can be expressed in terms of g and of $\mathrm{d}g/\mathrm{d}r$. The derivative $\mathrm{d}f/\mathrm{d}r$ can also be expressed in terms g and $\mathrm{d}g/\mathrm{d}r$, by means of (11) and the previous result. We then take the derivative of (12) with respect to r to obtain a second- order differential equation for the function g alone. One obtains

$$\frac{\mathrm{d}^2 g}{\mathrm{d}r^2} + \left(\frac{2}{r} + \frac{\mathrm{d}V/\mathrm{d}r}{E - V + m}\right)\frac{\mathrm{d}g}{\mathrm{d}r} + \left((E - V)^2 - m^2 - \frac{\ell(\ell+1)}{r^2} - \frac{\ell}{r}\frac{\mathrm{d}V/\mathrm{d}r}{(E - V + m)}\right) g = 0.$$

Let ε be the energy E from which the rest energy is subtracted, $E = m + \varepsilon$. In the limit of ε and V being small as compared to the rest energy m,

$$(E - V)^2 - m^2 \approx 2m(\varepsilon - V).$$

In this limit the second order differential equation for g reduces to

$$\frac{\mathrm{d}^2 g}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}g}{\mathrm{d}r} - \left(\frac{\ell(\ell+1)}{r^2} + 2m(V - \varepsilon)\right) g = 0. \tag{13}$$

The analogous analysis for $\kappa' = -\kappa - 1$ gives the same value of ℓ and leads to the same differential equation. The result (13) is identical with the nonrelativistic radial equation for the wave function $R_{\ell n}(r) = y_{n\ell}(r)/r$.

2.9. Making use of the identity

$$\boldsymbol{\ell} \cdot \boldsymbol{s} = \frac{1}{2}(\boldsymbol{j}^2 - \boldsymbol{\ell}^2 - \boldsymbol{s}^2)$$

the angular matrix element for the states $\ell = 1,\ j = 3/2$ or $j = 1/2$, is easily calculated,

$$\langle \boldsymbol{\ell} \cdot \boldsymbol{s} \rangle = \frac{1}{2}\left(j(j+1) - \ell(\ell+1) - \frac{3}{4}\right).$$

The former has $\kappa = -2$, the latter has $\kappa = 1$. In the nonrelativistic limit they have the same radial function (2.145) which, for a circular orbit reads

$$y_{n,n-1} = \frac{2^n}{n^{n+1}\sqrt{(2n-1)!}\,a_B^{n+1/2}} r^n \mathrm{e}^{-r/na_B}.$$

The expectation value of $1/r^3$ is calculated in an elementary way,

$$\langle 1/r^3 \rangle_{n,n-1} = \frac{2}{n^4(2n-1)(n-1)} \frac{1}{a_B^3}.$$

Thus, the fine structure splitting, when calculated in first order perturbation theory, is found to be

$$\Delta E = \frac{m(Z\alpha)^4}{2n^4(n-1)}. \tag{14}$$

On the other hand, taking the difference of (2.164) for $\kappa = n-1$ and for $\kappa = -n$ one finds the same expression (14). Thus, up to terms of higher order in $Z\alpha$ the relativistic result agrees with the perturbative estimate.

2.10. The strategy for solving this exercise is to use partial integration in such a way that the radial part of the Laplacian

$$\Delta(V_1 - V_2) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}(V_1 - V_2)}{\mathrm{d}r}\right)$$

appears in the integrand. By Poisson's equation this is then replaced by the difference of the corresponding charge densities. Accordingly, we transform the integral

$$\begin{aligned} I := & \int_0^\alpha \mathrm{d}r \,(ar^2 + br^3 + cr^4)(V_1 - V_2) \\ = & \int_0^\alpha r^2 \mathrm{d}r \left(\frac{a}{2}r^2 + \frac{b}{12}r^3 + \frac{c}{20}r^4\right) \Delta(V_1 - V_2). \end{aligned}$$

Making use of Poisson's equation $\Delta(V_1 - V_2) = -4\pi Ze(\rho_1 - \rho_2)$, the difference of the potentials is replaced by the difference of the charge distributions. The formula for I now contains the moments $\Delta\langle r^n \rangle = 4\pi \int r^2 \mathrm{d}r \, r^n$. This yields the desired formula for ΔE,

$$\Delta E = e \int \mathrm{d}^3x |\psi|^2 (V_1 - V_2) \approx 4\pi \mathrm{e} I.$$

The coefficients a, b, and c for the $2p$ state and for the $2s$ state are obtained by expanding the squared radial functions (2.145) in power of r, viz.

$$a^{(2p)} = b^{(2p)} = 0, \quad c^{(2p)} = \frac{1}{12a_B^5},$$

$$a^{(2s)} = \frac{1}{2a_B^3}, \quad b^{(2s)} = -\frac{1}{a_B^4}, \quad c^{(2s)} = \frac{7}{8a_B^5}.$$

2.11. Using the equations in momentum space $\not{p}u(p) = m\,u(p)$ and $\overline{u(p)}\not{p} = m\,\overline{u(p)}$ we calculate

$$\overline{u(p')}\,i\sigma_{\alpha\beta}(p-p')^{\beta}u(p) = -\frac{1}{2}\overline{u(p')}\,[\gamma_\alpha\not{p} - \gamma_\alpha\not{p}' - \not{p}\gamma_\alpha + \not{p}'\gamma_\alpha]u(p)$$
$$= -\overline{u(p')}\left[\gamma_\alpha\not{p} - p_\alpha + \not{p}'\gamma_\alpha - p'_\alpha\right]u(p)$$
$$= -2m\;\overline{u(p')}\,\gamma_\alpha u(p) + \left(p_\alpha + p'_\alpha\right)\overline{u(p')}\,u(p).$$

In the second step we have used $\gamma_\alpha\not{p}' = -\not{p}'\gamma_\alpha + 2p'_\alpha$ and $\not{p}\gamma_\alpha = -\gamma_\alpha\not{p} + 2p_\alpha$. Analogous relations hold with $u(p)$ replaced by $v(p)$ and $\overline{u(p')}$ by $\overline{v(p')}$. In view of weak interactions (Chaps. 3 and 4) it is instructive to derive analogous identities which hold when $\sigma_{\alpha\beta}$ is multiplied by γ_5.

Chapter 3: Weak Interactions and the Standard Model of Strong and Electroweak Interactions

3.1. Applying parity or charge conjugation to one of the indicated states obviously reproduces the same state. The eigenvalues are determined as follows. The parity of a quark–antiquark state contains a factor $(-)^{\ell}$ from the angular part $Y_{\ell m}$ of the orbital wave function, ℓ being the relative angular momentum, and a factor -1 from the relative intrinsic parity of quark and antiquark, thus $P = (-)^{\ell+1}$.

Charge conjugation interchanges quark and antiquark, hence the spins must be recoupled, giving a factor $(-)^{1/2+1/2-S}$. Regarding the orbital wave function, we note that interchanging q and $\bar{q}$ means relabeling coordinates in the relative coordinate $\boldsymbol{r}_1 - \boldsymbol{r}_2$. While the radial part does not change, the angular part changes by a factor $(-)^{\ell}$. Finally, charge conjugation gives an extra minus sign when applied to a doublet (with respect to an internal symmetry such as strong isospin); cf. Sect. 1.10. Therefore, $C = (-)^{\ell+S}$. If the ω meson is purely nonstrange, i.e., if $\omega = (u\bar{u} - d\bar{d})/\sqrt{2}$, then ϕ is a purely strange state, i.e., $\phi = s\bar{s}$. This is so because ϕ is an isoscalar and must be orthogonal to ω.

From the results above the P and C eigenvalues for low-lying mesons are as follows:

$$\pi_0,\ \eta\,(\ell = 0, S = 0): \qquad P = -1 \qquad C = +1$$
$$\rho_0, \omega, \phi\,(\ell = 0, S = 1): \qquad P = -1 \qquad C = -1$$
$$a_2\,(\ell = 1, S = 1): \qquad P = +1 \qquad C = +1$$

3.4. The generators fall into two sets, say T_k for SU(P) and S_j for SU(Q), which commute for all k and j, $[T_k, S_j] = 0$. Therefore, expressions such as (3.112) for the gauge potential and (3.121) for the field strength tensor will read explicitly

$$A_\alpha(x) = \mathrm{i}\left[e_P \sum B_\alpha^{(k)}(x) T_k + e_Q \sum C_\alpha^{(j)}(x) S_j\right],$$
$$F_{\alpha\beta}(x) = \mathrm{i}\left[e_P \sum G_{\alpha\beta}^{(k)}(x) T_k + e_Q \sum H_{\alpha\beta}^{(j)}(x) S_j\right].$$

In fact, a more precise way of writing the generators would be $\mathrm{T}_k \otimes \mathbb{1}$ and $\mathbb{1} \otimes S_j$, where the first factor acts on the internal space spanned by representations of $\mathrm{SU}(P)$, the second acts on the internal space pertaining to $\mathrm{SU}(Q)$. The construction of the gauge theory for the group $\mathrm{SU}(P) \times \mathrm{SU}(Q)$, along the lines of Sect. (3.3), goes through for any choice of the coupling constants e_P and e_Q. For example, one verifies that the Lagrangian for the pure gauge fields is just the sum of the corresponding Lagrangians for the $\mathrm{SU}(P)$ fields B_α and for the $\mathrm{SU}(Q)$ fields C_α,

$$-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = \frac{1}{4} G_{\alpha\beta} G^{\alpha\beta} - \frac{1}{4} H_{\alpha\beta} H^{\alpha\beta}.$$

Here we have used $\mathrm{tr}\,\{T_k \otimes \mathbb{1}\} = \mathrm{tr}\, T_k = 0$ and likewise for S_j.

3.5. The equation

$$-(\partial_\alpha g^{-1}(x)) g(x) = A_\alpha(x)$$

is interpreted as a differential equation for the gauge transformation $g(x)$, the inhomogeneity being $A_\alpha(x)$. Rewriting this equation

$$\partial_\alpha g^{-1}(x) = -A_\alpha(x) g^{-1}(x), \tag{15}$$

we obtain the integrability condition

$$(\partial_\beta \partial_\alpha - \partial_\alpha \partial_\beta) g^{-1}(x) = 0.$$

Inserting (15) we find

$$\begin{aligned}
&\partial_\alpha (A_\beta(x) g^{-1}(x)) - \partial_\beta (A_\alpha(x) g^{-1}(x)) \\
&\quad = (\partial_\alpha A_\beta - \partial_\beta A_\alpha) g^{-1} + A_\beta \partial_\alpha g^{-1} - A_\alpha \partial_\beta g^{-1} \\
&\quad = (\partial_\alpha A_\beta - \partial_\beta A_\alpha) g^{-1} + (-A_\beta A_\alpha + A_\alpha A_\beta) g^{-1} \\
&\quad = F_{\alpha\beta}(x) g^{-1}(x) = 0.
\end{aligned}$$

In the second step we have used (15) to replace $\partial_\alpha g^{-1}$, in the last step we have inserted the definition of the field strength tensor. As $g^{-1}(x)$ is not identically zero we conclude: $A_\alpha(x)$ is gauge equivalent to 0 if and only if $F_{\alpha\beta}(x)$ vanishes identically. Thus, $F_{\alpha\beta}$ is a measure which tells us to which extent the potential A_α cannot be gauged to zero.

3.6. With $G = \mathrm{SO}(3)$ the structure constants are $C_{ijk} = \varepsilon_{ijk}$. For simplicity consider a set of real scalar fields

$$\Phi(x) = \{\phi_1(x), \phi_2(x), \dots, \phi_M(x)\}$$

forming an M-dimensional representation of G. The invariant scalar product is

$$(\Phi, \Phi) = \sum_{k=1}^{M} \phi_k(x)\phi_k(x).$$

A *globally* invariant theory involving these fields could have the form

$$\mathscr{L}_0 = \frac{1}{2}(\partial_\alpha \Phi, \partial^\alpha \Phi) - m^2(\Phi, \Phi) + \lambda(\Phi, \Phi)^2.$$

This is turned into a *locally* gauge invariant theory by introducing gauge fields $A_\alpha(x) = \mathrm{ie}\sum_{i-1}^{3} T_i A_\alpha^{(i)}(x)$ in the adjoint representation of SO(3) and the covariant derivative $D(A)$ which acts on the scalar multiplet. Symbolically, we obtain

$$\mathscr{L} = -\frac{c}{4}(F^{\alpha\beta}, F_{\alpha\beta}) + \frac{1}{2}(D_\alpha \Phi, D^\alpha \Phi) - m^2(\Phi, \Phi) + \lambda(\Phi, \Phi)^2, \tag{16}$$

with $c = 1/(e^2\kappa)$ and $\mathrm{tr}(T_i T_k) = \kappa\delta_{ik}$. The second, third, and fourth terms are obvious. We work out the first one explicitly, viz.

$$\begin{aligned} -\mathscr{L}_A := {} & \frac{c}{4}(F^{\alpha\beta}, F_{\alpha\beta}) \\ = {} & \frac{1}{4}\sum_{i=1}^{3} f_{\alpha\beta}^{(i)} f^{(i)\alpha\beta} - \frac{\mathrm{e}}{2}\sum_{ijk} \varepsilon_{ijk} f_{\alpha\beta}^{(i)} A^{(j)\alpha} A^{(k)\beta} \\ & + \frac{\mathrm{e}^4}{4}\sum_{ikpq}\sum_{i} \varepsilon_{ijk}\varepsilon_{ipq} A_\alpha^{(j)} A_\beta^{(k)} A^{(p)\alpha} A^{(q)\beta} \qquad (17) \\ = {} & \frac{1}{4}\boldsymbol{f}_{\alpha\beta}\cdot\boldsymbol{f}^{\alpha\beta} - \frac{\mathrm{e}}{2}\boldsymbol{f}_{\alpha\beta}\cdot(\boldsymbol{A}^\alpha \times \boldsymbol{A}^\beta) \\ & + \frac{\mathrm{e}^2}{2}\{(\boldsymbol{A}_\alpha\cdot\boldsymbol{A}^\alpha)(\boldsymbol{A}_\beta\cdot\boldsymbol{A}^\beta) - (\boldsymbol{A}_\alpha\cdot\boldsymbol{A}^\beta)(\boldsymbol{A}_\beta\cdot\boldsymbol{A}^\alpha)\}. \end{aligned}$$

Here the boldface notation stands for $\boldsymbol{A}_\alpha = \left(A_\alpha^{(i)}, i = 1, 2, 3\right)$ and, likewise, for $\boldsymbol{f}_{\alpha\beta} = \left(f_{\alpha\beta}^{(i)} = \partial_\alpha A_\beta^{(i)} - \partial_\beta A_\alpha^{(i)}, i = 1, 2, 3\right)$. (Remember that the gauge fields and the field strengths belong to the adjoint representation of $G = \mathrm{SO}(3)$. Thus, they form a triplet and, therefore, are isomorphic to a vector in real, three-dimensional space $\mathbb{R}^3$.) We have made use of the identity

$$\sum_i \varepsilon_{ijk}\varepsilon_{ipq} = 2(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}),$$

and have introduced the well-known scalar and cross products of $\mathbb{R}^3$. This theory is perhaps the simplest, nontrivial example of a gauge theory. In the explicit form (16) and (17) it is easy to interpret in terms of interactions between the three gauge bosons and with the scalar fields. In the example studied here, G is both a spectrum symmetry and the structure group from which the gauge group is constructed. Without the gauge principle, i.e., without the geometric framework on which it rests, it would be difficult to guess the specific form (17) of the Lagrangian.

3.7. A priori the mixing matrix between any two generations is a unitary matrix $U \in U(2)$ and, hence, depends on four parameters: A common phase φ in $U = e^{i\varphi} U_S$ and three angles which parametrize the factor with determinant $+1$, $U_S \in SU(2)$. Clearly, the former is irrelevant and may be omitted from the start. In analogy to $D^{(1/2)}$, cf. Sect. 1.8.3.a, the latter must have the form, for the example of mixing between the first and the third generation,

$$U_S^{(1,3)} = \begin{pmatrix} \cos\alpha_{13} e^{i(\beta_{13}+\gamma_{13})} & 0 & \sin\alpha_{13} e^{i(\beta_{13}-\gamma_{13})} \\ 0 & 1 & 0 \\ -\sin\alpha_{13} e^{-i(\beta_{13}-\gamma_{13})} & 0 & \cos\alpha_{13} e^{-i(\beta_{13}+\gamma_{13})} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\beta_{13}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\beta_{13}} \end{pmatrix} \begin{pmatrix} \cos\alpha_{13} & 0 & \sin\alpha_{13} \\ 0 & 1 & 0 \\ -\sin\alpha_{13} & 0 & \cos\alpha_{13} \end{pmatrix} \begin{pmatrix} e^{i\gamma_{13}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\gamma_{13}} \end{pmatrix}.$$

What remains to be done is to multiply the three mixing matrices, $U_S^{(1,2)} U_S^{(2,3)} U_S^{(1,3)}$, and to realize that both the initial, unmixed, states, say $b^{(m)} \equiv (d, s, b)$, and the mixed states $d^{(n)}$ can each be multiplied by an unobservable phase, such that the mixing matrix

$$\operatorname{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}) U_S^{(1,2)} U_S^{(2,3)} U_S^{(1,3)} \operatorname{diag}(e^{i\psi_1}, e^{i\psi_2}, e^{i\psi_3})$$

is physically equivalent to $U_S^{(1,2)} U_S^{(2,3)} U_S^{(1,3)}$. This freedom in the choice of ϕ_i and ψ_i reduces the number of observable phases to four.

3.8. In this exercise, which we do not write out here, it is important to realize that the coupling of the charged W-bosons to the photon is fixed through the generalized kinetic term $(F_{\alpha\beta},\ F^{\alpha\beta})$, so that, in particular, the anomalous magnetic moment of the W can be identified. If the coupling had been constructed from the minimal coupling principle (1.200) of electrodynamics, the W would have obtained no anomalous magnetic moment.

3.9. The neutral partner of the Higgs doublet has $y = -2t_3$. The right-hand side of (3.173) then becomes

$$-i\left(gA_\alpha^{(3)} - g' A_\alpha^{(0)}\right)\ t_3\,\phi^0.$$

When this is inserted into (3.172) we obtain

$$\frac{1}{2}(\phi^0,\phi^0)\,t_3^2\left(gA_\alpha^{(3)}-g'A_\alpha^{(0)}\right)\,(gA^{(3)\alpha}-g'A^{(0)\alpha})\;\equiv\;\frac{1}{2}M_{ik}^2\,A_\alpha^{(i)}\,A^{(k)\alpha},\;i,k=0,3.$$

Thus the mass matrix for the neutral gauge bosons is given by

$$M^2=(\phi^0,\,\phi^0)\,t_3^2\begin{pmatrix} g'^2 & -gg' \\ -gg' & g^2 \end{pmatrix}. \tag{18}$$

Diagonalization of this matrix gives $m_\gamma^2=0$ and $m_Z^2=(\phi^0,\phi^0)t_3^2(g^2+g'^2)$. The result for m_w^2 is as given in Sect. 3.4.3c.

3.10. Return to (3.222) and write the second factor in terms of helicity projection operators, viz.

$$\gamma^\alpha-\lambda\gamma^\alpha\gamma_5=\frac{1+\lambda}{2}\,\gamma^\alpha(\mathbb{1}-\gamma_5)+\frac{1-\lambda}{2}\,\gamma^\alpha(\mathbb{1}+\gamma_5).$$

The longitudinal polarization of the outgoing neutrino is easily obtained from this;

$$P_l=\frac{(1-\lambda)-(1+\lambda)}{(1-\lambda)+(1+\lambda)}=-\lambda.$$

3.12. The calculation, which is a little lengthy, may be done along the lines of our calculation with $m_\tau=0$, as given in Sect 3.6.3. Alternatively, this might be a good example to try out an algebraic program such as REDUCE.

3.13. Inserting $c_A^{(e)}=0$ and $c_V^{(F\equiv\mu)}=0$ into (3.242) gives a vanishing asymmetry. Although the electromagnetic and the neutral weak (NC) couplings have negative relative parity, this is not seen in the asymmetry. This interference could only be seen by measuring a spin-momentum correlation.

Chapter 4: Beyond the Minimal Standard Model

4.2. A straightforward way of solving this exercise is to repeat the calculation of the isotropic spectrum along the lines of Sect. 4.1.2. and for the coupling $(\upsilon_\alpha-a_\alpha)(\upsilon^\alpha-a^\alpha)$, by singling out the electron neutrino and the muonic antineutrino, respectively, and integrating over the other two leptons in the final state. A more elegant way is the following: First, realize that the parameter ρ determines the intercept of the unpolarized spectrum at $x=1$, cf. Fig. 4.2. Recall that e^+ and $\bar{\nu}_\mu$ are right-handed, while ν_e is left-handed.

Suppose it is the electron neutrino ν_e that is measured. The point $x=1$ corresponds to the kinematic situation where ν_e moves in a given direction with maximal energy, while e^+ and $\bar{\nu}_\mu$ are parallel and move in the opposite direction. The three helicities adding up to 3/2 in the final state, this configuration is forbidden by angular momentum conservation. Hence, the parameter ρ must be zero, $\rho(\nu_e) = 0$.

Suppose now that one measures the $\bar{\nu}_\mu$. At $x = 1$ this neutrino moves in a given direction, and carries positive helicity. The helicities of e^+ and ν_e who move in the opposite direction, now add up to zero. Therefore, there is no obstacle for this configuration to occur. In fact, up to charge conjugation, this is exactly the situation drawn in Fig. 3.2. Regarding the weak couplings the electric charge of the leptons participating in a four-fermion process does not matter. The "V–A" interaction is invariant under Fierz reordering (possibly up to an overall sign) so that for the interaction term that is relevant here, we have

$$(\bar{\mu}\gamma_\alpha\,(\mathbb{1} - \gamma_5)\,\nu_\mu)\;(\bar{\nu}_e\,\gamma^\alpha\,(\mathbb{1} - \gamma_5)e) = -(\bar{\mu}\gamma_\alpha(\mathbb{1} - \gamma_5)e)\;(\bar{\nu}_e\,\gamma^\alpha(\mathbb{1} - \gamma_5)\nu_\mu).$$

Thus, the calculation of the $\bar{\nu}_\mu$ spectrum is exactly the same as the one of the positron. As we know that ρ was equal to 3/4 there, we conclude $\rho(\bar{\nu}_\mu) = 3/4$.

If the reaction (4.39b) did indeed occur, one would like to identify the electronic antineutrino $\bar{\nu}_e$ through inverse β-decay. Obviously, interpreting data for $\bar{\nu}_e$ in terms of the branching ratio (4.40) rests on the knowledge of the isotropic decay spectrum.

4.3. Let

$$I := \int \frac{\mathrm{d}^3k_1}{2E_1} \int \frac{\mathrm{d}^3k_2}{2E_2}\;(k_1 \cdot k_2)\;\delta\,(Q - k_1 - k_2). \tag{19}$$

As $k_1 + k_2 = Q, k_1^2 = k_2^2 = 0$ we have $(k_1 \cdot k_2) = Q^2/2$. The integral over $\boldsymbol{k}_2$ takes care of the spatial delta distribution. In a frame where $\boldsymbol{Q} = (0, 0, 0)$ and introducing polar coordinates for $\boldsymbol{k}_1$, i.e. $\mathrm{d}^3k_1 = E_1^2\mathrm{d}E_1\mathrm{d}\Omega$ we find

$$I = \frac{Q^2}{2}\pi \int_0^\infty \mathrm{d}E_1\,\delta(Q^0 - 2E_1) = \frac{\pi}{4}\,Q^2.$$

The integral (4.44b) is best decomposed into covariants which are then isolated by taking various contractions as follows. The momentum Q being the only variable, the integral must be a linear combination of the covariants $g^{\alpha\beta}$ and $Q^\alpha Q^\beta$ with Lorentz invariant coefficients. We set

$$\begin{aligned} I^{\alpha\beta} :&= \int \frac{\mathrm{d}^3k_1}{2E_1} \int \frac{\mathrm{d}^3k_2}{2E_2}\,\left\{k_1^\alpha\,k_2^\beta - (k_1 \cdot k_2)g^{\alpha\beta} + k_2^\alpha\,k_1^\beta\right\}\,\delta\,(Q - k_1 - k_2) \\ &\equiv AQ^\alpha Q^\beta + Bg^{\alpha\beta}\,Q^2, \end{aligned} \tag{20}$$

and calculate the invariant integrals

$$g_{\alpha\beta}I^{\alpha\beta} = (A + 4B)Q^2, \quad \text{and} \quad Q_\alpha Q_\beta I^{\alpha\beta} = (A + B)(Q^2)^2.$$

From the definition of the tensor integral $I^{\alpha\beta}$ we find its contraction with the metric tensor $g_{\alpha\beta}I^{\alpha\beta} = -2I$, with I as calculated above. Inserting $Q = k_1 + k_2$

in the integrand of $Q_\alpha I^{\alpha\beta} Q_\beta$, the second combination is zero. Thus $B = -A$, $(A + 4B)Q^2 = -\pi Q^2/2$, and, from these,

$$A = \frac{\pi}{6}, \quad B = -\frac{\pi}{6}. \tag{21}$$

4.4. Let k and k' denote the initial and final neutrino four-momenta, respectively, p the electron momentum, q the muon momentum. Neglecting terms quadratic in the electron mass the invariant kinematic variables s and t are expressed in terms of the laboratory energies as follows,

$$\begin{aligned} s &= (k+p)^2 \approx 2m_e E_\nu^{\text{lab}}, \\ t &= (k-q)^2 = (k'-p)^2 \approx -2m_e E_\nu'^{\text{lab}} \approx -2m_e \left(E_\nu^{\text{lab}} - E_\mu^{\text{lab}}\right) \approx -s(1-y). \end{aligned}$$

With $\mathrm{d}t/\mathrm{d}y = s$ and the fact that the cross section (4.69) is a quadratic function in the variable y, the integration is elementary. Note that it is the integral over y from some value y_{min} (that depends on the experimental arrangement), to its maximum $y = 1$ which is determined in the quoted experiments.

4.5. The calculation is similar to the one of exercise 4.3, except that now one of the neutrinos is massive, say, $k_2^2 = \lambda^2$. We first calculate the integral

$$I_0 := \int \frac{\mathrm{d}^3 k_1}{2E_1} \int \frac{\mathrm{d}^3 k_2}{2E_2} \, \delta\,(Q - k_1 - k_2), \tag{22}$$

say, in a frame where the spatial part of Q vanishes. The integral over $\mathrm{d}^3 k_2$ eliminates the spatial δ-distribution. In a frame where $\boldsymbol{Q} = (0, 0, 0)$, this leaves E_2 a function of E_1, so that we obtain

$$I_0 = \pi \int_0^\infty \mathrm{d}E_1 \, E_1 \, \frac{1}{E_2(E_1)} \, \delta(E_1 + E_2(E_1) - Q^0),$$

with $E_2(E_1) = \sqrt{\lambda^2 + E_1^2}$. The integral over E_1 is calculated using the well-known replacement rule for the delta distribution

$$\delta(g)(E_1)) \;\to\; \frac{1}{\left|g'\left(E_1^{(i)}\right)\right|} \, \delta(E_1^{(i)}),$$

where $E_1^{(i)}$ is a simple zero of the function $E_2(E_1)$. This gives

$$I_0 = \frac{\pi}{2} \, \frac{Q^2 - \lambda^2}{(Q^0)^2} \;\to\; I_0 = \frac{\pi}{2} \, \frac{Q^2 - \lambda^2}{Q^2}.$$

In the last step we returned to an arbitrary frame (where $\boldsymbol{Q}$ is possibly non-zero), using our knowledge that the integral I_0 must be a Lorentz scalar. As the neutrino number 2 is massive, the scalar product of the neutrino four-momenta becomes $(k_1 \cdot k_2) = (Q^2 - \lambda^2)/2$ and the integral (19) is given by

$$I = \frac{\pi}{4} \frac{(Q^2 - \lambda^2)}{Q^2}.$$

The tensor integral (20) is calculated as in exercise 4.3., making use of the result above. One finds

$$Q_\alpha I^{\alpha\beta} Q_\beta = (Q^2)^2 \, (A + B) = \lambda^2 I, \;\; g_{\alpha\beta} I^{\alpha\beta} = Q^2(A + 4B) = -2I,$$

and, from these relations,

$$A = \frac{\pi}{6} \frac{(Q^2 - \lambda^2)^2 \, (Q^2 + 2\lambda^2)}{(Q^2)^3}, \quad B = -\frac{\pi}{12} \frac{(Q^2 - \lambda^2)^2 \, (2Q^2 + \lambda^2)}{(Q^2)^3}. \tag{23}$$

Clearly, for $\lambda = 0$ the result (23) goes over into the result (21) of exercise 4.3.

4.6. It is simplest to use a Cartesian basis in isospin space and to drop all real factors such as $(2\pi)^{3/2}$ so that the ansatz reads

$$\left\langle 0 \,\middle|\, A_\alpha^{(i)}(0) \,\middle|\, \pi_j(q) \right\rangle = F \delta_{ij} q_\alpha. \tag{24}$$

Inserting first $T^{-1}T$ and then $P^{-1}P$ one obtains successively

$$\begin{aligned}
\left\langle 0 \,\middle|\, A_\alpha^{(i)}(0) \,\middle|\, \pi_j(q) \right\rangle &= \left\langle 0 \,\middle|\, T^{-1} \left(T A_\alpha^{(i)}(0)\, T^{-1} \right) \, T \,\middle|\, \pi_j\,(q) \right\rangle \\
&= (-)^{1+\delta_{\alpha 0}} \left\langle 0 \,\middle|\, A_\alpha^{(i)}(0) \,\middle|\, \pi_j(q_0 - \boldsymbol{q}) \right\rangle^* \eta_T \\
&= (-)^{1+\delta_{\alpha 0}} \left\langle 0 \,\middle|\, P \;\; A_\alpha^{(i)}(0)\, P^{-1} \right) \, P \,\middle|\, \pi_j(q_0 - \boldsymbol{q}) \right\rangle^* \eta_T \\
&= -\left\langle 0 \,\middle|\, A_\alpha^{(i)\dagger}(0) \,\middle|\, \pi_j(q_0, \; \boldsymbol{q}) \right\rangle^* \eta_T \eta_P \\
&= -F^* \delta_{ij} q_\alpha \eta_T \eta_P
\end{aligned}$$

Here η_P is the intrinsic parity of the pion, η_T is an analogous phase which appears when T is applied to a one-pion state. The pion has negative intrinsic parity. It is even with respect to charge conjugation, i.e. $\eta_C = 1$. With $\eta_P \eta_P \eta_C = 1$ we conclude that the product $\eta_P \eta_T$ is $+1$. Thus, F, as defined in (24) is pure imaginary, or, equivalently, f_π is pure real.

4.9. It is not difficult to check that this model for the pionic axial current yields the terms represented by the diagrams of Fig. 4.5 and, hence, that the structure terms vanish.

4.10. In the electronic decay mode $\pi \to e\nu\gamma$ the structure terms are found to be comparable in magnitude to the contributions from internal bremsstrahlung. The latter, however, contain the dynamic suppression factor $\propto m_e^2$ that we found in comparing $\pi \to e\nu$ to $\pi \to \mu\nu$. In the muonic decay mode $\pi \to \mu\nu\gamma$, the bremsstrahlung does not show this strong suppression while the structure terms, up to kinematic differences due to the muon mass, are the same as in the electronic mode. From this simple observation one draws two conclusions:

(i) as the contributions due to internal bremssstrahlung are dominant in the muonic mode, the branching ratio

$$\frac{\Gamma(\pi^+ \to e^+\nu_e\gamma)}{\Gamma(\pi^+ \to \mu^+\nu_\mu\gamma)}$$

must be practically the same as the branching ratio

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)}$$

(ii) the structure terms are very small, hence not easily detectable, in the muonic decay mode. The integral over the available phase space for $\pi \to \mu\nu\gamma$ is easily calculated using the formulae of Sect. 4.3.1 a for three-body decays. It is found markedly smaller than the surface of the triangle of Fig. 4.6 for $\pi \to e\nu\gamma$.

4.11. The calculation is analogous to the calculation of pion β-decay; cf. Sect. 4.2.2.

4.12. This problem is a little lengthier. We give only a few hints and a reference where further details will be found. A Feynman graph where the π^0 first converts to a single virtual photon which then decays via pair decay to the $e^+\ e^-$ final state gives no contribution because of conservation of C. Indeed, the neutral pion has $C(\pi^0) = +1$ while the photon has $C(\gamma) = -1$. Therefore, the diagram of lowest order is the triangular graph with π^0 decaying into a pair of virtual photons, and with the electron–positron pair being created via (virtual) Compton effect. It is not difficult to write down the analytic expression that corresponds to this diagram. It will contain the amplitude (4.126a) for $\pi^0 \to \gamma\gamma$, with the two photons off-shell, the electron and the positron as external legs, and the electron propagator joining the two vertices where the photons are annihilated. A lower limit on the branching ratio is obtained by means of the unitarity relation,

$$\mathrm{i}(T^\dagger - T) = (2\pi)^4 T^\dagger T,$$

taken between the initial state $|\pi_0\rangle$ and the final state $\langle e^+e^-|$. It is important to note that the product $T^\dagger T$, after insertion of a complete set of intermediate states, contains only *on-shell* amplitudes. The dominant contribution stems from the two-photon intermediate state,

$$\mathrm{i}\big(\big\langle e^+e^-|\pi_0\big\rangle^* - \big\langle\pi_0|e^+e^-\big\rangle\big) = (2\pi)^4\big\langle e^+e^-|\gamma\gamma\big\rangle^*\,\langle\gamma\gamma|\pi_0\rangle + \cdots,$$

with the two photons on their mass shell. This will be a lower limit because the unitarity relation yields only the imaginary part of the $\pi_0 \to e^+e^-$ amplitude. If this is worked out one finds the so-called *unitarity* limit

$$\Gamma(\pi^0 \to e^+e^-)|_{\text{unitarity}} \approx \frac{1}{2}(\alpha m_e/m_\pi)^2\Gamma(\pi^0 \to \gamma + \gamma).$$

This gives a lower limit on the branching ratio of 4.75×10^{-8} (cf., e.g., L.G. Landsberg, *Phys. Rep*. **128** (1985) 301). Compare this to the present experimental value for the branching ratio: $(7.5 \pm 2.0) \times 10^{-8}$ (Deshpande et al., *Phys. Rev. Lett*. **71** (1993) 27, McFarland et al., ibid. p.31.)

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