A Technical Appendix

A.1 Integration Theory

Definition A.1
A decomposition *$\mathcal{Z}$ of the interval $[a,b]$ is understood to be a set $\mathcal{Z} \stackrel{\text{def}}{=} \{t_0, t_1, \ldots, t_n\}$ of points t_j with $a = t_0 < t_1 < \ldots < t_n = b$. Through this the interval $[a,b]$ is decomposed into n sub-intervals $[t_k, t_{k+1}]$, where $k = 0, 1, 2, \ldots, n-1$. $|\mathcal{Z}| \stackrel{\text{def}}{=} \max_k(t_{k+1} - t_k)$, that is, the length of the largest resulting sub-interval and is referred to as the* refinement *of the decomposition $\mathcal{Z}$.*

Definition A.2
For a function $w : [a,b] \longrightarrow \mathbb{R}$ and a decomposition $\mathcal{Z} \stackrel{\text{def}}{=} \{t_0, t_1, \ldots, t_n\}$ one defines the variation *of w with respect to $\mathcal{Z}$ as:*

$$V(\mathcal{Z}) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} |w(t_{k+1}) - w(t_k)|$$

$V \stackrel{\text{def}}{=} \sup_{\mathcal{Z}} V(\mathcal{Z})$ is called the total variation *of w on $[a,b]$. If $V < \infty$ holds, then w is of* finite variation *on $[a,b]$.*

Theorem A.1
For a function $w : [a,b] \longrightarrow \mathbb{R}$ it holds that:

1. *w is of finite variation when w is monotone,*
2. *w is of finite variation when w is Lipschitz continuous,*
3. *w is bounded when w is of finite variation.*

Moreover, sums, differences and products of functions of finite variation are themselves of finite variation.

Definition A.3
Given the functions $f, w : [a,b] \rightarrow \mathbb{R}$ and a decomposition $\mathcal{Z}$, choose for

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$k = 0, 1, \ldots, n-1$ partitions $\tau_k \in [t_k, t_{k+1}]$ and form:

$$I(\mathcal{Z}, \boldsymbol{\tau}) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} f(\tau_k) \cdot \{w(t_{k+1}) - w(t_k)\}$$

If $I(\mathcal{Z}, \boldsymbol{\tau})$ converges for $|\mathcal{Z}| \to 0$ to a limiting value I, which does not depend on the chosen decomposition $\mathcal{Z}$ nor on the choice of the partitions τ_k, then I is called the **Riemann-Stieltjes integral** *of f. One writes:*

$$I = \int_a^b f(t) dw(t).$$

For $w(t) = t$ we get the **Riemann integral** *as a special case of the Stieltjes integrals.*

Theorem A.2 (Characteristics of the Riemann-Stieltjes Integral)

1. *If the corresponding integrals on the right hand side exist, then the* linearity characteristics *hold:*

$$\begin{aligned} \int_a^b (\alpha \cdot f + \beta \cdot g) \; dw &= \alpha \int_a^b f \; dw + \beta \int_a^b g \; dw \quad (\alpha, \beta \in \mathbb{R}) \\ \int_a^b f \; d(\alpha \cdot w + \beta \cdot v) &= \alpha \int_a^b f \; dw + \beta \int_a^b f \; dv \quad (\alpha, \beta \in \mathbb{R}) \end{aligned}$$

2. *If the integral $\int_a^b f dw$ and the integrals $\int_a^c f dw$ exist, then for $\int_c^b f dw$, $a < c < b$ it holds that:*

$$\int_a^b f dw = \int_a^c f dw + \int_c^b f dw$$

3. *If f is continuous on $[a, b]$ and w is of finite variation, then $\int_a^b f dw$ exists.*

4. *If f is continuous on $[a, b]$ and w is differentiable with a bounded derivative, then it holds that:*

$$\int_a^b f(t) dw(t) = \int_a^b f(t) \cdot w'(t) dt$$

5. Partial integration*: If $\int_a^b f dg$ or $\int_a^b g df$ exist, so does the other respective integral and it holds that:*

$$\int_a^b f dg + \int_a^b g df = f(b)g(b) - f(a)g(a)$$

6. *If w is continuous, it holds that* $\int_a^b dw(t) = w(b) - w(a)$

7. *If f is continuous on $[a,b]$ and w is step-wise constant with discontinuity points $\{c_k, k = 1,\ldots,m\}$, then:*

$$\int_a^b f dw = \sum_{k=1}^{m} f(c_k) \cdot \left\{ w(c_k^+) - w(c_k^-) \right\}$$

where c_k^+ (c_k^-) is the right (left) continuous limit and $w(c_k^+) - w(c_k^-)$ is the step height of w on $\{c_k\}$.

Theorem A.3 (Radon-Nikodym)
Let λ and μ be positive measures on $(\Omega, \mathcal{F})$ with

1. $0 < \mu(\Omega) < \infty$ *and* $0 < \lambda(\Omega) < \infty$
2. *λ is* absolutely continuous *with respect to μ, i.e. from $\mu(A) = 0$ it follows that $\lambda(A) = 0$ for all $A \in \mathcal{F}$ (written: $\lambda \ll \mu$).*

Then a non-negative $\mathcal{F}$-measurable function h exists on Ω, such that:

$$\forall A \in \mathcal{F}\colon\ \lambda(A) = \int_A h d\mu;$$

In particular, for all measurable functions f it holds that:

$$\int f d\lambda = \int f \cdot h \, d\mu.$$

Remark A.1
One often uses the abbreviation $\lambda = h \cdot \mu$ in the Radon-Nikodym theorem and refers to h as the density *of λ with respect to μ. Due to its construction h is also referred to as the* Radon-Nikodym derivative. *In this case one writes $h = \frac{d\lambda}{d\mu}$. One calls λ and μ equivalent measures if $\lambda \ll \mu$ and $\mu \ll \lambda$.*

An important tool in stochastic analysis is the transformation of measure, which is illustrated in the following example.

Example A.1
Let $Z_1, \ldots, Z_n$ be independent random variables with standard normal distributions on the measurable space $(\Omega, \mathcal{F}, \mathrm{P})$ and $\mu_1, \ldots, \mu_n \in \mathbb{R}$. Then by

$$\mathrm{Q}(d\omega) \stackrel{\text{def}}{=} \xi(\omega) \cdot \mathrm{P}(d\omega) \quad \textit{with} \quad \xi(\omega) \stackrel{\text{def}}{=} \exp\{\sum_{i=1}^{n} \mu_i Z_i(\omega) - \frac{1}{2}\mu_i^2\}$$

an equivalent probability measure Q *for* P *is defined. For the distribution of the $Z_1, \ldots, Z_n$ under the new measure* Q *it holds that:*

$$\begin{aligned} & \mathrm{Q}(Z_1 \in dz_1, \ldots, Z_n \in dz_n) \\ = \; & \exp\{\sum_{i=1}^{n}(\mu_i z_i - \frac{1}{2}\mu_i^2)\} \cdot \mathrm{P}(Z_1 \in dz_1, \ldots, Z_n \in dz_n) \\ = \; & \exp\{\sum_{i=1}^{n}(\mu_i z_i - \frac{1}{2}\mu_i^2)\} \cdot (2\pi)^{-\frac{n}{2}} \exp\{-\frac{1}{2}\sum_{i=1}^{n} z_i^2\} dz_1 \ldots dz_n \\ = \; & (2\pi)^{-\frac{n}{2}} \exp\{-\frac{1}{2}\sum_{i=1}^{n}(z_i - \mu_i)^2\} dz_1 \ldots dz_n, \end{aligned}$$

in other words $Z_1, \ldots, Z_n$ are, with respect to Q*, independent and normally distributed with expectations $\mathsf{E}_{\mathrm{Q}}(Z_i) = \mu_i$ and $\mathsf{E}_{\mathrm{Q}}[(Z_i - \mu_i)^2] = 1$. Thus the random variables $\widetilde{Z_i} \stackrel{\text{def}}{=} Z_i - \mu_i$ are independent random variables with standard normal distributions on the measurable space $(\Omega, \mathcal{F}, \mathrm{Q})$.*

Going from P to Q by multiplying with ξ changes the expectations of the normally distributed random variables, but the volatility structure remains notably unaffected.

The following Girsanov theorem generalises this method for the continuous case, that is, it constructs for a given P-Brownian motion W_t an equivalent measure Q and an appropriately adjusted process W_t^*, so that it represents a Q-Brownian motion. In doing so the ("arbitrarily" given) expectation μ_i is replaced by an ("arbitrarily" given) drift, that is, a stochastic process X_t.

Theorem A.4 (Girsanov)
Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, W_t a Brownian motion with respect to P*, $\mathcal{F}_t$ a filtration in $\mathcal{F}$ and X_t an adapted stochastic process. Then*

$$\xi_t \stackrel{\text{def}}{=} \exp(\int_0^t X_u dW_u - \frac{1}{2}\int_0^t X_u^2 du)$$

defines a martingal with respect to P *and $\mathcal{F}_t$. The process W_t^* defined by*

$$W_t^* \stackrel{\text{def}}{=} W_t - \int_0^t X_u du$$

is a Wiener process with respect to the filtration $\mathcal{F}_t$ *and*

$$Q \stackrel{\text{def}}{=} \xi_T \cdot P \tag{A.1}$$

is a P *equivalent probability measure* Q.

The Girsanov theorem thus shows that for a P-Brownian motion W_t an equivalent probability measure Q can be found such that W_t^*, as a Q-Brownian motion at time t, contains the drift X_t. In doing so (A.1) means that: $\int_\Omega \mathbf{1}(\omega \in A) dQ(\omega) = Q(A) \stackrel{\text{def}}{=} \int_\Omega \mathbf{1}(\omega \in A)\xi_T dP(\omega) = \mathsf{E}_P[\mathbf{1}(\omega \in A)\xi_T]$ for all $A \in \mathcal{F}$.

Remark A.2
With the relationships mentioned above ξ_t *is by all means a martingale with respect to* P *and* $\mathcal{F}_t$ *when the so-called* Novikov Condition

$$\mathsf{E}_P\Big[\exp(\int_0^t X_u^2 du)\Big] < \infty \qquad \textit{for all } t \in [0,T]$$

is met, that is, when X_t *does not vary too much.*

Another important tool used to derive the Black-Scholes formula by means of martingale theory is the martingale representation theory. It states that every Q-martingale under certain assumptions can be represented by a predetermined Q-martingale by means of a square-integrable process.

Theorem A.5 (Martingale Representation theorem)
Let M_t *be a martingale with respect to the probability measure* Q *and the filtration* $\mathcal{F}_t$*, for which the volatility process* $\sigma_t \neq 0$ *a.s. for all* $t \in [0,T]$*, where* $\sigma_t^2 = \mathsf{E}_Q[M_t^2|\mathcal{F}_t]$*. If* N_t *is another martingale with respect to* Q *and* $\mathcal{F}_t$*, then (uniquely defined) on* $\mathcal{F}_t$ *an adapted stochastic process* H_t *exists with* $\int_0^T H_t^2\sigma_t^2 dt < \infty$ *with:*

$$N_t = N_0 + \int_0^t H_s dM_s.$$

Example A.2
It is easy to show that the standard Wiener process W_t *with respect to the probability measure* P *is a martingale with respect to* P *and its corresponding filtration* $\mathcal{F}_t$*. If* X_t *is another martingale with respect to* P *and* $\mathcal{F}_t$*, then according to the previous theorem a* $\mathcal{F}_t$ *adapted stochastic process* H_t *exists, so that*

$$X_t = X_0 + \int_0^t H_s dW_s.$$

Remark A.3
Writing the last expression in terms of derivatives:

$$dX_t = H_t dW_t.$$

The example shows once again that a martingale cannot possess a drift.

A.2 Portfolio Strategies

The portfolio of an investor at time t, i.e., the market value of the single equities (contracts) in his portfolio at time t, is dependent on the development of the price $\{\boldsymbol{S}_s; s < t\}$, $\boldsymbol{S}_s = (S_s^1, \ldots, S_s^d)^\top \in \mathbb{R}^d$ up to time t, that is, on the information that is available at that particular time point. Given this, it is obvious that his strategy, i.e., the development of his portfolio's value over time, should be modelled as a $\mathcal{F}_t$ adapted d-dimensional stochastic process ϕ_t. In doing so $\phi_t^i(\omega)$ represents how much in state ω of the security i is in his portfolio at time t , where negative values indicate a short sell of the corresponding contract.

Definition A.4
Assume the following market model: $\mathcal{M} = (\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t, \boldsymbol{S}_t)$. A d-dimensional stochastic process ϕ_t adapted on the filtration $\mathcal{F}_t$ is called a portfolio strategy. The stochastic process $V(\phi_t)$ with $V(\phi_t) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{d} \phi_t^i S_t^i$ is called the value of the strategy ϕ.

Example A.3
In the Black-Scholes model two financial instruments are traded on the market: a risky security S (stock) and a riskless security B (zero bond). As in Chapter 5, the stock price S_t is assumed to follow a geometric Brownian motion, so that the following stochastic differential equation is satisfied:

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{A.2}$$

The price of the zero bond B_t satisfies the differential equation:

$$dB_t = rB_t dt$$

with a constant r. Without loss of generality it can be assumed that $B_0 = 1$, which leads to $B_t = exp(rt)$.

The corresponding market model is thus $\mathcal{M}_{BS} = (\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t, \boldsymbol{S}_t)$, where $\boldsymbol{S}_t \stackrel{\mathrm{def}}{=} (S_t, B_t)^\top \in \mathbb{R}^2$.

The two-dimensional stochastic process $\phi_t = (a_t, b_t)^\top$ now describes a portfolio strategy in which $a_t(\omega)$ gives the number of stocks and $b_t(\omega)$ gives the number of bonds in the portfolio at time t in state ω. The value of the portfolio at time t is then a random variable

$$V(\phi_t) = a_t S_t + b_t B_t.$$

A particularly important portfolio strategy is that once it is implemented it does not result in any cash flows over time, i.e., when the portfolio is re-balanced no payments are necessary. This means that eventual income (through selling securities, receiving dividends, etc.) is exactly offset by required payments (through buying additional securities, transaction costs, etc.) This is referred to as a self-financing strategy. For an outside observer the change in value of the portfolio only occurs as the price of the participating securities changes.

Definition A.5
Let $\mathcal{M} = (\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t, \boldsymbol{S}_t)$ be a market model and ϕ a portfolio strategy with the value $V(\phi_t)$. Then ϕ is called

1. self-financing, *when $dV(\phi_t) = \sum_{i=1}^{d} \phi_t^i dS_t^i$ holds (P-a.s.),*
2. admissible, *when $V(\phi_t) \geq 0$ holds (P-a.s.).*

Below the Black-Scholes model will be considered. The subsequent specification shows that arbitrage is not possible in such a market: There is no admissible self-financing strategy with a starting value of $V(\boldsymbol{\phi}_0) = 0$, whose end value $V(\boldsymbol{\phi}_T)$ is positive with a positive probability.

Lemma A.1
In the Black-Scholes model $\mathcal{M}_{BS} = (\Omega, \mathcal{F}, \mathrm{P}, \mathcal{F}_t, \boldsymbol{S}_t), \boldsymbol{S}_t = (S_t, B_t)^\top$, the portfolio strategy $\phi_t = (a_t, b_t)^\top$ is exactly self-financing when the discounted process $\tilde{V}_t$ with $\tilde{V}_t = e^{-rt} V_t$ satisfies the stochastic differential equation

$$d\tilde{V}_t = a_t d\tilde{S}_t,$$

where $\tilde{S}_t = e^{-rt} S_t$ describes the discounted stock price.

The explicit specification of the corresponding strategy can be left out when it is clear from the context and we write $V_t = V(\phi_t)$. With the help of the Girsanov theorem a P equivalent measure Q can be constructed, under which the process of the discounted stock prices is a martingale. Using (A.2) one obtains

$$d\tilde{S}_t = \tilde{S}_t\{(\mu - r)dt + \sigma dW_t\}. \tag{A.3}$$

By setting

$$X_t \stackrel{\text{def}}{=} -\frac{\mu - r}{\sigma}$$

the Novikov condition (see Remark A.2) is obviously fulfilled. Therefore, for Q with

$$\begin{aligned} \frac{dQ}{dP} = \xi_T &= \exp(\int_0^T X_u dW_u - \frac{1}{2}\int_0^T X_u^2 du) \\ &= \exp\{-\frac{\mu - r}{\sigma}W_T - \frac{1}{2}(\frac{\mu - r}{\sigma})^2 T\} \end{aligned}$$

$W_t^* \stackrel{\text{def}}{=} W_t + \frac{\mu - r}{\sigma}t$ is a Q-Brownian Motion according to the Girsanov theorem. Because of (A.3) and using the definition of W_t^* it holds that

$$d\tilde{S}_t = \tilde{S}_t \sigma dW_t^*. \tag{A.4}$$

According to Itô's lemma this becomes

$$\tilde{S}_t = \tilde{S}_0 \exp\left(\int_0^t \sigma dW_u^* - \frac{1}{2}\int_0^t \sigma^2 du\right)$$

and solves the stochastic differential equation (A.4). Since σ is constant, for all t the Novikov condition holds

$$\mathsf{E}\left[\exp\left(\int_0^t \sigma^2 du\right)\right] < \infty.$$

According to Remark A.2

$$\exp\left(\int_0^t \sigma dW_u^* - \frac{1}{2}\int_0^t \sigma^2 du\right),$$

that is $\tilde{S}_t$, is also a Q-martingale.

Q represents with respect to $\widetilde{S}_t$ a P equivalent martingale measure. It can be shown that given this form, it can be uniquely determined.

From the Definition of W_t^* and with the help of (A.2) one obtains

$$dS_t = S_t(rdt + \sigma dW_t^*),$$

i.e., under the measure Q the expected value of the risky securities is equivalent to the certain value of the riskless bonds. Because of this the martingale measure Q is also called the *risk neutral* probability measure and contrary to this P is called the *objective* or *physical* probability measure of the Black-Scholes markets.

As a result of the Q-martingale properties of $\widetilde{S}_t$, due to Lemma A.1, the discounted value of a self-financing strategy $\widetilde{V}_t$ is itself a Q-martingale. Consequently it holds that: If the starting value of an admissible self-financing strategy is equal to zero, then its value at all later time points t must also be equal to zero. Thus in using an admissible self-financing strategy, there is no riskless profit to be made: The Black-Scholes market is free of arbitrage.

The following theorem represents the most important tool used to value European options with the help of the Black-Scholes model. It secures the existence of an admissible self-financing strategy that duplicates the option, thus the value of which can be calculated using martingale theory.

Theorem A.6
Assume that the Black-Scholes model $\mathcal{M}_{BS}$ is given. The function X describes the value of a European option at the time to maturity T and is Q*-integrable.*

a) Then an admissible self-financing strategy $(a_t, b_t)^\top$ exists, which duplicates X and whose value V_t for all t is given by

$$V_t = \mathsf{E}_{\mathrm{Q}}[e^{-r(T-t)} X \mid \mathcal{F}_t]. \tag{A.5}$$

b) If the value V_t in a) is dependent on t and S_t and is written as a function $V_t = F(t, S_t)$ with a smooth function F, then it holds for the corresponding strategy that

$$a_t = \frac{\partial F(t, S_t)}{\partial S_t}.$$

Proof:

1. One defines V_t by (A.5), where the function defined follows from the Q-integrability of X. Due to

$$\widetilde{V}_t = e^{-rt} V_t = \mathsf{E}_{\mathrm{Q}}[e^{-rT} X \mid \mathcal{F}_t]$$

one identifies $\widetilde{V}_t$ as Q-martingale. One should notice that $e^{-rT}X$, exactly like X, is only dependent on the state at date T and thus it can be classified as a random variable on $(\Omega, \mathcal{F}_t, \mathrm{Q})$.

$\mathcal{F}_t$ represents, at the same time, the natural filtration for the process W^*, which, as was seen above, is also a Q-martingale. Therefore, according to Theorem A.5 using the martingale representation, a process H_t exists, adapted on $\mathcal{F}_t$ with $\int_0^T H_t^2\sigma^2 dt < \infty$ Q-almost sure, so that for all t it holds that:

$$\widetilde{V}_t = \widetilde{V}_0 + \int_0^t H_s dW_s^* = V_0 + \int_0^t H_s dW_s^*.$$

Thus one sets:

$$a_t \stackrel{\text{def}}{=} \frac{H_t}{\sigma \cdot \widetilde{S}_t} \quad , \quad b_t \stackrel{\text{def}}{=} \widetilde{V}_t - a_t\widetilde{S}_t.$$

Then after a simple calculation it holds that:

$$a_t S_t + b_t B_t = V_t$$

and $(a_t, b_t)^\top$ is a X duplicating strategy. Furthermore, with (A.4) it holds for all t:

$$a_t d\widetilde{S}_t = a_t \widetilde{S}_t \sigma dW_t^* = H_t dW_t^* = d\widetilde{V}_t,$$

i.e., $(a_t, b_t)^\top$ is according to Lemma A.1 self-financing. Due to the non-negativity of X and the definition of V_t, $(a_t, b_t)^\top$ is also admissible.

2. For $V_t = F(t, S_t)$ it holds using Itô's lemma:

$$\begin{aligned}
d\widetilde{V}_t &= d\{e^{-rt}F(t,S_t)\} \\
&= \frac{\partial\{e^{-rt}F(t,S_t)\}}{\partial S_t} dS_t + A(t,S_t)dt \\
&= \frac{\partial F(t,S_t)}{\partial S_t} e^{-rt} S_t(rdt + \sigma dW_t^*) + A(t,S_t)dt \\
&= \frac{\partial F(t,S_t)}{\partial S_t} \widetilde{S}_t \sigma dW_t^* + \widetilde{A}(t,S_t)dt \\
&= \frac{\partial F(t,S_t)}{\partial S_t} d\widetilde{S}_t + \widetilde{A}(t,S_t)dt.
\end{aligned}$$

Since not only $\widetilde{V}_t$ but also $\widetilde{S}_t$ are Q-martingales, the drift term $\widetilde{A}(t, S_t)$ must disappear. According to part a) of the theorem the corresponding strategy is self-financing and thus using Lemma A.1 the claim follows.

□

Remark A.4 *With the relationships of the preceding theorems, V_t is called the fair price for option X at date t, because at this price, according to the* *previous arguments, there is no arbitrage possible for either the buyer or the seller of the option. Equation (A.5) is called the risk neutral valuation formula, since it gives the fair price of the option as the (conditional) expectation of the (discounted) option value at maturity with respect to the risk neutral measure of the Black-Scholes model.*

The result obtained from the last theorem has already been formulated in Chapter 6 as equation (6.24).

Corollary A.1
The relationships of the preceding theorems hold. If the value X of the European option at date T is a function $X = f(S_T)$ dependent on the stock price S_T, then it holds that $V_t = F(t, S_t)$, where F for $x \in [0, \infty[$ and $t \in [0, T]$ is defined by:

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{+\infty} f\left\{ x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma y\sqrt{T-t}} \right\} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \ . \tag{A.6}$$

Proof:
With respect to Q, S_t contains the drift r and thus it holds that

$$S_t = S_0 \exp\{(r - \frac{\sigma^2}{2})t + \sigma W_t^*\}.$$

Thus S_T can be written in the following form:

$$S_T = S_t(S_T S_t^{-1}) = S_t \exp\{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T^* - W_t^*)\}.$$

Since S_t is measurable with respect to $\mathcal{F}_t$ and $W_T^* - W_t^*$ is independent of $\mathcal{F}_t$, one obtains

$$\begin{aligned} V_t &= \mathsf{E}_{\mathrm{Q}}[e^{-r(T-t)} f(S_T) \mid \mathcal{F}_t] \\ &= \mathsf{E}_{\mathrm{Q}}\left[e^{-r(T-t)} f(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W_T^*-W_t^*)}) \mid \mathcal{F}_t\right] \\ &= \mathsf{E}_{\mathrm{Q}}\left[e^{-r(T-t)} f(x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(W_T^*-W_t^*)})\right]_{x=S_t} \end{aligned}$$

From this it can be calculated that $V_t = F(t, S_t)$. □

Example A.4
Consider a European call $X = \max\{0, S_T - K\}$. Using (A.6) the value at date t is exactly the value given by the Black-Scholes formula in Chapter 6.

$$C(t, S_t) \stackrel{\text{def}}{=} V_t = S_T \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)$$

with

$$d_1 \stackrel{\text{def}}{=} \frac{\log(\frac{S_T}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 \stackrel{\text{def}}{=} \frac{\log(\frac{S_T}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

Frequently Used Notations

$x \stackrel{\mathrm{def}}{=} \ldots$ x is defined as ...
$\mathbb{R}$ real numbers
$\overline{\mathbb{R}} \stackrel{\mathrm{def}}{=} \mathbb{R} \cup \{\infty, \infty\}$
$A^\top$ transpose of matrix A
$X \sim D$ the random variable X has distribution D
$\mathsf{E}(X)$ expected value of random variable X
$\mathsf{Var}(X)$ variance of random variable X
$\mathsf{Cov}(X, Y)$ covariance of two random variables X and Y
$\mathrm{N}(\mu, \Sigma)$ normal distribution with expectation μ and covariance matrix Σ, a similar notation is used if Σ is the correlation matrix
Φ standard normal cummulative distribution function
φ standard normal density function
χ^2_p chi-squared distribution with p degrees of freedom
t_p t-distribution (Student's) with p degrees of freedom
W_t Wiener process
$\mathrm{P}[A]$ or $\mathrm{P}(A)$ probability of a set A
$\mathbf{1}$ indicator function
$[x]$ integer part of x
$(F \circ G)(x) \stackrel{\mathrm{def}}{=} F\{G(x)\}$ for functions F and G
$x \approx y$ x is approximately equal to y
$\alpha_n = \mathcal{O}(\beta_n)$ iff $\frac{\alpha_n}{\beta_n} \longrightarrow$ constant, as $n \longrightarrow \infty$
$\alpha_n = \mathcal{o}(\beta_n)$ iff $\frac{\alpha_n}{\beta_n} \longrightarrow 0$, as $n \longrightarrow \infty$
$\mathcal{F}_t$ is the information set generated by all information available at time t
Let A_n and B_n be sequences of random variables.
$A_n = \mathcal{O}_p(B_n)$ iff $\forall \varepsilon > 0\ \exists M,\ \exists N$ such that $\mathrm{P}[|A_n/B_n| > M] < \varepsilon$, $\forall n > N$.
$A_n = \mathcal{o}_p(B_n)$ iff $\forall \varepsilon > 0\ :\ \lim_{n\to\infty} \mathrm{P}[|A_n/B_n| > \varepsilon] = 0$.

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Index