# Concluding Remarks

Our goal in this book was to model counterparty credit exposure for all types of transactions. We saw that by appropriately choosing the fundamental quantities to model we can approach the problem in a modular way, dividing features and conquering products.

Price distributions are obtained using American Monte Carlo (AMC) techniques, allowing a valuation framework where modularity and flexibility are key. With the introduction of a booking language, PAL, we added a further layer of de-coupling and abstraction, enabling a system architecture that could address most of the problems faced by a counterparty exposure system dealing with large diverse portfolios.

The natural next step was to investigate how to manage counterparty exposure, both in static and dynamic ways. This led to the introduction of the so-called contingent credit default swap product, C-CDS, which replicates the cost of protection.

We now summarise the steps needed to compute and hedge credit exposure.

(i) *Translation.* First of all, all trades within the portfolio should be understood by the valuation engine. This means that each trade needs to be translated into the common trade representation language.

(ii) *Portfolio Valuation.* Once the first stage is completed, it is possible to model the underlying risk drivers, which have been recognised via the common trade representation language, and value each trade, along with its future price distributions. All trades are then aggregated together, including possible netting rules or break clauses, to finally arrive at the future distributions of the portfolio. If a collateral agreement exists, its logic should then be applied to the portfolio distributions.

(iii) *C-CDS Valuation.* The credit valuation adjustment, CVA, can be valued using the modified EPE profile of the portfolio and the counterparty credit spread curve. Using C-CDSs, however, we can compute not only the value of CVA, but also the CVA future price distribution.

(iv) *Sensitivities Computation and Replication.* As a final step, sensitivities can be computed from the C-CDS distribution, using either a regression-based approach, or a full revaluation (known in the industry as 'bumping' method), starting the process again from step (ii).

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(v) *Post-processing.* For purposes of risk control (e.g. to compute regulatory capital or compare PFE with limits), a post-processing of the price distribution may be needed. Examples of this include stress-testing and accounting for right-way/wrong-way risk.

The techniques we described can also be applied to other problems that large financial companies need to address. Examples are (i) computing the value of the so called *own credit* of a company, (ii) valuing *debt valuation adjustments* (DVA) of portfolios of transactions, (iii) addressing the problem of valuing the *cost of funding* and *cost of collateral*, (iv) computing potential values of transactions in different scenarios, (v) determining the value of *risk weighted assets* and of *regulatory capital*, or (vi) investigating various hedging strategies. All these problems deserve a thorough analysis which could be the subject of further research. It is interesting to note here that any solution to these questions will require, as fundamental feature, the capability of computing future distributions of prices. This is the feature at the heart of our work.

A final remark to conclude. What we described in this work is only a brief overview of the problem we try to solve. As we highlighted throughout this book, in many occasions we accepted compromises in our implementation and highlighted shortcuts. Many points can be improved, further explored and changed. We think, however, that at a general level, the framework and the ideas we provide are a viable solution to the modelling, pricing and hedging of counterparty credit exposure for large portfolios of different products.

# Appendix A
# Approximations

We summarise here some useful approximations of counterparty exposure computation, often used by practitioners. While they cannot provide satisfactory results in general, they may serve as a sanity check for more complex computations, and to help intuition. In some cases in the computation of Expected Positive Exposure (EPE) for some types of products, they are based on pricing information and give exact valuation. Some of the formulae we present are general and others can be used only for specific products. We consider here what we found useful in our day-to-day work.

## A.1 Maximum Likely Exposure

In general, the Potential Future Exposure profile (PFE) of a given product is a function of time. We call its maximum value Maximum Likely Exposure (MLE). In the following sections we provide some MLE estimate for simple products.

### *A.1.1 MLE of Equity and FX Products*

MLE values can be easily approximated in the case of options or forwards on assets that can be modelled as Geometric Brownian Motions assuming constant volatility and interest rate. Under these assumptions in fact the exposure profile reaches its maximum at maturity of the trade, where its value coincides with its intrinsic value. Thus, to compute the MLE, what is necessary is to estimate the potential value of the asset at maturity.

Consider for example an option on a stock $S$ with Black-Scholes volatility $\sigma$, interest rate $r$, and strike $K$. The maximum value of the exposure at maturity $T$ (within a 97.5% confidence interval) is given by

$$\mathrm{MLE} = S e^{(r-\frac{1}{2}\sigma^2)T + 1.96\sigma\sqrt{T}} - K. \tag{A.1}$$

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If we assume zero interest rate, stock returns normally distributed, and at the money products ($S = K$), we can simplify this formula as follows,

$$\text{MLE} = 1.96 S\sigma\sqrt{T}. \tag{A.2}$$

The main problem in these valuations is the choice of volatility. If the volatility is assumed to be constant, it is necessary to estimate the value that will best fit the terminal asset distribution. If the choice is to use implied volatilities, the at-the-money volatility is often the most suitable one to use. In practice if implied volatilities are not available historical volatilities are used.

### *A.1.2 MLE of Swaps*

Throughout this book we have seen several PFE profiles of interest-rate swaps. In general, when the product is vanilla, they show a typical bell shape, which starts from zero, increases over time and then decreases to reach zero again at maturity. This shape is driven by two factors, the declining duration (time to maturity) and the increasing variance of the swap. Assume that, at any time $t$, the duration is proportional to the remaining life of the swap via a constant $A_0 < 1$, and that the interest-rate volatility increases with the square root of time, $\sigma_N\sqrt{t}$.[1] We can write the volatility of the swap as

$$\text{Vol}_{Swap} = A_0(T-t)\sigma_N\sqrt{t}. \tag{A.3}$$

The peak exposure, i.e. the MLE, is reached at about one third of the life of the trade. We can see this by simply taking the first derivative of the volatility with respect to time, and imposing its value to be zero.

$$\frac{\partial \text{Vol}_{Swap}}{\partial t} = 0 \Longleftrightarrow -A_0 + A_0(T-t)\frac{1}{2t} = 0 \Longleftrightarrow t = \frac{T}{3}. \tag{A.4}$$

Using this result and assuming that the price distribution of an at-the-money swap is normally distributed, we can estimate the price distribution of a swap at time $T/3$,

$$\text{SwapDistribution}_{t=T/3} \approx A_0\frac{2}{3}T\sigma_N\sqrt{\frac{T}{3}}Z, \tag{A.5}$$

where $Z \sim N(0,1)$. If we want to value the MLE, i.e the peak PFE exposure at 97.5% confidence interval, we need to substitute $Z$ with 1.96. The present value of EPE can be computed by taking the expectation of the positive part of this distribution. Doing this we obtain

$$\text{EPE}_t^{PV} \approx \frac{1}{\sqrt{2\pi}}A_0(T-t)\sigma_N\sqrt{t} \approx 0.4A_0(T-t)\sigma_N\sqrt{t}. \tag{A.6}$$

[1] $\sigma_N$ is the volatility of a normal distribution. It is related to the log-normal (Black-Scholes) volatility $\sigma$ of the swap rate via the level of interest rate, $\sigma_N \approx r\sigma$.

## A.2 Expected Positive Exposure

The Expected Positive Exposure (EPE) computation is strongly related to pricing. In general, under pricing measure assumptions, the EPE of a transaction at time $t$ is the price of an option to enter in the transaction at time $t$. This is a very useful result, as it allows to approximate EPE computations using price information.

### *A.2.1 EPE and CVA of Equity Options*

As a first example consider an option on a stock or an FX currency. Under simplified assumptions, EPE can be written as

$$\mathrm{EPE}_t = \mathbb{E}[V_t^+], \tag{A.7}$$

where $V_t$ is the price distribution at time $t$. In the case where $V_t$ is always non-negative, as for example for options, this equation becomes

$$\mathrm{EPE}_t = \mathbb{E}[V_t^+] = \mathbb{E}[V_t] = \mathbb{E}[\mathbb{E}[e^{-r(T-t)}(S_T - K)^+|\mathscr{F}_t]] = V_0 e^{rt}, \tag{A.8}$$

where we have assumed constant interest rates and volatility. Thus,

$$\mathrm{EPE}_t = V_0 e^{rt}. \tag{A.9}$$

In other words the EPE of an option at time $t$ is the option premium increased at the risk-free rate.

The CVA can be computed as the discounted EPE multiplied by the spread (assumed to be constant) multiplied by time to maturity,

$$\mathrm{CVA} \approx V_0 s_0 T. \tag{A.10}$$

This formula holds for any product whose price distribution is non-negative and which does not pay intermediate cashflows. For example it can be used to compute CVA of a cash-settled swaption, while it cannot be applied in the case of a physically-settled swaption.

### *A.2.2 Relation between MLE, EPE*

If we assume zero interest rate we can approximate the price $V_0$ of an at-the-money $(S = K)$ option as,

$$V_0 = S\Phi(d_1) - K\Phi(d_2), \tag{A.11}$$

where $\Phi$ is the cumulative normal distribution, and

$$d_{1/2} = \frac{\ln(S/K)}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2} = \pm\frac{\sigma\sqrt{T}}{2}. \tag{A.12}$$

Using the following approximation

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x + O(x^3), \tag{A.13}$$

and assuming $\sigma\sqrt{T} \ll 1$ the above equation becomes,

$$V_0 = S\Phi\left(\frac{\sigma\sqrt{T}}{2}\right) - K\Phi\left(-\frac{\sigma\sqrt{T}}{2}\right) \approx \frac{1}{\sqrt{2\pi}}S\sigma\sqrt{T} \approx 0.4S\sigma\sqrt{T}. \tag{A.14}$$

We have seen that the EPE of an option can be computed as the option premium growing at risk free rate. Thus

$$\text{EPE} \approx 0.4\sigma S\sqrt{T}. \tag{A.15}$$

We can now compute a relation between EPE and the 97.5% MLE. Recall that,

$$\text{MLE} \approx 1.96\sigma S\sqrt{T}. \tag{A.16}$$

Thus, we obtain,

$$\frac{\text{EPE}}{\text{MLE}} \approx 0.2. \tag{A.17}$$

In other words, if the distribution of the portfolio is normal and centered around zero, then the 97.5% MLE is roughly five time larger than the EPE.

## A.3 CVA of Swaps

The EPE value at time $t$ of a swaps portfolio is often computed by practitioners as the value of a swaption, i.e. the value of an option to enter into a (portfolio of) swaps. This valuation is correct, however, only if the modified value of the EPE, as defined in Chaps. 12 and 14, is used. Often this valuation methodology is called *swaption approach*.

We can evaluate approximation of the CVA of a swap as follows.

$$\begin{aligned} \text{CVA}^{swap} &\approx \int_0^T \text{EPE}_u^{PV} s_0 du \approx s_0 \int_0^T 0.4A_0(T-u)\sigma_N\sqrt{u}du \\ &= s_0 0.4A_0\sigma_N T^{5/2}\frac{4}{15}, \end{aligned} \tag{A.18}$$

where $s_0$ is the CDS spread and $\text{EPE}^{PV}$ is the present value of the EPE. Recalling (A.5) we can approximate the peak value of the discounted EPE profile as

$$\text{EPE}^{PV}_{max} \approx 0.4 A_0 \sigma_N T^{3/2} \frac{2}{3\sqrt{3}}, \tag{A.19}$$

and thus,

$$\text{CVA} \approx s_0 T \frac{6\sqrt{3}}{15} \text{EPE}^{PV}_{max}. \tag{A.20}$$

Noting that the maximum value of the EPE profile of an at-the-money swap occurs at $t = T/3$ and using the 'swaption approach' we defined earlier, we get,

$$\text{CVA} \approx s_0 T \frac{2}{3} \text{Swaption}\left(\frac{T}{3}, T\right), \tag{A.21}$$

where we have approximated $6\sqrt{3}/15$ with $2/3$, and $\text{Swaption}(\frac{T}{3}, T)$ is the value of an option to enter at time $T/3$ into a swap of maturity $T$.

# Appendix B
# Results from Stochastic Calculus and Finance

This book is concerned with the pricing and hedging of risk borne by financial institutions when entering into transactions with other counterparties. Such risk arises from the random nature of the prices of products transacted as well as the possibility that the counterparty defaults, but its pricing and replication uses the same concepts as for other kinds of financial derivatives.

This appendix collects a few technical results that we will need throughout. We start by giving definitions for the basic stochastic processes we use, and then recall the concept of change of measure. We give also a brief overview of the fundamental theorem of asset pricing, which allows us to characterise the hedging portfolio for a traded derivative from martingale representation.

Derivation and analysis of these results can be found in standard finance books, such as Baxter & Rennie [10], Hunt & Kennedy [64], Karatzas & Shreve [68], Rogers & Williams [93, 94], Shreve [98], and Williams [106].

## B.1 Brownian Motion and Martingales

All our processes are defined relative to a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, where $(\mathscr{F}_t)_{t\geq 0}$ is a filtration in $\mathscr{F}$. The basic process we work with is Brownian Motion.

**Definition 1** A process $W \equiv (W_t)_{t\geq 0}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ is called Brownian Motion if

(i) $W_0(\omega) = 0$, for all paths $\omega \in \Omega$;
(ii) for each $\omega \in \Omega$, $W_t(\omega)$ is a continuous function of $t$;
(iii) for each $t, h \geq 0$, $W_{t+h} - W_t$ is independent of $W_t$, and has a Gaussian distribution with mean 0 and variance $h$.

Brownian Motion is an example of a martingale, the most important class of processes.

**Definition 2** A process $M$ is called a martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$ if

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(i) $M$ is adapted, that is $M_t$ is $\mathscr{F}_t$-measurable;
(ii) $\mathbb{E}[|M_t|] < \infty$;
(iii) if $s \leq t$, then $\mathbb{E}[M_t \mid \mathscr{F}_s] = M_s$.

$M$ is a *supermartingale* (resp. *submartingale*) if we replace equality in (iii) above by $\leq$ (resp. $\geq$). For proving general results, the class of martingales is not the right notion to work with, and one needs to consider *local martingales*. While all martingales are also local martingales, the converse is true only if certain conditions hold. The distinction will not be important for our purposes in this book.

At the heart of most of what we do is the idea of looking at various processes in a measure different from that of the given probability triple $(\Omega, \mathscr{F}, \mathbb{P})$. Indeed, if $Z$ is a non-negative random variable (that is, $\mathscr{F}$-measurable) then

$$\tilde{\mathbb{P}}(F) := \mathbb{E}[Z \mathbb{1}_F]/\mathbb{E}[Z], \quad F \in \mathscr{F} \tag{B.1}$$

defines a new probability measure $\tilde{\mathbb{P}}$ on $\mathscr{F}$ for which

$$\mathbb{P}[F] = 0 \Longrightarrow \tilde{\mathbb{P}}[F] = 0. \tag{B.2}$$

The last implication allows us to make the following definition:

**Definition 3** A probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathscr{F})$ is said to be *absolutely continuous* with respect to $\mathbb{P}$, denoted $\tilde{\mathbb{P}} \ll \mathbb{P}$, if for all $F \in \mathscr{F}$, (B.2) is true. If both $\tilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \tilde{\mathbb{P}}$ are true, then $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are said to be equivalent. In this case, $\mathbb{P}$ and $\tilde{\mathbb{P}}$ have the same sets of measure zero.

The converse to (B.1) is given by the Radon-Nikodym theorem.

**Theorem 1** *Let $\tilde{\mathbb{P}} \ll \mathbb{P}$ be a probability measure that is absolutely continuous with respect to $\mathbb{P}$. Then $\tilde{\mathbb{P}}$ can be characterised as in* (B.1) *for some non-negative random variable $Z$, which is then called the* Radon-Nikodym *derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write*

$$Z \equiv \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}. \tag{B.3}$$

The context in which we will most often see measure-change at work is when changing the drift of a Brownian Motion process. Given a $\mathbb{P}$-Brownian Motion $W$, if the process $\gamma \equiv (\gamma_t)_{t \geq 0}$ is such that

$$\zeta_t := \exp\left\{ \int_0^t \gamma_s dW_s - \frac{1}{2} \int_0^t \gamma_s^2 ds \right\} \tag{B.4}$$

is a martingale, then there exists a unique probability measure $\tilde{\mathbb{P}}$ such that

$$W_t - \int_0^t \gamma_s ds \tag{B.5}$$

is a $\tilde{\mathbb{P}}$-Brownian Motion. Moreover, the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ relative to $\mathbb{P}$ is given on every $\mathscr{F}_t$ by

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \zeta_t. \tag{B.6}$$

The above says that under $\tilde{\mathbb{P}}$, $W$ has a drift of $\gamma$. Equivalently,

$$\tilde{W} \text{ is a } \tilde{\mathbb{P}}\text{-martingale} \iff \zeta \tilde{W} \text{ is a } \mathbb{P}\text{-martingale}. \tag{B.7}$$

The change-of-measure technique is an indispensable device for simplifying calculations by removing from a process an unwanted drift term. We use it also to study price distributions under probability measures different to the ones in which they are simulated (see also Chap. 13).

## B.2 Replication of Contingent Claims: Martingale Representation

Consider an economy that puts at our disposal a number of assets $\mathbf{S}_t = (S_t^{(1)}, \ldots, S_t^{(n)})$, so that $S_t^{(i)}$ is the time-$t$ price of the $i$'th asset. There is a market for trading these assets. Thus, at any time $t$, a market participant, of wealth $V_t$ say, will have a proportion of wealth allocated to a portfolio $\boldsymbol{\theta}_t = (\theta_t^{(1)}, \ldots, \theta_t^{(n)})$, with the remainder held in some deposit account, so that

$$V_t = \varphi_t B_t + \boldsymbol{\theta}_t \cdot \mathbf{S}_t, \tag{B.8}$$

where $B_t$ is the value at $t$ of one unit invested in the deposit account at time zero, and $\varphi_t B_t$ is the wealth not invested in $\mathbf{S}$. Because any value kept in the deposit account grows at some positive rate, it is more useful to express asset prices in terms of $B$, writing $\tilde{V}_t \equiv B_t^{-1} V_t$, $\tilde{\mathbf{S}}_t \equiv B_t^{-1}\mathbf{S}_t$. The wealth equation (B.8) then becomes

$$\tilde{V}_t = \varphi_t + \boldsymbol{\theta}_t \cdot \tilde{\mathbf{S}}_t, \tag{B.9}$$

so that, as we expect, in any time interval where the holdings $\varphi$ and $\boldsymbol{\theta}$ are kept constant, the growth in discounted wealth $\tilde{V}$ derives only from growth in the discounted assets $\tilde{\mathbf{S}}$.

Of course, funds *may* be switched between the holdings in $\mathbf{S}$ and the deposit account, but it is natural to suppose that no *new* wealth can be injected, in which case the portfolio of holdings $(\varphi, \boldsymbol{\theta})$ is said to be *self-financing*. The consequence of $V$ being self-financing is then that

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \boldsymbol{\theta}_u \cdot d\tilde{\mathbf{S}}_u, \tag{B.10}$$

so that the *discounted wealth is the integral of the portfolio holdings against the discounted asset price process.*

The fundamental theorem of asset pricing, formalised by Harrison & Kreps [57] and Harrison & Pliska [58], and formulated in more general setting in the work of Delbaen & Schachermayer (for example, [34] and [35]), states that arbitrage is excluded if and only if there is some *equivalent martingale measure* under which discounted asset price processes are martingales. This implies that the price of a contingent claim can be computed as the expectation in the martingale measure of the discounted payoff of that claim. If the market is also *complete*, so that all claims can be replicated perfectly,[1] then the martingale measure (and hence the market price for any claim) is unique.

Now if $\tilde{\mathbb{P}}$ is a measure under which $\tilde{\mathbf{S}}$ is a martingale, and $Y = f(\tilde{\mathbf{S}}_T)$ is a contingent claim on $\tilde{\mathbf{S}}$, the discounted time-$t$ price of $Y$, $\tilde{\pi}_{t,T}$, say, being the price of a traded asset, is itself a $\tilde{\mathbb{P}}$ martingale. It follows that $\tilde{\pi}$ has a representation as

$$\tilde{\pi}_{t,T} = B_t^{-1}\pi_{t,T} = \tilde{\mathbb{E}}[B_T^{-1}Y|\mathscr{F}_t]. \tag{B.11}$$

In the absence of any other condition enforcing a unique price for the claim $Y$, there will be potentially as many prices $\tilde{\pi}$ for $Y$ as there are market agents, each price reflecting that agent's own risk aversion. If the market is *complete*, however, there *is* a price-enforcing mechanism: the price of $Y$ will be the cost $V_0^Y$ of setting up a portfolio worth

$$V^Y(0) = \varphi_0^Y + \boldsymbol{\theta}_0 \cdot \mathbf{S}_0 \tag{B.12}$$

at time zero and

$$V^Y(T) = Y \tag{B.13}$$

at time $T$.

The existence of a unique process $\theta^Y$ that makes the wealth equation (B.10) true is a consequence of the martingale property of the price processes $\tilde{\pi}_{t,T} = \tilde{V}^Y(t)$ and $\mathbf{S}_t$ and the *martingale representation theorem* (see Rogers & Williams [94]).

**Theorem 2** *Let $X$ be a local martingale on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, and assume that $(\mathscr{F}_t)$ is the filtration generated by $X$. Then, any local martingale $M$ adapted to $(\mathscr{F}_t)$ has a representation as*

$$M_t = M_0 + \int_0^t H_u dX_u \tag{B.14}$$

*where $H$ is previsible with respect to $(\mathscr{F}_t)$. Moreover, $H$ is unique up to sets of measure zero.*

Because the claim price process $\tilde{\pi}_{t,T}$ and the asset price process $\mathbf{S}$ are both $\tilde{\mathbb{P}}$-martingales, the martingale representation theorem shows the existence of a strategy with which to *hedge* the claim $Y$ by trading in the assets $\mathbf{S}$.

[1] By this we mean that for every time-$T$ claim $Y$ one can find a portfolio $\tilde{V}_t^Y = \tilde{V}_0 + \int_0^t \boldsymbol{\theta}_u \cdot d\tilde{\mathbf{S}}_u$ such that $V_T = Y$.

## B.3 Change of Numeraire

In writing the wealth equation (B.10) we defined $\tilde{\mathbf{S}}_t \equiv B_t^{-1}\mathbf{S}_t$ and $\tilde{V} \equiv B_t^{-1}V_t$ by expressing the prices of assets and the wealth $V$ in units of the deposit account. One says that the deposit account is being used as *numeraire*.

There is nothing that keeps us from using as numeraire the value of a different asset, and in fact changing numeraire is a powerful modelling and computational technique. Geman and Jamshidian were the first to employ this idea. Suppose $X$ is the price of any traded asset (scaled by its time-zero value); for reasons that will soon become obvious, we need to assume $X_t > 0$ for each $t$. Then, because by definition of $\tilde{\mathbb{P}}$ all discounted assets are $\tilde{\mathbb{P}}$-martingales, we have that

$$\frac{X_t}{B_t} \text{ is a } \tilde{\mathbb{P}}\text{-martingale.} \tag{B.15}$$

This allows us to define a new measure, $\mathbb{P}^X$ say, whose Radon-Nikodym derivative is given for every $t$ by

$$\zeta_t = \frac{X_t}{B_t} = \left.\frac{d\mathbb{P}^X}{d\tilde{\mathbb{P}}}\right|_{\mathscr{F}_t}. \tag{B.16}$$

It then follows, for any given process $M$, that

$$B_t^{-1}M_t \text{ is a } \tilde{\mathbb{P}}\text{-martingale} \iff X_t^{-1}M_t \text{ is a } \mathbb{P}^X\text{-martingale,} \tag{B.17}$$

so for any claim $Y$ maturing at $T$ we can write the equivalent expressions

$$\pi_{t,T} = \tilde{\mathbb{E}}\left[\frac{B^{-1}(T)}{B^{-1}(t)}Y\middle|\mathscr{F}_t\right] = \mathbb{E}^X\left[\frac{X^{-1}(T)}{X^{-1}(t)}Y\middle|\mathscr{F}_t\right], \tag{B.18}$$

where the first expectation happens under $\tilde{\mathbb{P}}$ and the second under $\mathbb{P}^X$. For example, if one takes for $X$ the price process of the bond maturing at time $T$, the price of any claim $Y$ received at $T$ is

$$\pi_{t,T} = \tilde{\mathbb{E}}\left[\frac{B^{-1}(T)}{B^{-1}(t)}Y\middle|\mathscr{F}_t\right] = D_{t,T}\mathbb{E}^T\left[Y\middle|\mathscr{F}_t\right], \tag{B.19}$$

where $D_{t,T}$ is the observed time-$t$ price of the $T$-bond, so that $D_{T,T} \equiv 1$, and where the expectation is now in the $T$-forward measure in which asset prices discounted by the $T$-bond are martingales. The price of $Y$ can now be computed as the expectation of $Y$ in the $T$-forward measure.

An in-depth account of martingale theory and stochastic processes, which we have used here, is Rogers & Williams [94]. Our description of self-financing portfolios closely follows the article of Rogers [92], which shows how ideas of economic equilibrium lead directly to the existence of equivalent martingale measures.

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