Bibliographical Notes

The history of theoretical stochastic processes can be found in books on theoretical probability or stochastic processes. Extensive notes on this are in [46, 61], and miscellaneous references are in the miscellaneous books [10, 18, 26, 37, 38, 41, 42, 62, 63, 84, 96, 102, 106, 114].

The history of applied stochastic processes parallels that of theoretical work. The developments in applied stochastic processes stem from advances in theoretical probability as well as from problems that cry out for solutions. Standard texts on applied stochastic processes are [10, 27, 42, 46, 62, 63, 76, 78, 93, 94, 98], and those with a more specialized focus are [2, 54, 59, 109, 116]. Much of the material on actual applications has been presented in technical reports or in specialized journals that is not conducive to a unified review. So I will only comment on some of the main themes and representative references while discussing the chapters.

Most of Chapter 1 on Markov chains is standard. Exceptions are the reflected random walk (a framework for several models), which is a discrete-time version of the Skorohod equation for reflected Brownian motion [64], and the network models are discrete-time versions of the continuous-time Jackson network models in Chapter 4. Further background and more intricate branching process models are discussed in [49, 58, 62, 63]. References on general Markov processes are the early work [75] and the more current books [38, 37, 41, 42, 61, 84, 96, 106]. Phase-type distributions and computational algorithms, which were not covered, are discussed in [85].

Chapter 2 on Renewal and Regenerative Processes is a fairly thorough coverage of the literature in the 1950s and 1960s, including the works [3, 42, 81, 107] and ending with [69] (only a few articles have appeared after this one). Regenerative processes were popularized with the work of Smith [107]; many applications continue to be based on analyzing processes at embedded times that may or may not be regeneration times. The examples here and in other chapters on reliability and maintenance, production systems, and insurance are indicative of those fields; see for instance [4, 5, 8] and [22, 48, 94].

The richness of the applications of Poisson processes led me to devote an entire chapter to it. Chapter 3 covers classical Poisson processes, but goes further to show the variety of Poisson processes in space and time and their transformations that yield tractable models for systems. Standard references on point processes are the early work of Poisson [87] and later works are [16, 30, 60, 66]; further examples are in [6, 40, 44, 73, 101, 103]. The section on batch-service queueing systems describes a classic Markov decision model [33] (or stochastic dynamic programming model). Dynamic programming is described in [77, 90, 109]. The Markov/Poisson particle model is an elementary example of independent particles [34]; interacting particle systems are discussed in [80]. The Grigelionis theorem [45] showing that Poisson processes are natural limits for sparse point processes, is analogous to the central limit theory for summation processes converging to Brownian motion covered in Chapter 5.

Queueing processes, input-output systems, and stochastic networks have been a main part of applied stochastic processes. This is reflected in Chapter 4 on Continuous-Time Markov Chains. In addition to covering the basics of CTMCs, the chapter provides numerous queueing and network models. Sample references on early work on queueing processes are [20, 28, 40, 44, 57, 68, 70, 71, 72, 82, 91], books on queueing are [2, 6, 12, 13, 15, 28, 29, 39, 47, 59, 74, 88, 108, 116], and works on stochastic networks are [7, 9, 23, 25, 67, 92, 95, 99, 101, 110, 111]. Palm probabilities, that began with Palm [86], are introduced to describe certain PASTA (Poisson actions see time averages) properties [116] of queues and other processes.

The final chapter on Brownian motion covers many of its properties, without getting into more advanced stochastic integration. Books that discuss Brownian motion include [10, 18, 46, 61, 64]. Several key works on Donsker's functional central limit theorem [35, 36] and based on ideas of Prohorov and Skorohod, followed by Billingsley, are [11, 35, 36, 104, 105]. Major applications in queueing and elsewhere are nicely reviewed in Whitt [55, 113]; also see the early work of Kingman 1961 and Iglehart and Whitt 1970 on heavy-traffic queues. During the last three decades, there has been a substantial stream of articles on heavy-traffic systems, Brownian approximations and fluid queues. Much of this research has been done by Harrison and his colleagues Bramson, Chen, Dai, Mandelbaum, Reiman, and Williams [14, 16, 24, 25, 30, 31, 50, 51, 52, 77, 83, 92, 112].

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Notation

$\mathbf{1}(\cdot)$	The indicator function that is 1 or 0 when $(\cdot)$ is true or false
$A(t) = t - T_{N(t)}$	Backward recurrence time
$B(t) = T_{N(t)+1} - t$	Forward recurrence time
$B,\ B(t)$	Standard Brownian motion process
$C_K^+(S)$	Set of continuous $f : S \to \mathbb{R}_+$ with compact support
$\delta_x(A) = \mathbf{1}(x \in A)$	Dirac measure with unit mass at 1
$\mathbb{D}$	Set of right-continuous, piece-wise constant functions $x : \mathbb{R} \to S$ with left-hand limits, and finite number of jumps in finite intervals
$D = D[0,1]$, $D[0,T]$, $D(\mathbb{R})$	Set of real valued functions on $[0,1]$, $[0,T]$, $\mathbb{R}$ that are right continuous with left-hand limits
DRI	Directly Riemann integrable
E	Expectation operator
$E[X\|Y],\ E[X\|A]$	Conditional expectations
$E[e^{sX}]$	Moment generating function of X
$\hat{F}(\alpha) = \int_{\mathbb{R}_+} e^{-\alpha t} dF(t)$	Laplace transform of F
$e_i = (0, \dots, 0, 1, 0 \dots, 0)$	ith unit vector
$\mathcal{F},\ \mathcal{F}_t,\ \mathcal{F}_n$	σ-field of events
$\mathcal{F}_\tau$	σ-field of events up to stopping time τ
$\mathcal{F}_t^Y$	σ-field of events generated by process Y
$f(t),\ g(t),\ h(t),\ H(t)$	Functions
$f(t) = f^+(t) - f^-(t)$	f equals its positive part minus its negative part
$F(t),\ G(t)$	Distribution functions
$F_e(x) = \frac{1}{\mu}\int_0^x [1 - F(s)]ds$	Equilibrium distribution of F
γ_i	Invariant measure of a CTMC
$H(t) = h(t) + F \star H(t)$	Renewal equation
$\lambda,\ \lambda(A)$	Arrival rate, and rate of entering A
$\mu,\ \mu(A)$	Service rate or mean, and measure of A
$M(t),\ M(A \times B)$	Counting process, or Poisson process

$M(t) = \max_{s \le t} B(s)$	Maxima of Brownian motion B
$N(t)$	Counting process, or renewal process
$N_{\mathcal{T}}$	Point process of $\mathcal{T}$-transitions of a CTMC
$N(\mu, \sigma^2)$	Normal random variable with mean μ and variance σ^2
$\mathrm{P} = (p_{ij})$	Matrix of Markov chain transition probabilities
Random element X in S	X is measurable map from a probability space to S
$P_{\mathcal{T}}(\cdot)$	Palm probability of a $\mathcal{T}$-transition of a CTMC
$p_{ij}(t)$	Transition probability of a CTMC
$p_i,\ p(x)$	Stationary distributions of a CTMC
q_i	Exponential sojourn rate in state i of a CTMC
$q_{ij},\ q(x,y)$	Transition rates of a CTMC
$Q = (q_{ij})$	Transition rate matrix of a CTMC
$\mathbb{R},\ \mathbb{R}_+$	The real numbers and nonnegative real numbers
S	A countable state space for Markov chains in Chapter 1
$S,\ \mathcal{S},\ \hat{\mathcal{S}}$	Polish space and its Borel sets and bounded Borel sets
$\tau,\ \tau_i,\ \tau_i(n)$	Stopping time, and entrance times to state i
$\mathcal{T}$	Subset of sample paths in D
$\mathcal{T}$-transition	A jump time t of a CTMC X with $S^t X \in \mathcal{T}$
T_n	Time of nth event occurrence, or nth renewal time, or time of nth jump in a CTMC
$U(t) = \sum_{n=0}^{\infty} F^{n\star}(t)$	Renewal function
$X_n,\ Y_n$	Markov chains or sequences of random variables
$X(t),\ Y(t),\ Z(t)$	Continuous-time stochastic processes
$\xi_n = T_n - T_{n-1}$	nth inter-renewal time, or time between event occurrences
$\xi_n = T_{n+1} - T_n$	Sojourn time in a CTMC
$W(t),\ W(A)$	Waiting time process or sojourn time in set A
$\mathbb{Z},\ \mathbb{Z}_+$	The integers and nonnegative integers

a.s.	Almost surely, meaning with probability one
$x \vee y = \max\{x, y\}$	Maximum of x and y
$x \wedge y = \min\{x, y\}$	Minimum of x and y
$X(t) \xrightarrow{d} Y$	$X(t)$ converges in distribution to Y as $t \to \infty$
$X \overset{d}{=} Y$	The distributions of X and Y are equal
$x^+ = \max\{0, x\}$	Positive part of x
$x^- = -\min\{0, x\}$	Negative part of x and $x = x^+ - x^-$
$\sum_{n=1}^{N(t)}(\cdot) = 0$	When $N(t) = 0$
$\prod_{k=1}^{x}(\cdot) = 1$	When $x = 0$
$\lfloor x \rfloor$	Largest integer $\le x$, or the integer part of x
$\lceil x \rceil$	Smallest integer $\ge x$
$f(t) = o(g(t))$ as $t \to t_0$	$\lim_{t \to t_0} f(t)/g(t) = 0$
$f(t) = O(g(t))$ as $t \to t_0$	$\limsup_{t \to t_0} \lvert f(t)\rvert / \lvert g(t)\rvert < \infty$
$a \Rightarrow b,\ a \Leftrightarrow a$	a implies b, and a is equivalent to b

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