Epilogue

In this monograph, we have reviewed the concept of chiral soliton models for baryons. In these models, the baryons emerge as (topological) defects of the chiral field. The elementary starting point is a chiral Lagrangian that fully contains the dynamics of the chiral field and/or other fields that parameterize meson degrees of freedom. Once it is set, in the sense that all pieces in the chiral Lagrangian are established, no further assumptions about the interaction are required. In particular, no additional information must be supplemented from outside, and any question on low-energy baryon properties or resonances has, in principle, a definite answer within the model. Even though reaching testable predictions may still involve complicated and lengthy computations and eventually simplifying approximations in a number of cases, this feature of straightforwardness nicely distinguishes the soliton picture from many other baryon models. A particularly fascinating feature is the fact that the soliton must indeed be quantized as a fermion. This result strictly emerged from reproducing the symmetries of QCD in the effective model.

We have encountered many applications of the soliton description. Starting from the baryon spectrum they reach from static baryon properties via nucleon resonances and deep inelastic scattering to even heavy ion collisions. Though we have only discussed some specific exemplary studies, which we will not itemize here, the reader may have recognized and appreciated the vast range of successful activities.

As physicists, we have the common interest of confronting theory with experiment. We do not expect the soliton picture to produce highly accurate results that survive comparison with data on the few percent level, or even better. After all in reality, $1/N_{\rm C}$, which is the rough expansion parameter within the soliton picture, is not small. Yet we certainly gain reliable qualitative insight into the physics of baryons. A particular example is the smallness of the singlet axial current matrix element which is nicely explained in the soliton picture of the nucleon. Simultaneously, we stress that this picture in principle represents a parameter-free approach once the chiral Lagrangian is set to correctly describe meson properties.

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In this monograph, we have discussed numerous successful applications of the soliton picture for baryons. Nevertheless, challenging problems remain that should be subjected to future investigations. Here is a short list containing those that are of general interest:

- We have seen that the nucleon–nucleon potential allows us to extract the pion–nucleon coupling $g_{\pi NN}$ for the boson exchange model stemming from Yukawa interactions. On the other hand, the soliton predictions for meson baryon scattering cannot be associated with simple Yukawa interactions. So, why can we map one sector of the model on the Yukawa theory but not the other?
- Certainly, we want to improve our understanding of quantum corrections to baryon properties, in particular the spectrum. After all, they contribute already at the next to leading order in the $1/N_{\rm C}$ expansion. This is also desirable to obtain reliable information about multi-baryon systems and eventually nuclei. In particular, it concerns their binding energies.
- We have discussed the computation of nucleon structure functions in the NJL chiral quark model. In that model, the Callan–Gross relation resulted naturally. This relation reflects the fermionic nature of the nucleon. After the gradient expansion, this information is contained in the Wess–Zumino term. Hence, it should also be possible to extract this relation from there.

Additionally, it is very likely that in future the soliton picture will find further specific applications so that the soliton picture prospectively promises to further enrich our understanding of the structure of baryons and their dynamics in the low and intermediate energy regimes.

In conclusion, the author hopes that the presented material not only comprehensively explains the soliton picture for baryons and leads to further insight but also initiates further progress on the subject.

Appendix A

Chiral Properties of Quark Bilinears

In this appendix we briefly summarize the behavior of quark bilinears under infinitesimal flavor and chiral transformations. These results can be employed to verify the invariance of the Lagrangian, (2.8), under these transformations.

Under flavor rotations the quark spinors as defined in (2.4) transform as

$$q \longrightarrow q' = \mathrm{e}^{(\mathrm{i}/2)\epsilon_a \lambda_a}\, q\,, \quad \bar{q} \longrightarrow \bar{q}' = \bar{q}\, \mathrm{e}^{-(\mathrm{i}/2)\epsilon_a \lambda_a}\,, \tag{A.1}$$

where $\bar{q} = q^\dagger \gamma_0$ and ϵ_a with $a = 0, \ldots, N_\mathrm{f}$ being the (infinitesimal) parameters that characterize the transformation. The Gell–Mann matrices λ_a are defined in Chap. 2. The sum $a = 1, \ldots, N_\mathrm{f}$ is implied in the notation of (A.1). Chiral rotations involve γ_5 in addition,

$$q \longrightarrow q' = \mathrm{e}^{(\mathrm{i}/2)\eta_a \lambda_a \gamma_5}\, q\,, \quad \bar{q} \longrightarrow \bar{q}' = \bar{q}\, \mathrm{e}^{(\mathrm{i}/2)\eta_a \lambda_a \gamma_5}\,, \tag{A.2}$$

with some different (infinitesimal) parameters η_a. The difference in the signs of the exponentials multiplying $\bar{q}$ in (A.1) and (A.2) arise from the anti-commutator $\gamma_0\gamma_5 + \gamma_5\gamma_0 = 0$. Note that under this transformation the chirality of a spinor is conserved, i.e., the properties $(1 \pm \gamma_5)\, q = 0$ are not affected. Transformations that act on purely left- and right-handed spinors as defined in (2.1) are characterized by $\epsilon_a = \eta_a$ and $\epsilon_a = -\eta_a$, respectively.

To summarize the transformation properties of quark bilinears under infinitesimal flavor and chiral rotations we first define the symmetric (d_{abc}) and anti-symmetric (f_{abc}) structure constants of the generators introduced in Chap. 2:

$$\lambda_a \lambda_b = \frac{2}{N_\mathrm{f}} \delta_{ab} + d_{abc}\lambda_c + \mathrm{i} f_{abc}\lambda_c\,. \tag{A.3}$$

This also includes the unit matrix in form of $\lambda_0 = \sqrt{2/N_\mathrm{f}}\, \mathbb{1}$. Though this is a notational convenience it brings into the game the complication that those

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components of d_{abc} with one or more of the subscripts being zero are not totally symmetric; using $i,j=1,\dots,N_\mathrm{f}^2-1$ we rather have

$$d_{000}=d_{00i}=d_{0i0}=0 \quad d_{0ij}=d_{i0j}=\sqrt{\frac{2}{N_\mathrm{f}}}\,\delta_{ij} \quad \text{but} \quad d_{ij0}=0\,. \tag{A.4}$$

In case none of the indices is zero, the d_{abc} are the well-known totally symmetric structure constants of $SU(N_\mathrm{f})$. The transformation properties are finally listed in Table A.1. Actually some of the flavor singlet ($a=0$) bilinears are invariant since $f_{0bc}=0$. With these transformation properties at hand it is straightforward to confirm that the combinations in (2.8) are indeed invariant under flavor and chiral rotations. As an example we consider the chiral transformation for the term multiplying G_1:

$$\begin{aligned}\Delta\sum_{a=0}^{N_\mathrm{f}^2-1}\left((\bar q\lambda_a q)^2-(\bar q\lambda_a\gamma_5 q)^2\right) &= \frac{4\mathrm{i}}{N_\mathrm{f}}\sum_{i=1}^{N_\mathrm{f}^2-1}\eta_i\left[\bar q\lambda_i q\,\bar q\gamma_5 q-\bar q\lambda_i\gamma_5 q\,\bar q q\right]\\ &\quad +2\,\mathrm{i}\sum_{\substack{a,b=0\\ i=1}}^{N_\mathrm{f}^2-1} d_{aib}\eta_i\left[\bar q\lambda_a q\,\bar q\lambda_b\gamma_5 q-\bar q\lambda_a\gamma_5 q\,\bar q\lambda_b q\right].\end{aligned} \tag{A.5}$$

Naïvely the symmetry of d_{abc} suggests the last term to vanish. As already mentioned in (A.4), the symmetry does not hold in case one or more indices are zero and we have to discuss the case $a=0$ separately,

$$\begin{aligned}\Delta\sum_{a=0}^{N_\mathrm{f}^2-1}\left((\bar q\lambda_a q)^2-(\bar q\lambda_a\gamma_5 q)^2\right) &= \frac{4\mathrm{i}}{N_\mathrm{f}}\sum_{i=1}^{N_\mathrm{f}^2-1}\eta_i\left[\bar q\lambda_i q\,\bar q\gamma_5 q-\bar q\lambda_i\gamma_5 q\,\bar q q\right]\\ &\quad +2\,\mathrm{i}\sqrt{\frac{2}{N_\mathrm{f}}}\sum_{\substack{b=0\\ i=1}}^{N_\mathrm{f}^2-1} d_{0ib}\eta_i\left[\bar q q\,\bar q\lambda_b\gamma_5 q-\bar q\gamma_5 q\,\bar q\lambda_b q\right].\end{aligned} \tag{A.6}$$

Table A.1. Infinitesimal variations of selected quark bilinears under flavor and chiral rotations. According to the summation convention in (A.1) and (A.2) we have $\epsilon_0=\eta_0=0$. In [1] a list may be found that does not utilize this condensed form including the $a=0$ components

quark bilinear	flavor rot. (A.1)	chiral rot. (A.2)
$\Delta\bar q\lambda_a q$	$-f_{abc}\epsilon_b\bar q\lambda_c q$	$\mathrm{i}\,\bar q\gamma_5\left(\frac{2}{N_\mathrm{f}}\eta_a+d_{abc}\eta_b\lambda_c\right)q$
$\Delta\bar q\gamma_5\lambda_a q$	$-f_{abc}\epsilon_b\bar q\gamma_5\lambda_c q$	$\mathrm{i}\,\bar q\left(\frac{2}{N_\mathrm{f}}\eta_b+d_{abc}\eta_b\lambda_c\right)q$
$\Delta\bar q\gamma_\mu\lambda_a q$	$-f_{abc}\epsilon_b\bar q\gamma_\mu\lambda_c q$	$-f_{abc}\eta_b\bar q\gamma_\mu\gamma_5\lambda_c q$
$\Delta\bar q\gamma_\mu\gamma_5\lambda_a q$	$-f_{abc}\epsilon_b\bar q\gamma_\mu\gamma_5\lambda_c q$	$-f_{abc}\eta_b\bar q\gamma_\mu\lambda_c q$

Inserting (A.4) for d_{0ib} finally shows that the right hand side vanishes. That is, the considered combination of quark bilinears is indeed invariant under chiral rotations.

Reference

1. G. Ripka, *Quarks Bound to Chiral Fields*. Clarendon Press, Oxford, 1997. 234

Appendix B

Functional Techniques

In field theory, we often encounter quadratic actions (cf. Chaps. 2 and 3)

$$S[\phi] = \int \mathrm{d}^4x\, \phi(x)\hat{O}\phi(x) \tag{B.1}$$

for some linear operator $\hat{O}$ in path integrals alike (2.11),

$$Z = \int [D\phi]\, \mathrm{e}^{-\mathrm{i}S[\phi]}\,. \tag{B.2}$$

Formally, we expand ϕ in terms of eigenfunctions of $\hat{O}$,

$$\phi(x) = \sum_n a_n \phi_n(x) \qquad \text{with} \qquad \hat{O}\phi_n(x) = \lambda_n \phi_n(x)\,. \tag{B.3}$$

Orthogonality of the ϕ_n and Gaußian integration for a_n yield

$$\begin{aligned} Z &\propto \left[\prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}a_n\right] \exp\left[-\mathrm{i}\int \mathrm{d}^4x \sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} a_k\phi_l(x)a_{k'}\lambda_{k'}\phi_{k'}(x)\right] \propto \prod_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} \\ &= N\left[\mathrm{Det}(\hat{O})\right]^{-\frac{1}{2}}\,. \end{aligned} \tag{B.4}$$

The normalization constant N is irrelevant to the logarithm

$$\ln\left(\frac{Z}{N}\right) = \ln\left[\mathrm{Det}(\hat{O})\right]^{-\frac{1}{2}} = -\frac{1}{2}\,\mathrm{Tr}\ln\left[\hat{O}\right]\,. \tag{B.5}$$

Commonly $\hat{O} = \hat{O}_0 + \hat{O}_{\mathrm{int}}$ where the second term contains the dynamics. Then the formal expansion

$$\begin{aligned} \ln\left[\hat{O}\right] &= \ln\left[\hat{O}_0\right] + \ln\left[1 + \hat{O}_0^{-1}\hat{O}_{\mathrm{int}}\right] \\ &= \ln\left[\hat{O}_0\right] + \hat{O}_0^{-1}\hat{O}_{\mathrm{int}} - \frac{1}{2}\hat{O}_0^{-1}\hat{O}_{\mathrm{int}}\hat{O}_0^{-1}\hat{O}_{\mathrm{int}} + \cdots \end{aligned} \tag{B.6}$$

defines the Feynman series. So far ϕ has been a real boson field. For complex boson fields, the integration space is twice as large, hence the factor $\frac{1}{2}$ on the right hand side of (B.5) is dropped in that case.

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In case of fermions, the a_n are anti-commuting Grassman variables,

$$a_n a_m = -a_m a_n \,, \tag{B.7}$$

which implies $a_n^2 = 0$. The basic integration rules

$$\int \mathrm{d}a_n = 0 \qquad \text{and} \qquad \int \mathrm{d}a_n a_m = \delta_{nm} \tag{B.8}$$

follow from translational invariance of the measure. The eventual normalization on the right hand is set to unity. To evaluate the path integral

$$Z[M] = \left[\prod_{n=1}^{\infty} \int \mathrm{d}\bar{a}_n\right] \left[\prod_{m=1}^{\infty} \int \mathrm{d}a_m\right] \exp\left[\bar{a} \cdot M \cdot a\right] \tag{B.9}$$

for the c-number valued matrix M, we expand the exponential function. The rules in (B.7) and (B.8) enforce that any matrix element M_{nm} appears exactly once. In addition the result must be totally anti-symmetric. Therefore,

$$Z[M] \propto \mathrm{Det}(M) = \exp\left\{\mathrm{Tr}\,\left[\ln(M)\right]\right\}, \tag{B.10}$$

which can easily be verified for a low-dimensional matrix M. In field theory, the above result turns into expressions alike (2.14).

The basic identification for the computation of the above-encountered functional traces is

$$\mathrm{Tr}\{\ldots\} \longrightarrow \mathrm{tr} \int \mathrm{d}^{\bar{4}}x \, \langle x|\{\ldots\}|x\rangle \,. \tag{B.11}$$

Here "tr" involves discrete indices only and $|x\rangle$ is an eigenstate of the position operator, i.e., $\hat{x}|x\rangle = x|x\rangle$. In this space, the unit operator reads

$$\mathbb{1} = \int \mathrm{d}^4x \, |x\rangle\langle x| = \int \frac{\mathrm{d}^4k}{(2\pi)^4} |k\rangle\langle k| \,, \tag{B.12}$$

with $|k\rangle$ being momentum state conjugate to $|x\rangle$: $\langle k|x\rangle = \mathrm{e}^{\mathrm{i}k\cdot x}$. Matrix elements of local functions are diagonal:

$$\langle x|\phi(\hat{x})|y\rangle = \phi(x)\delta^4(x-y) \,. \tag{B.13}$$

In this functional language, the Fourier transform of a field has the compact notation

$$\widetilde{\phi}(q-k) = \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}(q-k)}\phi(x) = \langle q|\phi(\hat{x})|k\rangle, \tag{B.14}$$

with $|k\rangle$ and $|q\rangle$ eigenstates of the momentum operator. Then the gap equation that has been introduced in Chap. 2 is computed via

$$\frac{\delta}{\delta M_{ij}(y)}\mathrm{Tr}\log\left(\mathrm{i}\partial\!\!\!/ - M\right) = N_\mathrm{C}\mathrm{tr}_\mathrm{D}\langle y|\left(\mathrm{i}\partial\!\!\!/ - M\right)^{-1}_{ij}|y\rangle$$
$$= 4N_\mathrm{C}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left(k^2 - MM^\dagger\right)^{-1}_{ij} . \qquad \text{(B.15)}$$

Note that right hand side of the first equation defines the quark condensate, $\langle\bar{q}(x)q(x)\rangle$. The trace in that equation only concerns the Dirac indices of the γ-matrices. For the regularized action, (2.17), one finds under the assumption that the solution to the gap equation is diagonal in flavor space

$$4N_\mathrm{C}\delta_{ij}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_{1/\Lambda^2}^{\infty}\mathrm{d}s\,\mathrm{e}^{-s\left(k^2+m_i^2\right)} = \frac{N_\mathrm{C}\delta_{ij}}{\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-sm_i^2} , \qquad \text{(B.16)}$$

since after Wick rotation the momentum integral is merely Gaußian. The result, (2.18), can be readily obtained from the definition of the incomplete Γ-function

$$\Gamma(u,x) = \int_x^\infty \mathrm{d}\tau\tau^{u-1}\mathrm{e}^{-\tau} , \qquad \text{(B.17)}$$

especially

$$\Gamma(0,x) = -\log x + \gamma + \mathcal{O}(x) \quad \text{for} \quad x\to 0^+ \qquad (\gamma = 0.57721\cdots) \qquad \text{(B.18)}$$

motivates the regularization prescription (2.17). Other Γ-functions are obtained from the recursion relation

$$\Gamma(a+1,x) = a\Gamma(a,x) + x^a\mathrm{e}^{-x} . \qquad \text{(B.19)}$$

In Chap. 3, Γ-functions with half-integer index occur. The above recursion relates them to the complementary error function

$$\Gamma(\tfrac{1}{2},x^2) = \sqrt{\pi}\,\mathrm{erfc}(|x|) . \qquad \text{(B.20)}$$

Next we expand the regularized real part of the Euclidean action up to quadratic order in the pseudoscalar meson fields ϕ_a, cf. (2.20), starting from

$$D\!\!\!\!/_\mathrm{E}^\dagger D\!\!\!\!/_\mathrm{E} = A_0 + A_1 + A_2 + \cdots . \qquad \text{(B.21)}$$

The subscripts indicate the order at which ϕ_a appears. Explicitly we have

$$A_0 = \partial^2 + m^2 , \quad A_1 = m\gamma_5\left[\partial\!\!\!/ \sum_a \phi_a\lambda_a\right] \quad \text{and} \quad A_2 = 0 , \qquad \text{(B.22)}$$

where m is the constituent quark mass, recall that flavor symmetry is assumed. Obviously, only derivatives of ϕ occur. This is a consequence of the chiral invariance of $\mathrm{Det}D\!\!\!\!/_\mathrm{E}^\dagger D\!\!\!\!/_\mathrm{E}$. The formal series

$$\begin{aligned}\mathcal{A}_{\rm F} &= -\frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{{\rm d}s}{s}\,{\rm Tr}\,\exp\left(-s\not{D}_{\rm E}^{\dagger}\not{D}_{\rm E}\right)\\ &= -\frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{{\rm d}s}{s}\,{\rm Tr}\,{\rm e}^{-sA_0}+\frac{1}{2}\int_{1/\Lambda^2}^{\infty}{\rm d}s\int_0^1{\rm d}\zeta\,{\rm Tr}\,{\rm e}^{-s\zeta A_0}A_2{\rm e}^{-s(1-\zeta)A_0}\\ &\quad -\frac{1}{2}\int_{1/\Lambda^2}^{\infty}{\rm d}s s\int_0^1{\rm d}\zeta\int_0^{1-\zeta}{\rm d}\eta\,{\rm Tr}\,{\rm e}^{-s\eta A_0}A_1{\rm e}^{-s(1-\zeta-\eta)A_0}A_1{\rm e}^{-s\zeta A_0}\\ &\quad +\mathcal{O}(\phi_a^3)\end{aligned} \tag{B.23}$$

allows us to systematically expand $\mathcal{A}_{\rm F}$. It is convenient to evaluate the functional trace in (Euclidian) momentum space since $\langle k|A_0|q\rangle = (-k^2+m^2)\,\delta(k-q)$. The matrix elements of the operators in $A_{1,2}$ that are local in coordinate space may be expressed in terms of the corresponding Fourier transformation (B.14). Due to the cyclic property of the trace, only the linear combination $\alpha = \zeta+\eta \in [0,1]$ occurs in the exponential functions and we may straightforwardly integrate over $\beta = \zeta-\eta \in [-\alpha,\alpha]$. The α-integral is simplified by the symmetry $\alpha \leftrightarrow 1-\alpha$,

$$\begin{aligned}\mathcal{A}_{\rm F}^{(2)} = -\frac{m^2N_{\rm C}}{4}\int_{1/\Lambda^2}^{\infty}{\rm d}s s\int_0^1{\rm d}\alpha\int\frac{{\rm d}^4q}{(2\pi)^4}\int\frac{{\rm d}^4k}{(2\pi)^4}\,{\rm tr}_{\rm DF}\\ \times{\rm e}^{-s\alpha(k^2+m^2)}\gamma_5\not{q}\,\widetilde{\phi}_a(q)\lambda_a{\rm e}^{-s(1-\alpha)((k-q)^2+m^2)}\gamma_5(-\not{q})\widetilde{\phi}_b(-q)\lambda_b\,.\end{aligned} \tag{B.24}$$

After the shift $k \to k-(1-\alpha)q$ that integral is computed as in (B.16) and the s-integral is expressed as an incomplete Γ-function. Final evaluation of the Dirac and flavor traces yields (2.21) and (2.22).

Let us round off this appendix by an outline for the computation of the pion matrix element of the axial current $\langle 0|\bar{q}(x)\gamma_\mu\gamma_5\frac{\tau^a}{2}q(x)|\pi^b(q)\rangle = if_\pi(q^2)q_\mu\delta_{ab}\,{\rm e}^{-{\rm i}qx}$. Here $a,b = 1,2,3$ refers to the isovector components of ϕ. This matrix element defines the pion decay constant, f_π, cf. Sect. 2.5. First we note that we formally get the expectation value of the axial current for prescribed meson fields (ϕ_a, etc.) from a functional derivative of the extended action

$$\langle\bar{q}(x)\gamma_\mu\gamma_5\frac{\tau^a}{2}q(x)\rangle = \frac{\delta}{\delta a^{a,\mu}(x)}\,{\rm Tr}_\Lambda\log\left[{\rm i}\not{D}+\boldsymbol{a}_\nu(\hat{x})\cdot\frac{\boldsymbol{\tau}}{2}\gamma_\nu\gamma_5\right]\Big|_{\boldsymbol{a}_\mu=0}, \tag{B.25}$$

where the need for regularization is indicated. This equation just tells us to expand the full action to linear order in the external source $\boldsymbol{a}_\mu(x)$ which can easily be accomplished by adding $\boldsymbol{a}_\mu(x)\gamma^\mu\gamma_5\frac{\boldsymbol{\tau}}{2}$ to the operator A_1 in (B.22). For the one-pion matrix element, it suffices to expand up to linear order in the pseudoscalar fields. Then the relevant part of the action will be similar to (B.24), with one of the $q_\nu\widetilde{\phi}_a\lambda_a$ replaced by $\widetilde{\boldsymbol{a}}_\nu(q)\cdot\frac{\boldsymbol{\tau}}{2}$. The remainder of the calculation proceeds as for (B.24). In the final matrix element with the pion state, the q-integral disappears. Finally $\frac{\delta}{\delta a^{a,\mu}(x)}\widetilde{a}_{b,\nu}(q) = g_{\mu\nu}\delta_{ab}\,{\rm e}^{{\rm i}qx}$ renders the functional form of the above definition. Note that the pion and ϕ fields are differently normalized, $\boldsymbol{\pi} = f_\pi\boldsymbol{\phi}$, since the propagator of a field operator associated with a one-particle state has unit residue. This similarity between the expansion of the action up to quadratic order in ϕ^a and the computation of the axial current matrix element clearly shows that (2.23) equals the pion decay constant (squared).

Appendix C

Baryon Current and Wess–Zumino Term

Here we will review the calculations showing that the topological current arises from the gradient expansion of the expectation value of a baryon source and as a symmetry current from the Wess–Zumino term. Furthermore we gauge the Wess–Zumino by photon fields and discuss the relevance for the decay of π^0.

C.1 Gradient Expansion of the Fermion Determinant with a Baryon Source

It is well known that the topological current arises as the leading term in the gradient expansion of the corresponding one-quark-loop expectation value [1]. Nevertheless we repeat the essential points of that calculation because this relation is essential for many of the arguments in the main text. In doing so, we apply the functional techniques of Appendix A to the approach of [2]. Essentially we only consider the coupling of the quarks to the chiral field U as in (2.20),

$$M(x) = mU(x) = g\left[\sigma(x) + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)\right] \quad \text{with} \quad \sigma^2(x) + \boldsymbol{\pi}^2(x) = f_\pi^2\,. \qquad \text{(C.1)}$$

We explicitly use neither the limitation to two flavors nor the unitary condition for σ and $\boldsymbol{\pi}$. However, we omit flavor symmetry breaking effects. We have defined the coupling constant $g = m/f_\pi$ because it is customary to parameterize the vacuum expectation value $\langle\sigma\rangle = f_\pi$ in the linear sigma model. For simplification we furthermore introduce

$$M_5(x) = g\left[\sigma(x) + \mathrm{i}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}(x)\right]\,. \qquad \text{(C.2)}$$

The expectation value of the baryon current solely concerns the fermion part of the generating functional, (2.14) and may be formally written as

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$$\begin{aligned}\langle B_\mu(y)\rangle &= \frac{1}{N}\int [D\psi]\,\left[D\bar{\psi}\right]\,\bar{\psi}(y)\gamma_\mu\frac{1}{N_\mathrm{C}}\psi(y)\\ &\quad\times\exp\left[\mathrm{i}\int \mathrm{d}^4x\,\bar{\psi}(x)\,(\mathrm{i}\partial\!\!\!/ - M_5(x))\,\psi(x)\right]\\ &= \frac{\mathrm{i}}{N_\mathrm{C}}\,\frac{\delta}{\delta s_\mu(y)}\ln\mathrm{Det}\,[\mathrm{i}\partial\!\!\!/ - M_5 - s\!\!\!/]_{s_\mu=0}\end{aligned} \tag{C.3}$$

since each quark carries baryon number $1/N_\mathrm{C}$. In the first equation the normalization factor N is merely the functional integral without the baryon current. In the second part it is taken care of by the logarithmic derivative. The expectation value on the left hand side is with respect to a prescribed configuration $M(x)$. The interaction via M_5 does not affect color degrees of freedom, hence that part of the determinant can be trivially computed and we write

$$\langle B_\mu(y)\rangle = -\mathrm{i}\mathrm{Tr}'\left[(\mathrm{i}\partial\!\!\!/ - M_5)^{-1}\,\gamma_\mu\delta^{(4)}(\hat{x}-y)\right]\,, \tag{C.4}$$

where Tr' denotes space–time integration (eventually in momentum space) and the sum over Dirac and flavor indices; the color trace, however, has already been performed. In (C.4) the hat (ˆ) indicates the position operator as defined after (B.11). It should not be confused with the symbol for a unit vector. To set up the gradient expansion we expand

$$M_5(x) = M_5^{(0)} + \delta M_5(x)\,, \tag{C.5}$$

where the first term on the right hand side has the properties

$$M_5^{(0)} = \mathrm{const} \quad \text{and} \quad M_5^{(0)}\left[M_5^{(0)}\right]^\dagger = m^2\,. \tag{C.6}$$

Note that the latter condition does not imply the chiral circle condition which would have to be imposed on the total field $M_5(x)$. The idea is to expand (C.4) in powers of $\delta M_5(x)$. Due to chiral symmetry, the result will be a function of only $M_5^{(0)}$ and derivatives of $\delta M_5(x)$. At the end we replace

$$M_5^{(0)} \longrightarrow M_5(x) \quad \text{and} \quad \partial_\mu\delta M_5(x) \longrightarrow \partial_\mu M_5(x)\,, \tag{C.7}$$

to obtain the gradient expansion approximation. Rather than going through the full calculation it is considerably simplifying to have an educated guess for the result. We know that the baryon current has the quantum numbers of the scalar–isoscalar ω meson which couples to three pions but neither to two pions nor to $\sigma\boldsymbol{\pi}$ due to isospin and G-parity invariance. Defining $\phi_a = (\boldsymbol{\pi},\sigma)_a$ we therefore expect

$$\langle B_\mu(y)\rangle = S(\phi_0^2)\,\epsilon_{abcd}\epsilon_{\mu\nu\rho\sigma}\,\phi_a^{(0)}\partial^\nu\left[\delta\phi_b\right]\partial^\rho\left[\delta\phi_c\right]\partial^\sigma\left[\delta\phi_d\right] + \cdots\,, \tag{C.8}$$

where $\phi_a^{(0)}$ and $\delta\phi_a$ are obtained from $M_5^{(0)}$ and $\delta M_5(x)$, respectively and $\phi_0^2 = \sum_{a=1}^4 \phi_a^{(0)}\phi_a^{(0)}$. Obviously the above formulation is consistent with the bosonic character of the (pseudo)scalar mesons. The main task now is to compute the coefficient function $S(\phi_0^2)$. Evidently the educated guess, (C.8) can only emerge from the contribution that is of third order in δM_5 in (C.3). Writing

$$(\mathrm{i}\partial\!\!\!/ - M_5)^{-1} = [1 - (\mathrm{i}\partial\!\!\!/ - M_5^{(0)})^{-1}\delta M_5]^{-1}(\mathrm{i}\partial\!\!\!/ - M_5^{(0)})^{-1} \tag{C.9}$$

the expansion of the factor in square brackets yields the unique third order contribution

$$\begin{aligned}\langle B_\mu(y)\rangle = -\mathrm{i}\mathrm{Tr}'\{&(\mathrm{i}\partial\!\!\!/ - M_5^{(0)})^{-1}\delta M_5(\mathrm{i}\partial\!\!\!/ - M_5^{(0)})^{-1}\delta M_5 \\ &\times(\mathrm{i}\partial\!\!\!/ - M_5^{(0)})^{-1}\delta M_5(\mathrm{i}\partial\!\!\!/ - M_5^{(0)})^{-1}\,\gamma_\mu\delta^{(4)}(\hat{x}-y)\} + \cdots\ .\end{aligned} \tag{C.10}$$

Without loss of generality we may chose the chiral basis such that $M_5^{(0)} = m\mathbb{1} = gf_\pi\mathbb{1}$ and $\delta M_5 = \mathrm{i}g\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5 = \mathrm{i}g\Pi\gamma_5$. In the last term we have defined a matrix in flavor space. The choice of this basis is helpful to sum over the Dirac indices because then the right hand side of (C.11) contains three factors of γ_5. To get $\epsilon_{\mu\nu\rho\sigma}$ we need to pick up four γ-matrices in (C.11), three of which must emerge from the propagators. As there are four propagators, one of them must deliver $M_5^{(0)}$ and there are four possible combinations to do so. This is most compactly presented in momentum space,

$$\begin{aligned}\langle B_\mu(y)\rangle = 4\mathrm{i}mg^3\epsilon_{\mu\nu\rho\sigma}&\int\frac{\mathrm{d}^4q_1}{(2\pi)^4}\frac{1}{q_1^2-m^2}\cdots\int\frac{\mathrm{d}^4q_4}{(2\pi)^4}\frac{1}{q_4^2-m^2} \\ &\times\{q_2^\nu q_3^\rho q_4^\sigma - q_1^\nu q_3^\rho q_4^\sigma + q_1^\nu q_2^\rho q_4^\sigma - q_1^\nu q_2^\rho q_3^\sigma\}\,\mathrm{e}^{i(q_4-q_1)y} \\ &\times\mathrm{tr}_\mathrm{F}\left[\widetilde{\Pi}(q_1-q_2)\widetilde{\Pi}(q_2-q_3)\widetilde{\Pi}(q_3-q_4)\right] + \cdots\ .\end{aligned} \tag{C.11}$$

The omitted lower index in any of the products of momenta within the curly brackets indicates the propagator that delivered $M_5^{(0)}$ and the alternating signs originate from various anti-commutators of γ_5 and $q\!\!\!/_i$. Furthermore $\widetilde{\Pi}$ denotes the Fourier transformation of Π according to (B.14). Using

$$\begin{aligned}\epsilon_{\mu\nu\rho\sigma}\{q_2^\nu q_3^\rho q_4^\sigma - q_1^\nu q_3^\rho q_4^\sigma + q_1^\nu q_2^\rho q_4^\sigma - q_1^\nu q_2^\rho q_3^\sigma\}& \\ = \epsilon_{\mu\nu\rho\sigma}\,(q_2-q_1)^\nu\,(q_3-q_2)^\rho\,(q_4-q_3)^\sigma&\end{aligned} \tag{C.12}$$

we observe that the momentum factors coincide with the arguments of the Fourier transforms. That is, we may, e.g., write

$$(q_2-q_1)^\nu\,\widetilde{\Pi}(q_1-q_2) = -\mathrm{i}\widetilde{[\partial^\nu\Pi]}(q_1-q_2)\,.$$

After appropriate redefinition of the momenta this yields

$$\langle B_\mu(y)\rangle = 4mg^3\epsilon_{\mu\nu\rho\sigma}\int\frac{\mathrm{d}^4r}{(2\pi)^4}\frac{\mathrm{d}^4s}{(2\pi)^4}\frac{\mathrm{d}^4t}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}(r+s+t)y}\int\frac{\mathrm{d}^4l}{(2\pi)^4}$$

$$\times \frac{\mathrm{tr_F}\,\widetilde{[\partial^\nu\Pi]}(r)\,\widetilde{[\partial^\rho\Pi]}(s)\,\widetilde{[\partial^\sigma\Pi]}(t)}{[l^2-m^2]\,[(l-r)^2-m^2]\,[(l-r-s)^2-m^2]\,[(l-r-s-t)^2-m^2]} + \dots \quad . \tag{C.13}$$

The gradient expansion is a power series in the external momenta r, s and t; these momenta are assumed small compared to m. In leading order we omit them in the denominator and replace

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,\frac{1}{\{\dots\}} \quad \xrightarrow[\text{expansion}]{\text{gradient}} \quad \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,\frac{1}{(l^2-m^2)^4} = \frac{\mathrm{i}}{96\pi^2 m^4} \tag{C.14}$$

where the imaginary unit stems from the Wick rotation. Without the complicated dependences on the external momenta, the Fourier transformations can easily be undone, they just yield the coordinate space analogs. Collecting pieces, we find the leading order (l.o.) gradient expansion to the baryon current,

$$\langle B_\mu(y)\rangle_{\mathrm{l.o.}} = \frac{\mathrm{i}g^3}{24\pi^2 m^3}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr_F}\,[\partial^\nu\Pi]\,(y)\,[\partial^\rho\Pi]\,(y)\,[\partial^\sigma\Pi]\,(y)\,. \tag{C.15}$$

This exactly is the Goldstone–Wilczek current [1]. Finally we want to express this result in a chiral invariant fashion in terms of U. We note that a chiral invariant quantity must contain as many Us as $U^\dagger$s and that

$$mU^\dagger\partial_\mu U = m\alpha_\mu = g\mathrm{i}\partial_\mu\Pi + \dots\,, \tag{C.16}$$

with α_μ defined after (2.40). This then uniquely yields the chirally invariant result

$$\langle B_\mu\rangle_{\mathrm{l.o.}} = \frac{1}{24\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr_F}\left\{\left[U^\dagger\partial^\nu U\right]\left[U^\dagger\partial^\rho U\right]\left[U^\dagger\partial^\sigma U\right]\right\}\,. \tag{C.17}$$

which is nothing but the topological or winding number current.

The Skyrme soliton picture of baryons makes extensive use of and strongly relies on the above identification of the baryon current. For the derivation of that relation spontaneous chiral symmetry breaking is essential as without it, $M^{(0)}$ vanishes and the gradient expansion in (C.14) remains undefined. For spontaneous chiral symmetry to occur it is necessary that $N_{\mathrm{f}} \geq 2$ since otherwise the symmetry is anyhow broken by the anomaly. Thus, even though the arguments put forward in favor of the soliton picture in Sect. 4.1 appear to be valid for any number of flavors, a cogent consideration requires at least two flavors.

C.2 Gauging the Wess–Zumino Term

In Sect. 2.4 we already encountered the non-local Wess–Zumino term, see (2.50). This time we want to start from that action together with the transformation properties on the meson fields, (2.30), to show that the winding number current emerges as symmetry current for the baryon charge from $\Gamma_{\rm WZ}$. A straightforward procedure to extract symmetry currents is to introduce gauge fields to elevate global symmetries to local ones. The terms in the gauged action that are linear in the gauge fields then determine the symmetry currents. Unfortunately it is not straightforward to gauge the Wess–Zumino term because the common description of introducing covariant derivatives only works for local action functionals. Rather we have to construct the gauge invariant action by a trial and error method. For our purposes (baryon current and electromagnetic π^0 decay) it is sufficient to only consider $U_{\rm V}(1)$ symmetries and thus abelian gauge fields. This simplifies matters considerably. The result for the general non-abelian case is given in the literature [3, 4].

First of all, we simplify the notation by introducing alternating differential forms, $\alpha = \alpha_\mu {\rm d}x^\mu$, ${\rm d} = \partial_\mu {\rm d}x^\mu$, $\alpha\beta = \alpha_\mu \beta_\nu\, {\rm d}x^\mu \wedge {\rm d}x^\nu$, etc. In that notation the Wess–Zumino term is compactly written as

$$\Gamma_{\rm WZ} = -\frac{{\rm i}N_{\rm C}}{240\pi^2}\int_5 {\rm tr}\,\left(\alpha^5\right)\,. \tag{C.18}$$

Here the integral is over a five-dimensional manifold with Minkowski space as boundary. With respect to vector symmetries, $L = R = 1 + {\rm i}\epsilon Q + \cdots$ in (2.30), the variation of the chiral field is given by the commutator,

$$\delta U = {\rm i}\epsilon[Q, U] \tag{C.19}$$

where ϵ parameterizes the infinitesimal transformation and Q is the generator of the considered $U_{\rm V}(1)$ symmetry. We assume ϵ to be a local quantity,

$$\delta\alpha = {\rm i}\epsilon[Q, \alpha] + {\rm i}{\rm d}\epsilon\left(U^\dagger Q U - Q\right)\,. \tag{C.20}$$

Note that $\delta\alpha$ is a differential one-form. When substituting into $\delta\int {\rm tr}(\alpha^5) = 5\int {\rm tr}(\delta\alpha\alpha^4)$ the first term of (C.20) does not contribute as it merely reflects the global symmetry. The variation due to the derivative term is

$$\delta\Gamma_{\rm WZ} = \frac{N_{\rm C}}{48\pi^2}\int_5 {\rm d}\epsilon\,{\rm tr}\,\left[\left(U^\dagger Q U - Q\right)\alpha^4\right] = -\frac{N_{\rm C}}{48\pi^2}\int_5 {\rm d}\epsilon\,{\rm tr}\,\left[Q\left(\alpha^4 - \beta^4\right)\right]\,. \tag{C.21}$$

where $\beta_\mu = U\partial_\mu U^\dagger = -U\alpha_\mu U^\dagger$. By pure definition we have $\alpha^4 = -{\rm d}(\alpha^3)$ and $\beta^4 = -{\rm d}(\beta^3)$. Thus

$$\delta\Gamma_{\rm WZ} = \frac{N_{\rm C}}{48\pi^2}\int_5 {\rm d}\epsilon\,{\rm d}\,{\rm tr}\,\left[Q\left(\alpha^3 - \beta^3\right)\right] = -\frac{N_{\rm C}}{48\pi^2}\int_4 {\rm d}\epsilon\,{\rm tr}\,\left[Q\left(\alpha^3 - \beta^3\right)\right] \tag{C.22}$$

by Stoke's theorem. Obviously the non-local Wess–Zumino term is not gauge invariant but the gauge variation is local. We introduce a gauge field, $\mathcal{A}_\mu$ to compensate for $\delta\Gamma_{\rm WZ}$,

$$\Gamma_{\rm WZ}^{(1)} = -\frac{{\rm i}N_{\rm C}}{240\pi^2}\int_5 {\rm tr}\,\left(\alpha^5\right) + \frac{N_{\rm C}}{48\pi^2}\int_4 \mathcal{A}\,{\rm tr}\left[Q\left(\alpha^3-\beta^3\right)\right]\,, \qquad ({\rm C.23})$$

with $\delta\mathcal{A}_\mu = \partial_\mu\epsilon$. Hence $\Gamma_{\rm WZ}^{(1)}$ is invariant to $\mathcal{O}(Q)$, but not completely as the explicit calculation exhibits

$$\begin{aligned}\delta\Gamma_{\rm WZ}^{(1)} &= \frac{{\rm i}N_{\rm C}}{24\pi^2}\int_4 \mathcal{A}\,{\rm d}\epsilon\,{\rm tr}\left[Q^2\left(\alpha^2-\beta^2\right)+Q{\rm d}UQ{\rm d}U^\dagger\right]\\ &= \frac{{\rm i}N_{\rm C}}{24\pi^2}\int_4 \epsilon\,{\rm d}\mathcal{A}\,{\rm tr}\left[Q^2\left(\alpha-\beta\right)+\tfrac{1}{2}\left(Q{\rm d}UQU^\dagger - QUQ{\rm d}U^\dagger\right)\right]\,. \qquad ({\rm C.24})\end{aligned}$$

Here it has, e.g., been used that $\int {\rm tr}\left[Q\alpha Q\alpha\right]=0$, as a reflection of the anti-symmetric nature of differential forms. In addition, the freedom in re-writing the $Q{\rm d}UQ{\rm d}U^\dagger$ term has been fixed by demanding invariance with respect to parity $U \leftrightarrow U^\dagger$ [4]. Again, we add a term, now quadratic in $\mathcal{A}_\mu$, to compensate for the variation $\delta\Gamma_{\rm WZ}^{(1)}$,

$$\begin{aligned}\Gamma_{\rm WZ}^{(2)} = &-\frac{{\rm i}N_{\rm C}}{240\pi^2}\int_5 {\rm tr}\,\left(\alpha^5\right) + \frac{N_{\rm C}}{48\pi^2}\int_4 \mathcal{A}\,{\rm tr}\left[Q\left(\alpha^3-\beta^3\right)\right]\\ &+\frac{{\rm i}N_{\rm C}}{24\pi^2}\int_4 \mathcal{A}\,{\rm d}\mathcal{A}\,{\rm tr}\left[Q^2\left(\alpha-\beta\right)+\tfrac{1}{2}\left(Q{\rm d}UQU^\dagger - QUQ{\rm d}U^\dagger\right)\right]\,. \qquad ({\rm C.25})\end{aligned}$$

It is now straightforward to verify that the last term in square brackets is indeed gauge invariant and so is $\Gamma_{\rm WZ}^{(2)}$. The term linear in $\mathcal{A}$ contributes to the Noether current; cf. last term in (5.46).

As indicated above, this result can be utilized to obtain the baryon current by setting $Q = \mathbb{1}/N_{\rm C}$, the baryon charge of a quark in a world with $N_{\rm C}$ color degrees of freedom. The term linear in the gauge field (second term in (C.25)) yields,

$$B_\mu = \frac{1}{48\pi^2}\epsilon_{\mu\nu\rho\sigma}{\rm tr}\left[\left(\alpha^\nu\alpha^\rho\alpha^\sigma - \beta^\nu\beta^\rho\beta^\sigma\right)\right] = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}{\rm tr}\left[\alpha^\nu\alpha^\rho\alpha^\sigma\right]\,, \quad ({\rm C.26})$$

which, as expected, is the same result as obtained from the quark loop in leading order, (C.17). This identity, of course, indirectly proves that the imaginary part of the Euclidean fermion determinant leads to the Wess–Zumino term in the effective meson theory. In (C.25) the terms quadratic in $\mathcal{A}$ vanish for $Q \propto \mathbb{1}$.

Note, however, that a kind of miracle occurred in deriving (C.26). For the $Q \propto \mathbb{1}$ there is no variation of U as in (C.19), hence one would not expect one for the Wess–Zumino term either. Nevertheless a variation occurred in (C.21) in form of a total derivative. In a local theory this would just be fine as such total derivatives would be discarded. Here it is not the case as this total derivative term of the non-local theory contributes locally in four dimensional Minkowski space. This is just one of many examples indicating that it is dangerous to use intuition from local theories to manipulate the Wess–Zumino term. As a rule of thumb, it is unavoidable to perform calculations explicitly. For this end it is occasionally helpful to write $U(x) = V(x)U_0(x)V^\dagger(x)$

with all matrices defined on the five-dimensional manifold. Using that parameterization and applying once again Stoke's theorem the Wess–Zumino term becomes [5]

$$\begin{aligned}\Gamma_{\rm WZ}[U] &= \frac{-{\rm i}N_{\rm C}}{240\pi^2}\int_5 {\rm tr}[\upsilon + U_0(\alpha_0-\upsilon)U_0^\dagger]^5 \\ &= \Gamma_{\rm WZ}[U_0] - \frac{{\rm i}N_{\rm C}}{48\pi^2}\int_4 {\rm tr}\left[\alpha_0^3\upsilon - \upsilon^3\alpha_0 - \frac{1}{2}(\alpha_0\upsilon)^2 + U_0(\alpha_0-\upsilon)^3U_0^\dagger\upsilon \right. \\ &\qquad \left. -\upsilon^3U_0(\alpha_0-\upsilon)U_0^\dagger - \frac{1}{2}[\upsilon U_0(\alpha_0-\upsilon)U_0^\dagger]^2\right] \\ &= \Gamma_{\rm WZ}[U_0] - \frac{{\rm i}N_{\rm C}}{48\pi^2}\int_4 {\rm tr}[\alpha_0^3(\upsilon + U_0^\dagger \upsilon U_0)] + \mathcal{O}(\upsilon^2)\,, \end{aligned} \tag{C.27}$$

where $\alpha_0^\mu = U_0^\dagger\partial^\mu U_0$ and $\upsilon^\mu = V^\dagger\partial^\mu V$. A significant simplification occurs when $V(x)$ depends on only a single coordinate, as it happens to be the case in the collective coordinate approach to the $SU(3)$ hedgehog, cf. (6.4). Then only terms linear in υ_μ can contribute due to the anti-symmetric structure of the alternating differential forms. This is indicated in the last line of (C.27).

C.3 Wess–Zumino Term in the Bound State Approach

In this section we prove that in the bound state approach the Wess–Zumino term emerges exactly as shown in the last term of (6.39). For this end we will show that the equation of motion (4.44) is the same as applying Euler–Lagrange to (6.39). This is sufficient because the Wess–Zumino term has mainly been introduced to provide that equation of motion. It is obvious that the Skyrme term contributions may be ignored for this discussion. Nevertheless, there is one caveat. Equation (4.44) is only one form of the equation of motion. Equivalently we may write

$$-\frac{f_\pi^2}{2}\,\partial_\mu\beta^\mu + 5{\rm i}\lambda\epsilon_{\mu\nu\rho\sigma}\beta^\mu\beta^\nu\beta^\rho\beta^\sigma = 0 \tag{C.28}$$

since $\beta_\mu = -U\alpha_\mu U^\dagger$ and thus $\partial_\mu\beta^\mu = -U\partial_\mu\alpha^\mu U^\dagger$. We have to find the proper combination to identify the variation of (6.39). Using the parameterization of (6.38) the term linear in Z from (4.44) reads

$$\begin{aligned}&\frac{f_\pi^2}{2}\left\{\partial^2 Z - 2{\rm i}\,[v_\mu,\partial^\mu Z] + [\partial_\mu r^\mu, Z] + [l_\mu,[r^\mu,Z]]\right\} \\ &\qquad -40\lambda\epsilon_{\mu\nu\rho\sigma}\,[p^\mu p^\nu p^\rho, \partial^\sigma Z + [r^\mu, Z]] = 0\,, \end{aligned} \tag{C.29}$$

with $r_\mu = \xi^\dagger\partial_\mu\xi$ and $l_\mu = \xi\partial_\mu\xi^\dagger$. Furthermore p_μ and v_μ are defined as in Sect. 6.6, i.e., (4.62) with $\xi \to \xi_\pi$. This implies $p_\mu = \frac{\rm i}{2}(l_\mu - r_\mu)$ and $v_\mu = \frac{\rm i}{2}(l_\mu + r_\mu)$. From (C.28) we find a similar equation of motion, with l_μ and r_μ exchanged. Combining these two equations yields,

$$\frac{f_\pi^2}{2}\Big\{\partial^2 Z - 2\mathrm{i}\,[v_\mu, \partial^\mu Z] - \mathrm{i}\,[\partial_\mu v^\mu, Z] + \frac{1}{2}\,[l_\mu,[r^\mu, Z]] + \frac{1}{2}\,[r_\mu,[l^\mu, Z]]\Big\}$$
$$-40\lambda\epsilon_{\mu\nu\rho\sigma}\,[p^\mu p^\nu p^\rho, \partial^\sigma Z - \mathrm{i}\,[v^\mu, Z]] = 0\,. \qquad \text{(C.30)}$$

In the isospin reduction we take the same symbols r_μ, etc., to merely denote their (non-zero) $SU(2)$ entries. Since $l \cdot r + r \cdot l = 2(p^2 - v^2)$ the equation of motion for the isospinor K reads,

$$\partial^2 K - 2\mathrm{i}v_\mu\partial^\mu K - \mathrm{i}(\partial_\mu v^\mu)K + \left(p_\mu^2 - v_\mu^2\right)K$$
$$-\frac{80\lambda}{f_\pi^2}\epsilon_{\mu\nu\rho\sigma}p^\mu p^\nu p^\rho\left(\partial^\sigma - \mathrm{i}v^\sigma\right)K = 0\,. \qquad \text{(C.31)}$$

In the isospin reduction p_μ^2 and $\epsilon_{\mu\nu\rho\sigma}p^\mu p^\nu p^\rho$ are both pure isoscalar expressions and we may replace them by the respective traces divided by 2. With $p_\mu = -\frac{\mathrm{i}}{2}\xi\alpha_\mu\xi^\dagger$ we finally obtain

$$\partial^2 K - 2\mathrm{i}v_\mu\partial^\mu K - \mathrm{i}(\partial_\mu v^\mu)K - v_\mu^2 K + \frac{1}{2}\mathrm{tr}\left(p_\mu^2\right)K$$
$$+\frac{\mathrm{i}N_\mathrm{C}}{2f_\pi^2}B_\mu\left(\partial^\mu - \mathrm{i}v^\mu\right)K = 0\,, \qquad \text{(C.32)}$$

where we have inserted $\lambda = \frac{-N_\mathrm{C}}{240\pi^2}$ and used the definition of the baryon current, (C.26). Similarly an equation of motion for $K^\dagger$ can be extracted from (C.30); it is merely the hermitean conjugate of (C.32). It is now a matter of simple algebra to verify that the Euler–Lagrange equations from (6.39) give the same equation of motion (apart from Skyrme and mass terms). This proves the form of the Wess–Zumino term in (6.39) correct. We had to combine the equations of motion in the two equivalent forms, (4.44) and (C.28) simply because the variation for obtaining the kaon equation of motion requires an infinitesimal axial transformation, $\delta U(x) = \{\epsilon(x), U(x)\}$, to maintain its pseudoscalar nature. A right transformation only as in (4.41) in general supplements scalar modes.

C.4 π^0 Decay

On the fundamental level of quark–photon interactions the π^0 decay is described by the Feynman diagrams in Fig. C.1. In these diagrams the pion is represented by an axial current via the PCAC relation, cf. Sect. 2.5. The two fermion axial current vertices contain γ_5 and any symmetry-conversing regularization of the fermion loop momentum shows that the two diagrams do not cancel each other as one would naïvely find if the diagrams were superficially finite. This anomalous non-cancellation gives a non-zero result for the π^0 decay width. We will now work out how such a result emerges in the effective meson theory from the Wess–Zumino term.

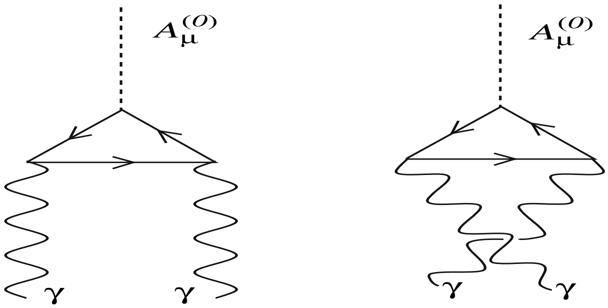

Fig. C.1. Feynman diagrams that describe the π^0 decay. Here $A_\mu^{(0)}$ is the (electrically neutral) axial current that represents (derivative of) the pion field

From the last term in (C.25) we compute the width for the decay $\pi^0 \to \gamma\gamma$ because it contains the coupling of a pseudoscalar meson to two gauge bosons. To gauge with respect to the electromagnetic interaction we first conclude from the baryon number of a single quark $B_\mathrm{q} = \frac{1}{N_\mathrm{C}}$ that the charges of the up and down quarks are $Q_\mathrm{u} = \frac{e}{2}\left(\frac{1}{N_\mathrm{C}} + 1\right)$ and $Q_\mathrm{d} = \frac{e}{2}\left(\frac{1}{N_\mathrm{C}} - 1\right)$ where e is the elementary electric charge. In the case of two light flavors[1] we therefore set

$$Q = \frac{e}{2}\left(\tau_3 + \frac{1}{N_\mathrm{C}}\mathbb{1}\right) . \tag{C.34}$$

We expand the chiral field in powers of the physical pion field $U = 1 + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}/f_\pi$ and find the relevant interaction Lagrangian after integration by parts

$$\mathcal{L}_{\pi\gamma\gamma} = -\frac{N_\mathrm{C}e^2}{24\pi^2 f_\pi}\epsilon_{\mu\nu\rho\sigma}\partial^\mu\mathcal{A}^\nu\partial^\rho\mathcal{A}^\sigma \mathrm{tr}\left[3Q^2\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right] = -\frac{\pi^0}{8\pi^2 f\pi}\epsilon_{\mu\nu\rho\sigma}\partial^\mu\mathcal{A}^\nu\partial^\rho\mathcal{A}^\sigma . \tag{C.35}$$

Using standard techniques of second quantization we obtain the transition matrix element for the decay of the neutral pion into two photons,

$$\begin{aligned}\mathcal{M}_\mathrm{fi} &= \frac{e^2}{4\pi^2 f_\pi}\epsilon_{\mu\nu\rho\sigma}k_1^\mu\epsilon^{*\nu}(\boldsymbol{k}_1,\lambda_1)k_2^\rho\epsilon^{*\sigma}(\boldsymbol{k}_2,\lambda_2)(2\pi)^4\delta(p-k_1-k_2)\\ &=: (2\pi)^4\delta(p-k_1-k_2)\mathcal{T}_\mathrm{fi} .\end{aligned} \tag{C.36}$$

---

[1] For $N_\mathrm{f} = 3$ we furthermore have $Q_\mathrm{s} = \frac{e}{2}\left(\frac{1}{N_\mathrm{C}} - 1\right)$ for the charge of the strange quark and hence

$$Q = \frac{e}{2}\left[\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8 + \left(\frac{1}{N_\mathrm{C}} - \frac{1}{3}\right)\mathbb{1}\right] . \tag{C.33}$$

With $\lambda_8/\sqrt{3} = Y = N_\mathrm{C}\hat{B}/3 + \hat{S}$ and $\mathbb{1} = N_\mathrm{C}\hat{B}$ we have $Q = (2\hat{I}_3 + \hat{B} + \hat{S})/2$ and the charge of any hadron is unambiguously determined by its flavor quantum numbers as for N_C=3.

The pion momentum is p and the photons have momenta k_1 and k_2 with polarizations λ_1 and λ_2, respectively, and $\epsilon^{*\nu}(\boldsymbol{k}_i, \lambda_i)$ are the corresponding polarization vectors. The decay width Γ is computed by squaring $\mathcal{T}_{\rm fi}$, summing over the polarizations of the final photons, taking care of total momentum conservation and integrating over the available phase space of the two photons. Note that a factor 1/2 arises to comply with Bose statistics of the decay products,

$$\Gamma = \frac{1}{2m_\pi}\frac{1}{2}\sum_{\lambda_1\lambda_2}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2k_1^0}\,\frac{\mathrm{d}^3k_2}{(2\pi)^3 2k_2^0}(2\pi)^4\delta^4(p-k_1-k_2)|\mathcal{T}_{\rm fi}|^2 . \qquad \text{(C.37)}$$

The integrals are most conveniently evaluated in the pion rest frame that is defined by setting the pion momentum to $p^\mu = (m_\pi, \mathbf{0})^\mu$, and implies $k_{1,2}^\mu = (\omega, \pm\boldsymbol{k})^\mu$ for the photon momenta. Hence

$$\Gamma = \frac{\alpha^2 m_\pi^3}{32\pi^2 f_\pi}\int \mathrm{d}\omega\,\delta\,(m_\pi - 2\omega)\int\frac{\mathrm{d}^2k}{4\pi^2} = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi}\,, \qquad \text{(C.38)}$$

where $\alpha = e^4/4\pi = 1/137$ is the QED fine structure constant. Inserting numerical values yields $\Gamma \approx 7.6\,\mathrm{eV}$. This compares reasonably well with the experimental value, $\Gamma = (8.2 \pm 0.6)\,\mathrm{eV}$ [6]. This is, of course, one of the most striking empirical evidences for the relevance of the Wess–Zumino term.

The final result, (C.38), obtained by gauging the Wess–Zumino term, coincides with that deduced from the triangle anomaly in the microscopic fermion theory in conjunction with PCAC to identify the pion field [7]. The corresponding Feynman diagrams are depicted in Fig. C.1. For this identification, the coefficient of the Wess–Zumino term is crucial, in particular its $N_{\rm C}$ dependence that arises from $N_{\rm C}$ fermions running through the loop. In reversing the line of arguments, Witten [3] concluded that the integer in (4.49) had to equal $N_{\rm C}$.

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Appendix D

$SU(3)$ Euler Angles

In this appendix, we display the explicit forms of the right $SU(3)$ generators R_a ($a = 1, .., 8$) in terms of differential operators with respect to $SU(3)$ "Euler angles" [1]. The current presentation briefly summarizes the calculations of [2, 3].

An appropriate definition of the $SU(3)$ Euler angles is given by parameterizing the collective flavor rotations (6.4) via

$$A = \mathrm{e}^{-\mathrm{i}(\alpha/2)\lambda_3}\mathrm{e}^{-\mathrm{i}(\beta/2)\lambda_2}\mathrm{e}^{-\mathrm{i}(\gamma/2)\lambda_3}\mathrm{e}^{-\mathrm{i}\nu\lambda_4} \times\mathrm{e}^{-\mathrm{i}(\alpha'/2)\lambda_3}\mathrm{e}^{-\mathrm{i}(\beta'/2)\lambda_2}\mathrm{e}^{-\mathrm{i}(\gamma'/2)\lambda_3}\mathrm{e}^{-\mathrm{i}(\rho/\sqrt{3})\lambda_8} \,. \tag{D.1}$$

This is merely the $SU(3)$ generalization of (5.12). The group manifold is completely covered by varying the angles $\alpha, \beta, ..., \rho$ according to

$$0 \le \alpha, \gamma, \alpha', \gamma' < 2\pi,\ 0 \le \beta, \beta' < \pi,\ 0 \le \nu < \pi/2,\ 0 \le \rho < 3\pi. \tag{D.2}$$

Since the $SU(3)$ generators are linear operators, they may in general be written as linear combinations of differential operators [4, 5]

$$R_a = \mathrm{i} d_{ba}(\boldsymbol{\alpha})\frac{\partial}{\partial \alpha_b}, \tag{D.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_8) = (\alpha, \beta, ..., \rho)$ compactly refers to the eight "Euler angles." The coefficient functions $d_{ab}(\boldsymbol{\alpha})$ will be extracted from the defining equation of the $SU(3)$ algebra

$$A R_a A^\dagger = \frac{1}{2} A \lambda_a A^\dagger = \frac{1}{2}\lambda_b D_{ba}(\boldsymbol{\alpha}). \tag{D.4}$$

As in (5.32), D_{ab} denotes the adjoint representation of the rotation matrix A. The explicit computation of the derivatives defines

$$M_{bc}(\boldsymbol{\alpha}) = \frac{1}{2}\mathrm{tr}\left[\lambda_b A \mathrm{i}\frac{\partial}{\partial\alpha_c} A^\dagger\right] \,. \tag{D.5}$$

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We use this to compute the left hand side of (D.4) and read off the coefficients

$$d_{ab}(\boldsymbol{\alpha}) = \left(M^{-1}(\boldsymbol{\alpha})\right)_{ac} D_{cb}(\boldsymbol{\alpha}) \tag{D.6}$$

and substitute them into (D.3). Then the explicit expressions for the generators are

$$\begin{aligned}
R_1 &= \mathrm{i}\frac{\cos\gamma'}{\sin\beta'}\frac{\partial}{\partial\alpha'} - \mathrm{i}\sin\gamma'\frac{\partial}{\partial\beta'} - \mathrm{i}\cos\gamma'\cot\beta'\frac{\partial}{\partial\gamma'}\,, \\
R_2 &= -\mathrm{i}\frac{\sin\gamma'}{\sin\beta'}\frac{\partial}{\partial\alpha'} - \mathrm{i}\cos\gamma'\frac{\partial}{\partial\beta'} + \mathrm{i}\sin\gamma'\cot\beta'\frac{\partial}{\partial\gamma'}\,, \\
R_3 &= -\mathrm{i}\frac{\partial}{\partial\gamma'}\,, \\
R_4 &= -\,\mathrm{i}\sin\left(\gamma-\rho+\tfrac{\alpha'-\gamma'}{2}\right)\frac{\sin\frac{\beta'}{2}}{\sin\beta\sin\nu}\frac{\partial}{\partial\alpha} - \mathrm{i}\cos\left(\gamma-\rho+\tfrac{\alpha'-\gamma'}{2}\right)\frac{\sin\frac{\beta'}{2}}{\sin\nu}\frac{\partial}{\partial\beta} \\
&\quad - \mathrm{i}\left[2\sin\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\frac{\cos\frac{\beta'}{2}}{\sin 2\nu} - \sin\left(\gamma-\rho+\tfrac{\alpha'-\gamma'}{2}\right)\cot\beta\frac{\sin\frac{\beta'}{2}}{\sin\nu}\right]\frac{\partial}{\partial\gamma} \\
&\quad - \frac{\mathrm{i}}{2}\cos\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\cos\frac{\beta'}{2}\frac{\partial}{\partial\nu} - \frac{3\mathrm{i}}{4}\sin\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\tan\nu\cos\frac{\beta'}{2}\frac{\partial}{\partial\rho} \\
&\quad + \frac{\mathrm{i}}{2}\sin\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\left[\cos\frac{\beta'}{2}\tan\nu + \frac{\cot\nu}{\cos\frac{\beta'}{2}}\right]\frac{\partial}{\partial\alpha'} \\
&\quad + \mathrm{i}\cos\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\cot\nu\sin\frac{\beta'}{2}\frac{\partial}{\partial\beta'} + \frac{\mathrm{i}}{2}\sin\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\frac{\cot\nu}{\cos\frac{\beta'}{2}}\frac{\partial}{\partial\gamma'}, \\
R_5 &= \mathrm{i}\cos\left(\gamma-\rho+\tfrac{\alpha'-\gamma'}{2}\right)\frac{\sin\frac{\beta'}{2}}{\sin\beta\sin\nu}\frac{\partial}{\partial\alpha} - \mathrm{i}\sin\left(\gamma-\rho+\tfrac{\alpha'-\gamma'}{2}\right)\frac{\sin\frac{\beta'}{2}}{\sin\nu}\frac{\partial}{\partial\beta} \\
&\quad - \mathrm{i}\left[2\cos\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\frac{\cos\frac{\beta'}{2}}{\sin 2\nu} + \cos\left(\gamma-\rho+\tfrac{\alpha'-\gamma'}{2}\right)\cot\beta\frac{\sin\frac{\beta'}{2}}{\sin\nu}\right]\frac{\partial}{\partial\gamma} \\
&\quad + \frac{\mathrm{i}}{2}\sin\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\cos\frac{\beta'}{2}\frac{\partial}{\partial\nu} - \frac{3\mathrm{i}}{4}\cos\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\tan\nu\cos\frac{\beta'}{2}\frac{\partial}{\partial\rho} \\
&\quad + \frac{\mathrm{i}}{2}\cos\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\left[\cos\frac{\beta'}{2}\tan\nu + \frac{\cot\nu}{\cos\frac{\beta'}{2}}\right]\frac{\partial}{\partial\alpha'} \\
&\quad - \mathrm{i}\sin\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\cot\nu\sin\frac{\beta'}{2}\frac{\partial}{\partial\beta'} + \frac{\mathrm{i}}{2}\cos\left(\rho+\tfrac{\alpha'+\gamma'}{2}\right)\frac{\cot\nu}{\cos\frac{\beta'}{2}}\frac{\partial}{\partial\gamma'}, \\
R_6 &= -\mathrm{i}\sin\left(\gamma-\rho+\tfrac{\alpha'+\gamma'}{2}\right)\frac{\cos\frac{\beta'}{2}}{\sin\beta\sin\nu}\frac{\partial}{\partial\alpha} - \mathrm{i}\cos\left(\gamma-\rho+\tfrac{\alpha'+\gamma'}{2}\right)\frac{\cos\frac{\beta'}{2}}{\sin\nu}\frac{\partial}{\partial\beta} \\
&\quad + \mathrm{i}\left[2\sin\left(\rho+\tfrac{\alpha'-\gamma'}{2}\right)\frac{\sin\frac{\beta'}{2}}{\sin 2\nu} + \sin\left(\gamma-\rho+\tfrac{\alpha'+\gamma'}{2}\right)\cot\beta\frac{\cos\frac{\beta'}{2}}{\sin\nu}\right]\frac{\partial}{\partial\gamma}
\end{aligned}$$

$$+ \frac{\mathrm{i}}{2}\cos\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\sin\frac{\beta'}{2}\frac{\partial}{\partial\nu} + \frac{3\mathrm{i}}{4}\sin\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\tan\nu\sin\frac{\beta'}{2}\frac{\partial}{\partial\rho}$$

$$- \frac{\mathrm{i}}{2}\sin\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\left[\sin\frac{\beta'}{2}\tan\nu + \frac{\cot\nu}{\sin\frac{\beta'}{2}}\right]\frac{\partial}{\partial\alpha'}$$

$$+ \mathrm{i}\cos\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\cot\nu\cos\frac{\beta'}{2}\frac{\partial}{\partial\beta'} + \frac{\mathrm{i}}{2}\sin\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\frac{\cot\nu}{\sin\frac{\beta'}{2}}\frac{\partial}{\partial\gamma'}\,,$$

$$R_7 = \mathrm{i}\cos\Big(\gamma - \rho + \tfrac{\alpha'+\gamma'}{2}\Big)\frac{\cos\frac{\beta'}{2}}{\sin\beta\sin\nu}\frac{\partial}{\partial\alpha} - \mathrm{i}\sin\Big(\gamma - \rho + \tfrac{\alpha'+\gamma'}{2}\Big)\frac{\cos\frac{\beta'}{2}}{\sin\nu}\frac{\partial}{\partial\beta}$$

$$+ \mathrm{i}\left[2\cos\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\frac{\sin\frac{\beta'}{2}}{\sin 2\nu} - \cos\Big(\gamma - \rho + \tfrac{\alpha'+\gamma'}{2}\Big)\cot\beta\frac{\cos\frac{\beta'}{2}}{\sin\nu}\right]\frac{\partial}{\partial\gamma}$$

$$- \frac{\mathrm{i}}{2}\sin\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\sin\frac{\beta'}{2}\frac{\partial}{\partial\nu} + \frac{3\mathrm{i}}{4}\cos\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\tan\nu\sin\frac{\beta'}{2}\frac{\partial}{\partial\rho}$$

$$- \frac{\mathrm{i}}{2}\cos\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\left[\sin\frac{\beta'}{2}\tan\nu + \frac{\cot\nu}{\sin\frac{\beta'}{2}}\right]\frac{\partial}{\partial\alpha'}$$

$$- \mathrm{i}\sin\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\cot\nu\cos\frac{\beta'}{2}\frac{\partial}{\partial\beta'} + \frac{\mathrm{i}}{2}\cos\Big(\rho + \tfrac{\alpha'-\gamma'}{2}\Big)\frac{\cot\nu}{\sin\frac{\beta'}{2}}\frac{\partial}{\partial\gamma'}\,,$$

$$R_8 = -\frac{\mathrm{i}\sqrt{3}}{2}\frac{\partial}{\partial\rho}\,. \tag{D.7}$$

Here we also want to outline how the eigenvalue problem for the collective Hamiltonian (6.26) reduces to coupled differential equations for functions which only depend on the strangeness changing angle ν. Up to the normalization, a suitable decomposition of the baryon wave functions is given by [1]

$$\Psi(I, I_3, Y; J, J_3, Y_\mathrm{R}) = \sum_{M_\mathrm{L}, M_\mathrm{R}} D^{(I)*}_{I_3,M_\mathrm{L}}(\alpha,\beta,\gamma) f^{(I,Y;J,Y_\mathrm{R})}_{M_\mathrm{L},M_\mathrm{R}}(\nu) \times \mathrm{e}^{\mathrm{i}Y_\mathrm{R}\rho} D^{(J)*}_{M_\mathrm{R},-J_3}(\alpha',\beta',\gamma'). \tag{D.8}$$

The D-functions refer to $SU(2)$ Wigner functions. It is important to note that the sums over the intrinsic spins ($M_\mathrm{R} = -J, -J+1, ..., J$) and isospins ($M_\mathrm{L} = -I, -I+1, ..., I$) are subject to the constraint $M_\mathrm{L} - M_\mathrm{R} = (Y - Y_\mathrm{R})/2$. Using the explicit forms for the $SU(3)$ generators (D.7), the action of the quadratic Casimir operator $C_2 = \sum_{a=1}^{8} R_a^2$ on the baryon wave function (D.8) is found to be

$$C_2\Psi(I, I_3, Y; J, J_3, Y_\mathrm{R}) = \sum_{M_\mathrm{L}, M_\mathrm{R}} D^{(I)*}_{I_3,M_\mathrm{L}}(\alpha,\beta,\gamma)\mathrm{e}^{\mathrm{i}Y_\mathrm{R}\rho} D^{(J)*}_{J_3,M_\mathrm{R}}(\alpha',\beta',\gamma')$$

$$\times\Bigg\{ -\frac{1}{4}\left[\frac{\mathrm{d}^2}{\mathrm{d}\nu^2} + (3\cot\nu - \tan\nu)\frac{\mathrm{d}}{\mathrm{d}\nu}\right] + \frac{I^2+J^2}{\sin^2\nu} + \frac{M_\mathrm{L}^2}{\cos^2\nu}$$

$$+\frac{M_{\rm R}^2}{4}\left(3+\frac{1}{\cos^2\nu}\right)-\frac{1+\cos^2\nu}{\sin^2\nu\cos^2\nu}M_{\rm L}M_{\rm R}+\frac{3Y_{\rm R}M_{\rm L}}{2\cos^2\nu}$$
$$\left.-3\frac{1+\cos^2\nu}{4\cos^2\nu}Y_{\rm R}M_{\rm R}+\left(\frac{3}{4}+\frac{9}{16}\tan^2\nu\right)Y_{\rm R}^2\right\}f^{(I,Y;J,Y_{\rm R})}_{M_{\rm L},M_{\rm R}}(\nu)$$
$$-\frac{\cos\nu}{\sin^2\nu}\sqrt{(I+M_{\rm L}+1)(I-M_{\rm L})(J+M_{\rm R}+1)(J-M_{\rm R})}\;f^{(I,Y;J,Y_{\rm R})}_{M_{\rm L}+1,M_{\rm R}+1}(\nu)$$
$$-\frac{\cos\nu}{\sin^2\nu}\sqrt{(I-M_{\rm L}+1)(I+M_{\rm L})(J-M_{\rm R}+1)(J+M_{\rm R})}\;f^{(I,Y;J,Y_{\rm R})}_{M_{\rm L}-1,M_{\rm R}-1}(\nu)\,. \tag{D.9}$$

Obviously, the dependence on the angles other than ν can be factorized leaving a set of coupled ordinary differential equations in the variable ν. This becomes even more transparent by displaying the ν dependence of the dominating symmetry breaking term in the collective Hamiltonian (6.26):

$$1-D_{88}=\frac{3}{2}\sin^2\nu. \tag{D.10}$$

Equation (D.9) also illustrates how the intrinsic functions $f^{(I,Y;J,Y_{\rm R})}_{M_{\rm L},M_{\rm R}}(\nu)$ depend on the spin and isospin quantum numbers.

The eigenvalue equation $C_2\Psi=\mu\Psi$ yields the flavor symmetric $SU(3)$ D-functions, which correspond to states in irreducible representations. As an example, we display the non-vanishing intrinsic isoscalar functions for the baryon octet with $Y_{\rm R}=1$ and $\mu=3$ [5].

$$N:\; f^{\frac12,1;\frac12,1}_{\frac12,\frac12}(\nu)=\cos^2\nu\,,\quad f^{\frac12,1;\frac12,1}_{-\frac12,-\frac12}(\nu)=\cos\nu\,;$$
$$\Sigma:\; f^{1,0;\frac12,1}_{0,\frac12}(\nu)=\frac{1}{\sqrt2}\cos\nu\sin\nu\,,\quad f^{1,0;\frac12,1}_{-1,-\frac12}(\nu)=\sin\nu\,;$$
$$\Lambda:\; f^{0,0;\frac12,1}_{0,\frac12}(\nu)=\sqrt{\frac32}\sin\nu\cos\nu\;;\qquad \Xi:\; f^{\frac12,-1;\frac12,1}_{-\frac12,\frac12}(\nu)=\sin^2\nu\,. \tag{D.11}$$

All other octet isoscalar functions vanish because of the constraint $M_{\rm L}-M_{\rm R}=(Y-1)/2$. The normalization is always such that

$$\int_0^{\frac{\pi}{2}}{\rm d}\nu\sin2\nu\sin^2\nu\sum_{M_{\rm L}M_{\rm R}}\left[f^{(I,Y;J,Y_{\rm R})}_{M_{\rm L},M_{\rm R}}(\nu)\right]^2=\frac{(2J+1)(2I+1)}{16}\,. \tag{D.12}$$

Obviously, none of these wave functions vanish except at the boundaries $\nu=0,\pi/2$. This is, of course, a special feature of the ground states, which reside in the octet representation. The isoscalar wave functions associated with baryons in higher dimensional representations, which carry the same physical quantum numbers (I,J,Y), may well develop nodes.

When the eigenvalue problem is augmented by symmetry breaking terms, the intrinsic function deviate from (D.11) such that they get more pronounced at small ν, i.e., rotations into the direction of strangeness are suppressed. This can also be deduced from Fig. D.1, where the dependencies of the nucleon

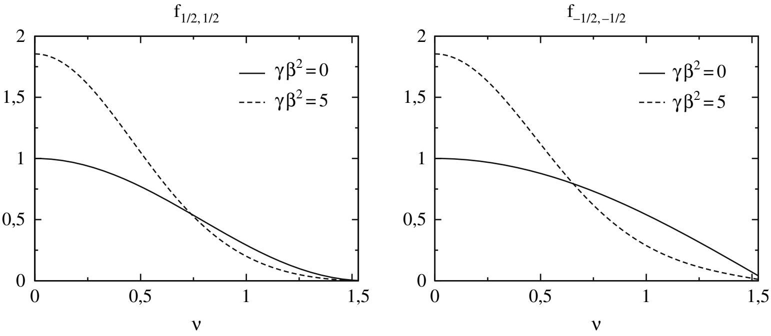

Fig. D.1. The dependencies of the nucleon scalar functions on the strangeness changing angle ν for two values of the symmetry breaking. The case $\gamma\beta^2 = 0$ should be compared with the expressions for (N) in (D.11)

isoscalar functions $f^{\frac{1}{2},1;\frac{1}{2},1}_{\pm\frac{1}{2},\pm\frac{1}{2}}(\nu)$ are displayed for the symmetric case, $\gamma\beta^2 = 0$, as well as for sizable symmetry breaking $\gamma\beta^2 = 5$. We note that this diagonalization approach can indeed be generalized to arbitrary (odd) $N_{\rm C}$ by implementing the constraint $Y_{\rm R} = \frac{N_{\rm C}}{3}$ [6] in (D.9).

Finally, we add a few comments on the treatment of the slow rotator discussed in Sect. 6.5. In a first step, the explicit form (D.9) for the Casimir operator is used in order to express the Hamiltonian, (6.37), as a second-order differential equation for the isoscalar functions $f^{(I,Y;J,Y_{\rm R})}_{M_{\rm L},M_{\rm R}}(\nu)$, which are defined in equation (D.8). These coupled differential equations are then integrated by standard means. In order to evaluate matrix elements, one again employs the decomposition equation (D.8) to reduce them to expressions which only contain functions of the strangeness changing angle ν and $f^{(I,Y;J,Y_{\rm R})}_{M_{\rm L},M_{\rm R}}(\nu)$. The final result is obtained by integrating with respect to the measure, cf. (D.12),

$$\int_0^{\frac{\pi}{2}} {\rm d}\nu \, \sin 2\nu \, \sin^2\nu \left\{ \cdots \right\} . \tag{D.13}$$

As an example, we present the V_1 contribution in (7.7) to the proton magnetic moment,

$$\begin{aligned} \mu_p = &-\frac{8\pi}{3} M_N \int_0^{\frac{\pi}{2}} {\rm d}\nu \, \sin 2\nu \, \sin^2\nu \; m_1(\nu) \\ &\times \left\{ \frac{2}{3} \sin^2\nu \left(\left(f^{\frac{1}{2},1;\frac{1}{2},1}_{\frac{1}{2},\frac{1}{2}}(\nu) \right)^2 - \left(f^{\frac{1}{2},1;\frac{1}{2},1}_{-\frac{1}{2},-\frac{1}{2}}(\nu) \right)^2 \right) \right. \\ &- \frac{2}{9} \Big[\left(1 + \cos^2\nu\right) \left(\left(f^{\frac{1}{2},1;\frac{1}{2},1}_{\frac{1}{2},\frac{1}{2}}(\nu) \right)^2 + \left(f^{\frac{1}{2},1;\frac{1}{2},1}_{-\frac{1}{2},-\frac{1}{2}}(\nu) \right)^2 \right) \\ &\left. + 8 \cos\nu f^{\frac{1}{2},1;\frac{1}{2},1}_{\frac{1}{2},\frac{1}{2}}(\nu) f^{\frac{1}{2},1;\frac{1}{2},1}_{-\frac{1}{2},-\frac{1}{2}}(\nu) \Big] \right\} . \end{aligned} \tag{D.14}$$

The ν dependence of m_1 is purely due to the implicit dependence of the chiral angle $F = F(r, \nu)$:

$$m_1(\nu) = \int_0^\infty \mathrm{d}r r^2 \sin^2 F \left[f_\pi^2 + \frac{1}{e^2} \left(F'^2 + \frac{\sin F}{r^2} \right) + \frac{2}{3}(f_K^2 - f_\pi^2) \cos F \right] . \tag{D.15}$$

Here a prime indicates a derivative with respect to the radial coordinate, i.e., $F' = \partial F(r, \nu)/\partial r$.

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Appendix E

Matrix Elements of Momentum Eigenstates

In this appendix we review the evaluation of matrix elements between momentum eigenstates of the soliton that are needed to compute the current matrix elements in Chap. 7. Mostly we will follow the discussion of [1].

E.1 Momentum Eigenstates from Collective Coordinates

In Chapters 5 and 6 we have extensively discussed how to generate states with good flavor quantum numbers from the soliton by means of collective coordinate quantization. However, this is not sufficient to compute form factors. To this end we also need eigenstates with good momentum. The conjugate coordinate is the translation of the soliton and we again introduce respective collective variables, $\boldsymbol{X}(t)$ to parameterize the time-dependent chiral field,

$$U(\boldsymbol{x},t) = U_0\left(\boldsymbol{x} - \boldsymbol{X}(t)\right) , \tag{E.1}$$

where U_0 refers to the hedgehog configuration of (4.23). Due to translational invariance, the Lagrangian that emerges from substituting the configuration, (E.1), will depend on $X(t)$ only through its time derivative $\boldsymbol{V} = \dot{\boldsymbol{X}}(t)$. For the time being and simplicity we omit collective coordinates for the flavor orientation of the soliton for the following reason. As long as we consider effective meson Lagrangians with at most two time derivatives, the coupling between translational and rotational modes can only be proportional to $\boldsymbol{V}\cdot\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the angular velocity defined in (5.16). Such a coupling is odd under parity and thus disallowed. The Lagrange function obtained for the configuration (E.1) has the general structure

$$L(\boldsymbol{V}) = -E_{\rm cl} + \frac{1}{2}M_{\rm trans}\boldsymbol{V}^2 , \tag{E.2}$$

where $E_{\rm cl}$ is the classical soliton mass and $M_{\rm trans}$ is also a functional of the soliton profiles. In case of the Skyrme model one finds $M_{\rm trans} = \frac{2}{3}\left(M_2 + 2M_4\right)$

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where M_2 and M_4 are the contributions to $E_{\rm cl}$ from $\mathcal{L}_{\rm nl\sigma}$, (4.22), and $\mathcal{L}_{\rm Sk}$, (4.26), respectively. In the absence of the mass term, simple scaling arguments [2] show that the stationary condition for the soliton implies $M_2 = M_4 = E_{\rm cl}/2$ and thus $M_{\rm trans} = E_{\rm cl}$. In a more general framework we note that (E.1) is merely the non-relativistic form of the Lorentz boost $U_H(\Lambda(\boldsymbol{V}) \cdot x)$ with $\Lambda(\boldsymbol{V})$ being the 4×4 matrix that parameterizes the boost of a four-vector such as $x = (t, \boldsymbol{x})$. Since the Lagrangian density is a Lorentz scalar, the only effect of $\boldsymbol{V} \neq 0$ on L arises from the volume integration yielding a factor $\sqrt{1-\boldsymbol{V}^2}$ on $E_{\rm cl}$. Expansion with respect to $\boldsymbol{V}$ immediately shows $M_{\rm trans} = E_{\rm cl}$, an identity that we will henceforth adopt.

The conjugate momentum is

$$\boldsymbol{P} = \frac{\partial L(\boldsymbol{V})}{\partial \boldsymbol{V}} = E_{\rm cl}\boldsymbol{V} \,. \tag{E.3}$$

The quantization of $\boldsymbol{P}$ is just that of a free non-relativistic particle with the dispersion relation $E = E_{\rm cl} + \frac{\boldsymbol{P}^2}{2E_{\rm cl}}$. The full wave function for a baryon B becomes

$$\langle B; \boldsymbol{P}; I, J, \ldots | \boldsymbol{X}; A\rangle = \mathcal{N} {\rm e}^{-{\rm i}\boldsymbol{X}\cdot\boldsymbol{P}} D_{I,J,\ldots}(A)\,, \tag{E.4}$$

where we have reintroduced the rotational degrees of freedom and $\mathcal{N}$ is a normalization factor that must be chosen compatible with the normalization of the spinor in (7.4). Its value depends on whether we take $A \in SU(2)$ or $A \in SU(3)$. Here we do not further specify $\mathcal{N}$ nor the Wigner D-functions.

To compute form factors we have to evaluate matrix elements of the form

$$\langle B; \boldsymbol{P}; I, J, \ldots | v(0) | B', \boldsymbol{P}'; I', J', \ldots\rangle\,, \tag{E.5}$$

where $v(0)$ is some current operator to be evaluated at the origin $x = (\boldsymbol{x}, t) = 0$ because the translational piece of matrix element is commonly factored out in the definition of form factors [3]. Within soliton models, the parameterization (5.13) (or its $SU(3)$ generalization, (6.4)) and (E.2) formally yield the current operator

$$v(\boldsymbol{x}, t) = \sum_i f_i\left(\boldsymbol{x} - \boldsymbol{X}(t)\right)\mathcal{O}_i(A)\,, \tag{E.6}$$

that is, a sum of products in which one factor depends on the spatial and the other on the rotational coordinates. We thus find

$$\begin{aligned}
&\langle B; \boldsymbol{P}; I, J, \ldots | v(0) | B', \boldsymbol{P}'; I', J', \ldots\rangle = \\
&\int {\rm d}^3X {\rm d}^3X' \int {\rm d}A \int {\rm d}A' \langle B; \boldsymbol{P}; I, J, \ldots\rangle\langle X, A|v(0)|X'A'\rangle\langle B', \boldsymbol{P}'; I', J', \ldots\rangle \\
&= \mathcal{N}\mathcal{N}' \sum_i \int {\rm d}^3X {\rm e}^{{\rm i}\boldsymbol{X}\cdot\boldsymbol{q}} f_i\left(-\boldsymbol{X}(t)\right) \int {\rm d}A D^*_{I,J,\ldots}(A)\mathcal{O}_i(A) D_{I',J',\ldots}(A)\,,
\end{aligned} \tag{E.7}$$

where $\boldsymbol{q} = \boldsymbol{P} - \boldsymbol{P}'$ is the momentum transfer. The second integral (over ${\rm d}A$) concerns the spin-flavor degrees of freedom and is processed with techniques described in Chapters 5 and 6, as well as the previous appendix. We read off quite a simple recipe to handle the linear momentum part of the matrix element: just take the Fourier transformation with respect to (minus) the momentum transfer of the coordinate-dependent factors in the decomposition of the current operators after substituting the soliton configuration. Of course that is precisely reflected by the spherical Bessel function appearing, e.g., in (7.7). In general we may choose any frame to do these calculation. However, it turns out that the Breit frame with,

$$\boldsymbol{P} = -\boldsymbol{P}' = \frac{\boldsymbol{q}}{2} \qquad \text{and} \qquad q^0 = 0 \tag{E.8}$$

is particularly suited not only because it properly reflects the zero energy transfer onto an infinitely heavy (large N_C) soliton but also because it directly connects the electric form factor, G_E and the magnetic form factor, G_M to the time and spatial components of the electromagnetic current, respectively. Specifically, for baryons with spin $\frac{1}{2}$ we find the Sachs form factors from the matrix elements

$$\begin{aligned} \langle B'|J^0(0)|B\rangle &= G_\mathrm{E}(\boldsymbol{q}^2)\langle s_3'|s_3\rangle \\ \langle B'|J^i(0)|B\rangle &= \frac{-\mathrm{i}}{M_\mathrm{B}} G_\mathrm{M}(\boldsymbol{q}^2)\epsilon^{ijk} q^j \langle s_3'|S_k|s_3\rangle \,. \end{aligned} \tag{E.9}$$

Here $\boldsymbol{S}$ is the spin operator and $|s_3\rangle$ indicates the state with spin projection quantum number $s_3 = \pm\frac{1}{2}$.

E.2 Relativistic Recoil Corrections

The formalism discussed above does not take into account recoil corrections. However, they should be significant for momentum transfers of the order of the nucleon mass and larger. Here we will sketch an approach to incorporate these corrections by means of a covariant formulation [4]. The resulting physics is discussed in Sect. 7.2.

The basic idea is to construct a classical moving soliton from the static solution by a Lorentz boost. This is rendered possible because of the covariance of the field equations. As results we will (i) gain a relativistic generalization of (E.3) and (ii) compute the current operators for the relativistically moving soliton. It is convenient to choose a frame such that the boosted coordinates are

$$x' = \gamma(x - Vt)\,, \quad y' = y \quad \text{and} \quad z' = z\,, \tag{E.10}$$

with $\gamma = 1/\sqrt{1-V^2}$. With regard to the discussion in Sect. E.1, Vt should be considered as the collective coordinate. For simplicity of presentation we again refrain from making the rotational coordinates explicit. As already mentioned, the Lagrange function $L = \int \mathrm{d}^3x\mathcal{L}$ for the boosted soliton

$$U(\boldsymbol{x}, t) = U_0(\boldsymbol{x}') \tag{E.11}$$

acquires the factor $1/\gamma$ when compared to the $V = 0$ case. Hence the momentum is

$$\boldsymbol{P} = \gamma \boldsymbol{V} E_{\rm cl} \quad \text{with} \quad \boldsymbol{V} = V \boldsymbol{e}_x \,. \tag{E.12}$$

Upon quantization $\boldsymbol{P}$ is elevated to an operator. The analog of $f(\boldsymbol{X}(t))$ in (E.7) for the isoscalar density is (at $t = 0$ as required for matrix elements)

$$J^0(\boldsymbol{X}) = \gamma J^0_{\rm nr}(\gamma X, Y, Z) \tag{E.13}$$

where the subscript indicates the functional form of the unboosted (non-relativistic) isoscalar density. The computation of matrix elements as in (E.9) becomes complicated because $J^0(\boldsymbol{X})$ is a highly non-linear function of the momentum operator $\boldsymbol{P}$ which induces operator ordering ambiguities. Here again the use of the Breit frame is helpful because γ only depends on $\boldsymbol{P}^2$ and thus its application on $|B\rangle$ and $|B'\rangle$ gives identical results. Therefore it is well justified to replace γ by a c-number [4]. In particular we find

$$1 - V^2 = \frac{1}{1 + \frac{\boldsymbol{q}^2}{4E_{\rm cl}}} \,. \tag{E.14}$$

With these tools at hand we compute

$$\begin{aligned} \langle B'|J^0|B\rangle &\sim G_{\rm E}(q_x^2) \\ &= \int {\rm d}^3X {\rm e}^{{\rm i}q_x X} \gamma J^0_{\rm nr}(\gamma X, Y, Z) = \int {\rm d}^3X {\rm e}^{{\rm i}q_x X/\gamma} J^0_{\rm nr}(X, Y, Z) \\ &= G_{\rm E,nr}\left(\frac{q_x^2}{\gamma^2}\right) \,, \end{aligned} \tag{E.15}$$

where we have ignored constants arising from the normalization. Next we consider the spatial components of the current, J^i. Since in the chosen frame the velocity is along the X axis and we quantize the baryons to be eigenstates of S_3 it is obvious that only

$$J^2(\boldsymbol{X}) = (\gamma X) J^2_{\rm nr}(\gamma X, Y, Z) S_3 \,, \tag{E.16}$$

has a non-vanishing matrix element. Hence we find

$$\begin{aligned} q_x G_{\rm M}(q_x^2) \sim \int {\rm d}^3X {\rm e}^{{\rm i}q_x X} (\gamma X) J^2_{\rm nr}(\gamma X, Y, Z) = \frac{1}{\gamma} \int {\rm d}^3X {\rm e}^{{\rm i}q_x X/\gamma} X J^2_{\rm nr}(X, Y, Z) \\ \sim \frac{1}{\gamma}\left(\frac{q_x}{\gamma}\right) G_{\rm M,nr}\left(\frac{q_x^2}{\gamma^2}\right) \,. \end{aligned} \tag{E.17}$$

Using (E.14) and returning to a frame-independent formulation suggests the identifications [4]

$$G_{\rm E}(Q^2) = G_{\rm E,nr}\left(\frac{Q^2}{1 + \frac{Q^2}{4M^2}}\right)$$

$$G_{\mathrm{M}}(Q^2) = \frac{1}{1+\frac{Q^2}{4M^2}}\, G_{\mathrm{M,nr}}\left(\frac{Q^2}{1+\frac{Q^2}{4M^2}}\right) . \tag{E.18}$$

We have replaced the classical soliton mass by the baryon mass, which is a self-suggesting approximation to find a model-independent mapping of the non-relativistic (rest frame) form factors to the relativistic ones.

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Appendix F

Recoupling Coefficients in Adiabatic Scattering

In this appendix, some technicalities concerning the computation of the S-matrix for meson fluctuations about the soliton are discussed.

F.1 Adiabatic Recoupling Coefficients

We first derive the recoupling coefficients that transform the intrinsic S-matrix to the laboratory frame, as used in Sect. 8.1. We present that derivation here because it merely relies on the conservation of the grand spin (see the explanations after (8.4) for detailed definitions)

$$\boldsymbol{G} = \boldsymbol{J} + \boldsymbol{I} = \boldsymbol{L} + \boldsymbol{S} + \boldsymbol{I} \tag{F.1}$$

in the intrinsic frame. This conservation reflects a symmetry of QCD at large $N_{\rm C}$ and is thus more general than the soliton models [1]. Soliton models must satisfy that property because they are consistent with large $N_{\rm C}$ QCD. We will first consider the most simple case of pion–baryon scattering in flavor $SU(2)$ and then describe obvious generalizations.

We start by considering the fluctuations as in (8.1) to relate fluctuations in the intrinsic frame to those in the laboratory frame:

$$\begin{aligned} U(\boldsymbol{x},t) &= A \exp\left[\mathrm{i}\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}F(r) + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\eta}(\boldsymbol{x},t)\right] A^\dagger \\ &= \exp\left[\mathrm{i}A\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}F(r)A^\dagger + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\xi}(\boldsymbol{x},t)\right] . \end{aligned} \tag{F.2}$$

This implies

$$\eta_\nu(\boldsymbol{x},t) = \sum_{\nu'} D^1_{\nu,\nu'}(A)\xi_{\nu'}(\boldsymbol{x},t)\,, \tag{F.3}$$

where $D^1_{\nu,\nu'}$ is the (iso)spin one representation of the collective rotations A.

Pion fluctuations are labeled by angular momentum (L,m) and isospin projection ν. In $SU(2)$, the soliton models describe baryons with identical spin and isospin. For the time being we call that quantum number s. The respective

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projection quantum numbers (σ, τ) may assume different values. We have to couple pion and baryon states to total spin (J, M) and isospin (I, I_3),

$$|(Ls)JM;(1s)II_3\rangle = \sum_{m\sigma\nu\tau} C^{JM}_{Lm,s\sigma} C^{II_3}_{1\nu,s\tau} |Lm,\nu\rangle_{\mathrm{L}} \sqrt{\frac{2s+1}{8\pi^2}}\,(-1)^{s+\tau}\, D^s_{\sigma,-\tau}, \tag{F.4}$$

because the pion has unit isospin. The C's are Clebsch–Gordan coefficients and the last factor represents the Wigner D-function for the baryon. This function arises from quantizing the collective coordinates as discussed in Sect. 5.3. The subscript on the pion points out that it is in the laboratory frame. The pion state in the intrinsic (or body-fixed) frame, that is related to the pion state in the laboratory frame by the rotation (F.3)

$$|Lm,\nu\rangle_{\mathrm{B}} = \sum_{\nu'} D^1_{\nu,\nu'} |Lm,\nu'\rangle_{\mathrm{L}}, \tag{F.5}$$

carries the same quantum numbers. We still have to relate these states to the fluctuations that appear in the differential equations (8.3). The latter possess good grand spin (F.1). It arises from coupling the fluctuations' angular momentum and isospin so that the grand spin states are given by

$$|GG_3, L\rangle = \sum_{m\nu} C^{GG_3}_{Lm,1\nu} |Lm,\nu\rangle_{\mathrm{B}}\,. \tag{F.6}$$

The states with good total spin and isospin that we obtain from the grand spin states are therefore

$$|L(GI)JM;II_3\rangle = \sum_{G_3 I_3'} C^{JM}_{GG_3,II_3'} |GG_3,L\rangle D^I_{I_3',-I_3}\,. \tag{F.7}$$

The main task in finding the recoupling coefficients is to relate the states in (F.4) and (F.7). This is mainly a matter of arranging Clebsch–Gordan coefficients and the result is [2]

$$\langle L(GI)JM;II_3|(Ls)JM;(1s)II_3\rangle = (-1)^{L+s+J}\hat{G}\hat{s} \left\{ \begin{matrix} I & 1 & s \\ L & J & G \end{matrix} \right\}, \tag{F.8}$$

where we have defined $\hat{G} = \sqrt{2G+1}$, etc. The object in curly brackets denotes a 6–j symbol that summarizes Clebsch–Gordan coefficients, cf. [3]. We obtain the above recoupling coefficient from (8.8) when we set $s_\phi = 0$ and $I_\phi = 1$. To derive the corresponding generalization from (F.8), we need to (i) replace the orbital angular momentum of the meson by its spin j, where $\boldsymbol{j} = \boldsymbol{L} + \boldsymbol{s}_\phi$, and (ii) introduce a label (I_ϕ) to identify its isospin. Hence, we need to consider the coupling scheme

$$\left\{ \left[(Ls_\phi)_j\, s \right]_J (I_\phi s)_I \right\}_G$$

in the laboratory frame. The subscripts refer to the quantum number to which the quantities in the respective parenthesis are coupled. Equation (F.6) tells us that the grand spin should now be the vector sum $\boldsymbol{j} + \boldsymbol{I}_\phi$. Hence, we need to generalize (F.7) to the coupling scheme

$$\left\{\left[(Ls_\phi)_j\, I_\phi\right]_G I\right\}_J$$

in the intrinsic frame. As before, the baryon quantum numbers do not explicitly appear in this scheme. This must be so because the corresponding S-matrix is computed from (8.3) which does not contain them either. Thus, the recoupling coefficient from (F.8) turns into

$$\left\langle \left\{\left[(Ls_\phi)_j\, s\right]_J (I_\phi s)_I\right\}_G \middle| \left\{\left[(Ls_\phi)_j\, I_\phi\right]_G I\right\}_J \right\rangle = (-1)^{L+s+J}\hat{G}\hat{s}\begin{Bmatrix} I & I_\phi & s \\ j & J & G \end{Bmatrix}. \tag{F.9}$$

Though this is the final result, it is not of the form encountered in (8.8). This is due to a different intermediate coupling scheme that introduces the total spin $\boldsymbol{S}_\mathrm{t} = \boldsymbol{s} + \boldsymbol{s}_\phi$ rather than $\boldsymbol{j}$ in the laboratory frame as well as the intermediate grand spin $\boldsymbol{K} = \boldsymbol{L} + \boldsymbol{I}_\phi$ in the intrinsic frame. That representation is straightforwardly written in terms of the above basis states, as it merely involves the definition and symmetries of 6–j symbols:

$$\begin{aligned}\left[(ss_\phi)_{S_\mathrm{t}}\, L\right]_J &= \sum_j (-1)^{2s+s_\phi+L+j}\, \hat{S}_\mathrm{t}\hat{j}\begin{Bmatrix} s & s_\phi & S_\mathrm{t} \\ L & J & j \end{Bmatrix}\left[(s_\phi L)_j\, s\right]_J, \\ \left[(LI_\phi)_K\, s_\phi\right]_G &= \sum_j (-1)^{2I_\phi+s_\phi}\, \hat{K}\hat{j}\begin{Bmatrix} I_\phi & L & K \\ s_\phi & G & j \end{Bmatrix}\left[(Ls_\phi)_j\, I\right]_G .\end{aligned} \tag{F.10}$$

Putting (F.9) and (F.10) together and utilizing the identity [3]

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{Bmatrix} = \sum_s (-1)^s \hat{s}\begin{Bmatrix} a & b & c \\ f & j & s \end{Bmatrix}\begin{Bmatrix} d & e & f \\ b & s & h \end{Bmatrix}\begin{Bmatrix} g & h & j \\ s & a & d \end{Bmatrix} \tag{F.11}$$

yields the result quoted in the main text (8.8).

Though we have merely considered two-flavor soliton models, the generalization to arbitrary isospin of the scattering meson grasps kaons as well. The only condition is that the target baryon has identical spin and isospin. In particular, we may read off the recoupling coefficient for kaon–nucleon scattering from (F.9) with $j = L$ because the kaons carry spin zero and $I_K = \frac{1}{2}$. These coefficients are important when computing the width of exotic baryons [4], which is discussed at length in Sect. 9.3.

F.2 Jost Function for Intrinsic Fluctuations

We will take the opportunity to sketch one out of several numerical techniques for the computation of the intrinsic S-matrix, $\widetilde{S}_G$, that appears in (8.7). We start from the second-order differential equation for radial functions $\eta_{GJ}(r)$

in the grand spin decomposition (8.5). We define the vector $\widetilde{\eta}_G$ whose entries are the radial functions $\eta_{GJ}(r)$, with all possible J values for a given G. This vector obeys a matrix differential equation of the general form

$$\left\{\mathbb{1}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}D_G^{(1)}(r)\frac{\mathrm{d}}{\mathrm{d}r}-\frac{1}{r^2}K_G+V_G(r)+k^2M(r)\right\}\widetilde{\eta}_G=0\,. \tag{F.12}$$

The coefficient functions are $n\times n$ matrices when n is the number of possible J values. These matrices stem from the Euler–Lagrange equation (8.3). They are block-diagonal with respect to the parity associated with a given channel, η_{GJ}. The radial dependences originate from the soliton profile that acts as background potential about which the fluctuations scatter. The matrices $D^{(1)}$ and M approach unity asymptotically. The matrix $K_G = \mathrm{diag}\left(J_1(J_1+1),\ldots,J_n(J_n+1)\right)$ contains the angular momentum part so that $V_G(r)$ vanishes faster than $1/r^2$ as $r\to\infty$. Furthermore, k is the momentum associated with the dispersion relation for the conserved energy, $\omega=\sqrt{k^2+m^2}$, where m is the meson mass.[1] The appearance of the coordinate-dependent metric function M is unconventional. In (8.13), we have argued it to cause the ever-rising phase shifts at large momenta in the Skyrme model of only pseudoscalar mesons [5]. In the baryon number zero sector, i.e., when all soliton profiles take their vacuum expectation values, we have $D^{(1)}\equiv\mathbb{1}$, $M\equiv\mathbb{1}$ and $V_G\equiv 0$.

In the next step, we elevate the n-component column vector $\widetilde{\eta}_G$ to an $n\times n$ matrix, $\mathcal{N}_G$: its columns contain the linearly independent solutions of the second-order differential equation (F.12). In particular, we may consider those that asymptotically behave like an incoming spherical wave in a given channel,

$$\mathcal{H}_G(kr)=\mathrm{diag}\left(h_{J_1}^{(2)}(kr),\ldots,h_{J_n}^{(2)}(kr)\right), \tag{F.13}$$

where $h_\ell^{(2)}(kr)$ are spherical Hankel functions associated with orbital angular momentum ℓ. Asymptotically, they behave as $h_\ell^{(2)}(z)\to(i^{\ell+1}/z)\mathrm{e}^{-\mathrm{i}z}$ when $z\to\infty$ [6]. Of course, $\mathcal{H}_G(kr)$ is the free solution and we may take it as a starting point for the exact solution by parameterizing,

$$\mathcal{N}_G(r)=\mathcal{F}_G(r)\cdot\mathcal{H}_G(kr)\,. \tag{F.14}$$

Essentially $\mathcal{F}_G(r)$ is the matrix analog of the Jost function in scattering theory. This $n\times n$ matrix is subject to the second-order differential equation

$$\frac{\mathrm{d}^2\mathcal{F}_G}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}\mathcal{F}_G}{\mathrm{d}r}\left(\mathbb{1}+rL_G\right)+\frac{2}{r}\left(D_G^{(1)}-\mathbb{1}\right)\left(\frac{\mathrm{d}\mathcal{F}_G}{\mathrm{d}r}+\mathcal{F}_GL_G\right)$$
$$-\frac{1}{r^2}\left[K_G,\mathcal{F}_G\right]+V_G\mathcal{F}_G+k^2\left(M-\mathbb{1}\right)\mathcal{F}_G(r)=0\,. \tag{F.15}$$

[1] If mesons with different masses ($m_1<m_2$) are involved, e.g., pions and vector mesons, k^2 must be considered as a diagonal matrix $\mathrm{diag}\left(\omega^2-m_1^2,\ldots,\omega^2-m_2^2\right)$. In the regime $m_1<\omega<m_2$, the associated k values are imaginary and the corresponding wave functions are those of bound states, thereby describing closed channels.

Here $L_G = \left(\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{H}_G\right) \cdot \mathcal{H}_G^{-1}$ contains the logarithmic derivatives of the Hankel functions on the diagonal. Equation (F.15) seems complicated but it is easy to see that $\mathcal{F}_G = \mathbb{1}$ is an asymptotic solution and also a solution when the background potentials vanish.

When the boundary condition $\mathcal{F}_G \to \mathbb{1}$ as $r \to \infty$ is imposed, the matrix $\mathcal{N}_G$ contains the solutions that behave as incoming spherical waves. Furthermore, the coefficient functions in the differential equation (F.12) are all real; thus the complex conjugate $\mathcal{N}_G^*$ is a solution as well. So n of the $2n$ linearly independent solutions are contained in $\mathcal{N}_G$ and the remaining n are in $\mathcal{N}_G^*$. Asymptotically, $\mathcal{N}_G^*$ are outgoing spherical waves. This suggests to parameterize the scattering solution as

$$\mathcal{N}_G^{(\mathrm{sc})} = \mathcal{N}_G + \mathcal{N}_G^* \cdot \widetilde{S}_G \,. \tag{F.16}$$

Obviously, $\widetilde{S}_G$ is the scattering matrix for the intrinsic fluctuations. We compute it from the wave functions in $\mathcal{F}_G$ by requiring that $\mathcal{N}_G^{(\mathrm{sc})}$ is regular at the origin, $r \to 0$. Hence,

$$\widetilde{S}_G = -\lim_{r\to 0}\left[\left(\mathcal{H}_G^*\right)^{-1} \cdot \left(\mathcal{F}_G^*\right)^{-1} \cdot \mathcal{F}_G \cdot \mathcal{H}_G\right] \tag{F.17}$$

provides the entry of (8.7). Technically, we integrate the differential equation (F.15) from sufficiently large $r = R_\infty$ with the boundary conditions $\mathcal{F}_G\big|_{r=R_\infty} = \mathbb{1}$ and $\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{F}_G\big|_{r=R_\infty} = 0$ to the inside and read off $\lim_{r\to 0}\mathcal{F}_G$ from the numerical integration to compute $\widetilde{S}_G$ via (F.17). The method described here is a variant of the variable phase approach, which is exhaustively discussed in [7].

For the application to the vacuum polarization energy in Sect. 8.6, we note that the leading term of $h_\ell^{(2)}(kr)$ as $r \to 0$ is purely imaginary. Therefore, $\lim_{r\to 0}\mathcal{H}_G(kr) \cdot \left(\mathcal{H}_G^*(kr)\right)^{-1} = -1$, and in each grand spin channel, the sum of eigen phase shifts that appears in (8.57) is obtained from

$$\delta_G^{\mathrm{tot}}(k) = \frac{1}{2\mathrm{i}}\lim_{r\to 0}\,\mathrm{tr}\left[\left(\mathcal{F}_G^*\right)^{-1} \cdot \mathcal{F}_G\right] \,. \tag{F.18}$$

Finally, let us mention that the approach based on the differential equation (F.15) is well suited to set up the Born series [8, 9]. To do so, let us introduce an artificial order parameter λ whose deviation from unity labels the interaction strength, i.e.,

$$D_G^{(1)} - \mathbb{1} = \mathcal{O}(\lambda)\,, \quad V_G = \mathcal{O}(\lambda) \quad \text{and} \quad M - \mathbb{1} = \mathcal{O}(\lambda)\,. \tag{F.19}$$

Then we expand

$$\mathcal{F}_G = \mathbb{1} + \lambda\mathcal{F}_G^{(1)} + \lambda^2\mathcal{F}_G^{(2)} + \cdots \tag{F.20}$$

and solve (F.15) order by order in λ. The boundary conditions are that all $\mathcal{F}_G^{(n)}$ and their derivatives vanish at spatial infinity. In this manner, the pieces in (F.19) act as source terms for $\mathcal{F}_G^{(1)}$, which in turn induces $\mathcal{F}_G^{(2)}$ and so on. At the end, we substitute the expansion (F.20) into (F.17) to extract the nth order of the Born series for the scattering matrix as the coefficient of λ^n in (F.17).

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