A

Principal Bundles, Vector Bundles and Connections

We recall in this Appendix some of the main definitions and concepts of the theory of principal bundles and their associated vector bundles, including the theory of connections in principal and vector bundles, exterior covariant derivatives, etc. which we shall need in order to introduce the Clifford and spin-Clifford bundles and to discuss some other issues in the main text. Propositions are in general presented without proofs, which can be found, e.g. in [1, 2, 3, 4, 5, 6, 7, 8, 9].

A.1 Fiber Bundles

Definition 535 *A fiber bundle over M with Lie group G will be denoted by (E, M, π, G, F). E is a topological space called the total space of the bundle, $\pi : E \to M$ is a continuous surjective map, called the canonical projection and F is the typical fiber. The following conditions must be satisfied:*

(a) *$\pi^{-1}(x)$, the fiber over x, is homeomorphic to F.*

(b) *Let $\{U_i,\ i \in \mathfrak{I}\}$, where $\mathfrak{I}$ is an index set, be a covering of M, such that:*

- *Locally a fiber bundle E is trivial, i.e. it is diffeomorphic to a product bundle, i.e. $\pi^{-1}(U_i) \simeq U_i \times F$ for all $i \in \mathfrak{I}$.*
- *The diffeomorphisms $\Phi_i : \pi^{-1}(U_i) \to U_i \times F$ have the form*

$$\Phi_i(p) = (\pi(p), \phi_{i,x}(p)) \ , \tag{A.1}$$

$$\phi_i\big|_{\pi^{-1}(x)} \equiv \phi_{i,x} : \pi^{-1}(x) \to F \text{ is onto}, \tag{A.2}$$

The collection $\{(U_i, \Phi_i)\}$, $i \in \mathfrak{I}$, are said to be a family of local trivializations for E.

- *The group G acts on the typical fiber. Let $x \in U_i \cap U_j$. Then,*

$$\phi_{i,x} \circ \phi_{j,x}^{-1} : F \to F \ , \tag{A.3}$$

must coincide with the action of an element of G for all $x \in U_i \cap U_j$ and $i, j \in \mathfrak{I}$.

W. A. Rodrigues and E. Capelas de Oliveira: *Principal Bundles, Vector Bundles and Connections*, Lect. Notes Phys. **722**, 407–431 (2007)
DOI 10.1007/978-3-540-71293-0_14

- *We call transition functions of the bundle the continuous induced mappings* (*see* Fig. A.1)

$$g_{ij} : U_i \cap U_j \to G, \text{ where } g_{ij}(x) = \phi_{i,x} \circ \phi_{j,x}^{-1}. \tag{A.4}$$

For consistence of the theory the transition functions must satisfy the cocycle condition

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x). \tag{A.5}$$

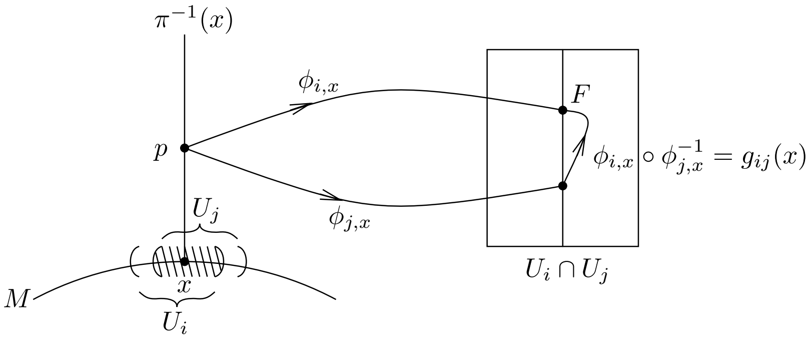

Fig. A.1. Transition functions on a fiber bundle

Definition 536 $(P, M, \pi, G, F \equiv G) \equiv (P, M, \pi, G)$ *is called a* (PFB) *if all conditions in* Definition 535 *are fulfilled and moreover, if there is a right action of* G *on elements* $p \in P$*, such that:*

(a) *the mapping* (*defining the right action*) $P \times G \ni (p, g) \mapsto pg \in P$ *is continuous.*

(b) *given* $g, g' \in G$ *and* $\forall p \in P$, $(pg)g' = p(gg')$.

(c) $\forall x \in M, \pi^{-1}(x)$ *is invariant under the action of* G*, i.e. each element of* $p \in \pi^{-1}(x)$ *is mapped into* $pg \in \pi^{-1}(x)$*, i.e. it is mapped into an element of the same fiber.*

(d) G *acts free and transitively on each fiber* $\pi^{-1}(x)$*, which means that all elements within* $\pi^{-1}(x)$ *are obtained by the action of all the elements of* G *on any given element of the fiber* $\pi^{-1}(x)$*. This condition is, of course necessary for the identification of the typical fiber with* G*.*

Definition 537 *A bundle* $(E, M, \pi_1, G = \mathrm{Gl}(m, \mathcal{F}), F = \mathbf{V})$*, where* $\mathcal{F} = \mathbb{R}$ *or* $\mathbb{C}$ (*respectively the real and complex fields*), $\mathrm{Gl}(m, \mathcal{F})$ *is the linear group, and* $\mathbf{V}$ *is an* m*-dimensional vector space over* $\mathcal{F}$ *is called a vector bundle.*

Definition 538 *A vector bundle* (E, M, π_1, G, F) *denoted* $E = P \times_\rho F$ *is said to be associated with a* PFB *bundle* (P, M, π, G) *by the linear representation* ρ *of* G *in* $F = \mathbf{V}$ (*a linear space of finite dimension over an appropriate* *field, which is called the* carrier space *of the representation) if its transition functions are the images under ρ of the corresponding transition functions of the* PFB (P, M, π, G)*. This means the following: consider the following local trivializations of P and E respectively*

$$\Phi_i : \pi^{-1}(U_i) \to U_i \times G, \tag{A.6}$$

$$\Xi_i : \pi_1^{-1}(U_i) \to U_i \times \boldsymbol{V}, \tag{A.7}$$

$$\Xi_i(q) = (\pi_1(q), \chi_i(q)) = (x, \chi_i(q)), \tag{A.8}$$

$$\chi_i|_{\pi_1^{-1}(x)} \equiv \chi_{i,x} : \pi_1^{-1}(x) \to \boldsymbol{V}, \tag{A.9}$$

where $\pi_1 : P \times_\rho \boldsymbol{V} \to M$ is the projection of the bundle associated with (P, M, π, G). Then, for all $x \in U_i \cap U_j$, $i, j \in \mathfrak{I}$, we have

$$\chi_{j,x} \circ \chi_{i,x}^{-1} = \rho(\phi_{j,x} \circ \phi_{i,x}^{-1}) . \tag{A.10}$$

In addition, the fibers $\pi^{-1}(x)$ are vector spaces isomorphic to the representation space $\boldsymbol{V}$.

Definition 539 *Let (E, M, π, G, F) be a fiber bundle and $U \subset M$ an open set. A local section of the fiber bundle (E, M, π, G, F) on U is a mapping*

$$s : U \to E \quad \text{such that} \quad \pi \circ s = Id_U. \tag{A.11}$$

If $U = M$ we say that s is a global section.

Remark 540 *There is a relation between sections and local trivializations for principal bundles. Indeed, each local section s, (on $U_i \subset M$) for a principal bundle (P, M, π, G) determines a local trivialization $\Phi_i : \pi^{-1}(U) \to U \times G$, of P by setting*

$$\Phi_i^{-1}(x, g) = s(x)g = pg = R_g p. \tag{A.12}$$

Conversely, Φ_i determines s since

$$s(x) = \Phi_i^{-1}(x, e). \tag{A.13}$$

Proposition 541 *A principal bundle is trivial, if and only if, it has a global cross section.*

Proposition 542 *A vector bundle is trivial, if and only if, its associated principal bundle is trivial*

Proposition 543 *Any fiber bundle (E, M, π, G, F) such that M is a paracompact manifold and the fiber F is a vector space admits a cross section*

Remark 544 *Then, any vector bundle associated with a trivial principal bundle has non zero global sections. Note however that a vector bundle may admit a non zero global section even if it is not trivial. Indeed, any Clifford bundle* *possess a global identity section, and some spin-Clifford bundles admits also identity sections once a trivialization is given* (*see* Chap. 6).

Definition 545 *The structure group G of a fiber bundle (E, M, π, G, F) is reducible to G' if the bundle admits an equivalent structure defined with a subgroup G' of the structure group G. More precisely, this means that the fiber bundle admits a family of local trivializations such that the transition functions takes values in G', i.e. $g_{ij} : U_i \cap U_j \to G'$.*

A.1.1 Frame Bundle

The tangent bundle TM to a differentiable n-dimensional manifold M is an associated bundle to a principal bundle called the frame bundle $F(M) = \bigcup_{x \in M} F_x M$, where $F_x M$ is the set of frames at $x \in M$. The structure group of $F(M)$ is $\mathrm{Gl}(n, \mathbb{R})$. Let $\{x^i\}$ be the coordinates associated with a local chart (U_i, φ_i) of the maximal atlas of M. Then, $T_x M$ has a natural basis $\{ \frac{\partial}{\partial x^i}\big|_x \}$ on $U_i \subset M$.

Definition 546 *A frame at $T_x M$ is a set $\boldsymbol{\Sigma}_x = \{ e_1|_x, ..., e_n|_x \}$ of linearly independent vectors such that*

$$e_i|_x = F_i^j \left. \frac{\partial}{\partial x^j} \right|_x , \tag{A.14}$$

and where the matrix (F_i^j) with entries $A_i^j \in \mathbb{R}$, belongs to the the real general linear group in n dimensions $\mathrm{Gl}(n, \mathbb{R})$. *We write $(F_i^j) \in \mathrm{Gl}(n, \mathbb{R})$.*

A local trivialization $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathrm{Gl}(n, \mathbb{R})$ of $F(M)$ is defined by

$$\phi_i(f) = (x, \boldsymbol{\Sigma}_x), \ \pi(f) = x \, . \tag{A.15}$$

The action of $a = (a_i^j) \in \mathrm{Gl}(n, \mathbb{R})$ on a frame $f \in F(U)$ is given by $(f, a) \to fa$, where the new frame $fa \in F(U)$ is defined by $\phi_i(fa) = (x, \boldsymbol{\Sigma'}_x)$, $\pi(f) = x$, and

$$\begin{aligned} \boldsymbol{\Sigma'}_x &= \{ e'_1|_x , ..., e'_n|_x \} , \\ e'_i|_x &= e_j|_x \, a_i^j \, . \end{aligned} \tag{A.16}$$

Conversely, given frames $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma'}_x$ there exists $a = (a_i^j) \in \mathrm{Gl}(n, \mathbb{R})$ such that (A.16) is satisfied, which means that $\mathrm{Gl}(n, \mathbb{R})$ acts on $F(M)$ actively.

Let $\{x^i\}$ and $\{\bar{x}^i\}$be the coordinates associated with the local charts (U_i, φ_i) and (U'_i, φ'_i) and of the maximal atlas of M. If $x \in U_i \cap U_j$ we have

$$\begin{aligned} e_i|_x &= F_i^j \left. \frac{\partial}{\partial x^j} \right|_x = \bar{F}_i^j \left. \frac{\partial}{\partial \bar{x}^j} \right|_x , \\ (F_i^j), (\bar{F}_i^j) &\in \mathrm{Gl}(n, \mathbb{R}) \, . \end{aligned} \tag{A.17}$$

Since $F_i^j = \bar{F}_k^j \left(\frac{\partial x^k}{\partial \bar{x}^i}\right)\Big|_x$ we have that the transition functions are

$$g_i^k(x) = \left(\frac{\partial x^k}{\partial \bar{x}^i}\right)\Bigg|_x \in \mathrm{Gl}(n, \mathbb{R}) \, . \tag{A.18}$$

Remark 547 *Given $U \subset M$ we shall also denote by $\boldsymbol{\Sigma} \in \sec F(U)$ a section of $F(U) \subset F(M)$. This means that given a local trivialization $\phi : \pi^{-1}(U) \to U \times \mathrm{Gl}(n, \mathbb{R})$, $\phi(\boldsymbol{\Sigma}) = (x, \boldsymbol{\Sigma}_x)$, $\pi(\boldsymbol{\Sigma}) = x$. Sometimes, we also use the sloppy notation $\{e_i\} \in \sec F(U)$ or even $\{e_i\} \in \sec F(M)$ when the context is clear. Moreover, we recall that a section of $F(U)$ is also called a moving frame for $\mathcal{H}(U)$, the module of differentiable vector fields on U.*

A.1.2 Orthonormal Frame Bundle

Suppose that the manifold M is equipped with a metric field $\mathbf{g} \in \sec T_2^0 M$ of signature (p, q), $p+q = n$. Then, we can introduce *orthonormal* frames in each $T_x U$, . In this case we denote an orthonormal frame by $\boldsymbol{\Sigma}_x = \{\, \mathbf{e}_1|_x \,, ..., \, \mathbf{e}_n|_x\}$ and

$$\mathbf{e}_i|_x = h_i^j \left.\frac{\partial}{\partial x^j}\right|_x \, , \tag{A.19}$$

$$\mathbf{g}(\,\mathbf{e}_i|_x \,, \, \mathbf{e}_j|_x)\big|_x = \mathrm{diag}(1, 1, ...1, -1, ..., -1), \tag{A.20}$$

with $(h_i^j) \in \mathrm{O}_{p,q}$, the real orthogonal group in n dimensions. In this case we say that the frame bundle has been reduced to the *orthonormal frame bundle*, which will be denoted by $\mathbf{P}_{\mathrm{O}n}(M)$. A section $\boldsymbol{\Sigma} \in \sec \mathbf{P}_{\mathrm{O}n}(U)$ is called a *vierbein.*

Remark 548 *The principal bundle of oriented orthonormal frames $\mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$ over a Lorentzian manifold modeling spacetime and its covering bundle called spin bundle $\mathbf{P}_{\mathrm{Spin}_{1,3}^e}(M)$ discussed in* Chap. 6 *play an important role in this book. Also, vector bundles associated these bundles are very important. Associated with $\mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$ we have the tensor bundle, the exterior bundle and the Clifford bundle. Associated with $\mathbf{P}_{\mathrm{Spin}_{1,3}^e}(M)$ we have several spinor bundles, in particular the spin-Clifford bundle, whose sections are the DHSF. All those bundles and their relationship are studied in* Chap. 6.

Remark 549 *In complete analogy to the construction of orthonormal frame bundle we may define an orthonormal coframe bundle that may be denoted by $P_{\mathrm{O}n}(M)$. Since to each given frame $\boldsymbol{\Sigma} \in \sec \mathbf{P}_{\mathrm{O}n}(M)$ there is a natural coframe field $\Sigma \in \sec P_{\mathrm{O}n}(M)$, the one where the covectors are the duals of the vectors of the frame. It follows that $P_{\mathrm{O}n}(M) \simeq \mathbf{P}_{\mathrm{O}n}(M)$. In particular $P_{\mathrm{SO}_{1,3}^e}(M) \simeq \mathbf{P}_{\mathrm{SO}_{1,3}^e}(M)$.*

A.2 Product Bundles and Whitney Sum

Given two vector bundles $(E, M, \pi, G, \boldsymbol{V})$ and $(E', M', \pi', G', \boldsymbol{V}')$ we have the definitions:

Definition 550 *The product bundle $E \times E'$ is a fiber bundle whose basis space is $M \times M'$, the typical fiber is $\boldsymbol{V} \oplus \boldsymbol{V}'$, the structural group of $E \times E'$ acts separately as G and G' in each one of the components of $\boldsymbol{V} \oplus \boldsymbol{V}'$ and the projection $\pi \times \pi'$ is such that $E \times E' \overset{\pi\times\pi'}{\to} M \times M'$.*

Definition 551 *Given two vector bundles over the same basis space, i.e. $(E, M, \pi, G, \boldsymbol{V})$ and $(E', M, \pi', G', \boldsymbol{V}')$, the Whitney sum bundle $E \oplus E'$ is the pullback of $E \times E'$ by $h : M \to M \times M$, $h(p) = (p, p)$.*

Definition 552 *Given vector bundles $(E, M, \pi, G, \boldsymbol{V})$ and $(E', M, \pi', G', \boldsymbol{V}')$ over the same basis space, the tensor product bundle $E \otimes E'$ is the bundle obtained from E and E' by assigning the tensor product of fibers $\pi_x^{-1} \otimes \pi_x'^{-1}$ for all $x \in M$.*

Remark 553 *With the above definitions we can easily show that given three vector bundles, say, E, E', E'' we have*

$$E \oplus (E' \otimes E'') = (E \otimes E') \oplus (E \otimes E'') \,. \tag{A.21}$$

A.3 Connections

A.3.1 Equivalent Definitions of a Connection in Principal Bundles

To define the concept of a *connection* on a PFB (P, M, π, G), we recall that since $\dim(M) = m$, if $\dim(G) = n$, then $\dim(P) = n + m$. Obviously, for all $x \in M$, $\pi^{-1}(x)$ is an n-dimensional submanifold of P diffeomorphic to the structure group G and π is a submersion, $\pi^{-1}(x)$ is a closed submanifold of P for all $x \in M$.

The tangent space T_pP, $p \in \pi^{-1}(x)$, is an $(n + m)$-dimensional vector space and the tangent space $V_pP \equiv T_p(\pi^{-1}(x))$ to the fiber over x at the same point $p \in \pi^{-1}(x)$ is an n-dimensional linear subspace of T_pP called the *vertical subspace* of T_pP[1].

[1] Here we may be tempted to realize that as it is possible to construct the vertical space for all $p \in P$ then we can define a horizontal space as the complement of this space in respect to T_pP. Unfortunately this is not so, because we need a smoothly association of a horizontal space in every point. This is possible only by means of a connection.

Now, roughly speaking a connection on P is a rule that makes possible a *correspondence* between any two fibers along a curve $\sigma : \mathbb{R} \supseteq I \to M, t \mapsto \sigma(t)$. If p_0 belongs to the fiber over the point $\sigma(t_0) \in \sigma$, we say that p_0 is parallel translated along σ by means of this *correspondence*.

Definition 554 *A horizontal lift of* σ *is a curve* $\hat{\sigma} : \mathbb{R} \supseteq I \to P$ *(described by the parallel transport of* p*).*

It is intuitive that such a transport takes place in P along directions specified by vectors in T_pP, which do not lie within the vertical space V_pP. Since the tangent vectors to the paths of the basic manifold passing through a given $x \in M$ span the entire tangent space T_xM, the corresponding vectors $Y_p \in T_pP$ (in whose direction parallel transport can generally take place in P) span a n-dimensional linear subspace of T_pP called the *horizontal space* of T_pP and denoted by H_pP. Now, the mathematical concept of a connection can be presented. This is done through three equivalent definitions given below which encode rigorously the intuitive discussion given above. We have the following definitions:

Definition 555 *A connection on a* PFB (P, M, π, G) *is an assignment to each* $p \in P$ *of a subspace* $H_pP \subset T_pP$*, called the horizontal subspace for that connection, such that* H_pP *depends smoothly on* p *and the following conditions hold:*

(i) $\pi_* : H_pP \to T_xM$, $x = \pi(p)$, *is an isomorphism.*
(ii) H_pP *depends smoothly on* p*.*
(iii) $(R_g)_* H_pP = H_{pg}P, \forall g \in G, \forall p \in P$.

Here we denote by π_* the *differential* of the mapping π and by $(R_g)_*$ the differential of the mapping $R_g : P \to P$ (the right action) defined by $R_g(p) = pg$.

Since $x = \pi(\hat{\sigma}(t))$ for any curve in P such that $\hat{\sigma}(t) \in \pi^{-1}(x)$ and $\hat{\sigma}(0) = p_0$, we conclude that π_* maps all vertical vectors in the zero vector in T_xM, i.e. $\pi_*(V_pP) = 0$ and we have[2],

$$T_pP = H_pP \oplus V_pP \ . \tag{A.22}$$

Then every $\mathbf{Y}_p \in T_pP$ can be written as

$$\mathbf{Y}_p = \mathbf{Y}_p^h + \mathbf{Y}_p^v, \qquad \mathbf{Y}_p^h \in H_pP, \qquad \mathbf{Y}_p^v \in V_pP \ . \tag{A.23}$$

Therefore, given a vector field Y over M it is possible to lift it to a horizontal vector field over P, i.e. $\pi_*(\mathbf{Y}_p) = \pi_*(\mathbf{Y}_p^h) = Y_x \in T_xM$ for all $p \in P$ with $\pi(p) = x$. In this case, we call $\mathbf{Y}_p^h$ the *horizontal lift* of Y_x. We say moreover that $\mathbf{Y}$ is a horizontal vector field over P if $\mathbf{Y}^h = Y$.

[2] We also write $TP = HP \oplus VP$.

Definition 556 *A* connection *on a* PFB (P, M, π, G) *is a mapping* $\varGamma_p : T_xM \to T_pP$*, such that* $\forall p \in P$ *and* $x = \pi(p)$ *the following conditions hold:*
(i) $\varGamma_p$ *is linear.*
(ii) $\pi_* \circ \varGamma_p = Id_{T_xM}$.
(iii) *the mapping* $p \mapsto \varGamma_p$ *is differentiable.*
(iv) $\varGamma_{R_gp} = (R_g)_*\varGamma_p$*, for all* $g \in G$*.*

We need also the concept of parallel transport. It is given by the following definition:

Definition 557 *Let* $\sigma : \mathbb{R} \supset I \to M$*,* $t \mapsto \sigma(t)$ *with* $x_0 = \sigma(0) \in M$*, be a curve in* M *and let* $p_0 \in P$ *such that* $\pi(p_0) = x_0$*. The parallel transport of* p_0 *along* σ *is given by the curve* $\hat{\sigma} : \mathbb{R} \supset I \to P, t \mapsto \hat{\sigma}(t)$ *defined by*

$$\frac{d}{dt}\hat{\sigma}(t) = \varGamma_p(\frac{d}{dt}\sigma(t)) \ , \tag{A.24}$$

with $p_0 = \hat{\sigma}(0)$ *and* $\hat{\sigma}(t) = p_{\|t}$*,* $\pi(p_{\|t}) = x$*.*

In order to present yet a *third* definition of a connection we need to know more about the nature of the vertical space V_pP. For this, let $\mathfrak{Y} \in T_eG = \mathfrak{G}$ be an element of the Lie algebra $\mathfrak{G}$ of G. The vector $\mathfrak{Y}$ is the tangent to the curve produced by the exponential map

$$\mathfrak{Y} = \left.\frac{d}{dt}\left(\exp(t\mathfrak{Y})\right)\right|_{t=0} . \tag{A.25}$$

Then, for every $p \in P$ we can attach to each $\mathfrak{Y} \in T_eG = \mathfrak{G}$ a unique element $\mathbf{Y}_p^v \in V_pP$ as follows: let $f : (-\varepsilon, \varepsilon) \to P$, $t \mapsto p \exp t\mathfrak{Y}$ be a curve on P. Observe that it is obtained by right translation and then $\pi(p) = \pi(p \exp t\mathfrak{Y}) = x$ and so the curve lies in $\pi^{-1}(x)$ the fiber over $x \in M$. Next let $\mathfrak{F} : P \to \mathbb{R}$ be a smooth function. Then we define

$$\mathbf{Y}_p^v\mathfrak{F}(\mathfrak{p}) = \left.\frac{d}{dt}\mathfrak{F}\left(p\exp(t\mathfrak{Y})\right)\right|_{t=0} . \tag{A.26}$$

By this construction we attach to each $\mathfrak{Y} \in T_eG = \mathfrak{G}$ a unique vector field over P, called the fundamental field corresponding to this element. We then have the canonical isomorphism

$$\mathbf{Y}_p^v \longleftrightarrow \mathfrak{Y}, \quad \mathbf{Y}_p^v \in V_pP, \quad \mathfrak{Y} \in T_eG = \mathfrak{G} \ , \tag{A.27}$$

from which we get

$$V_pP \simeq \mathfrak{G} \ . \tag{A.28}$$

Definition 558 *A connection on a* PFB (P, M, π, G) *is a* 1*-form field* $\boldsymbol{\omega}$ *on* P *with values in the Lie algebra* $\mathfrak{G} = T_eG$ *such that* $\forall p \in P$ *we have,*
(i) $\boldsymbol{\omega}_p(\mathbf{Y}_p^v) = \mathfrak{Y}$ *and* $\mathbf{Y}_p^v \longleftrightarrow \mathfrak{Y}$*, where* $\mathbf{Y}_p^v \in V_pP$ *and* $\mathfrak{Y} \in T_eG = \mathfrak{G}$*.*
(ii) $\boldsymbol{\omega}_p$ *depends smoothly on* p*.*
(iii)$\boldsymbol{\omega}_p[(R_g)_*\mathbf{Y}_p] = (Ad_{g^{-1}}\boldsymbol{\omega}_p)(\mathbf{Y}_p)$*, where* $Ad\boldsymbol{\omega}_p = g^{-1}\boldsymbol{\omega}_p g$*.*

It follows that if $\{\mathcal{G}_a\}$ is a basis of $\mathfrak{G}$ and $\{\theta^i\}$ is a basis for T^*P then

$$\boldsymbol{\omega}_p = \omega_p^a \otimes \mathcal{G}_a = \omega_i^a(p)\theta_p^i \otimes \mathcal{G}_a \ , \tag{A.29}$$

where ω^a are 1-forms on P.

Then the horizontal spaces can be defined by

$$H_pP = \ker(\boldsymbol{\omega}_p) \ , \tag{A.30}$$

which shows the equivalence between the definitions.

A.3.2 The Connection on the Base Manifold

Definition 559 *Let $U \subset M$ and*

$$s : U \to \pi^{-1}(U) \subset P, \qquad \pi \circ s = Id_U \ , \tag{A.31}$$

be a local section of the PFB (P, M, π, G).

Definition 560 *Let $\boldsymbol{\omega}$ be a connection on P. The 1-form $s^*\boldsymbol{\omega}$ (the pullback of $\boldsymbol{\omega}$ under s) given by*

$$(s^*\boldsymbol{\omega})_x(Y_x) = \boldsymbol{\omega}_{s(x)}(s_*Y_x), \ Y_x \in T_xU, \ s_*Y_x \in T_pP, \quad p = s(x) \ , \tag{A.32}$$

is called the local gauge potential.

It is quite clear that $s^*\boldsymbol{\omega} \in \sec T^*U \otimes \mathfrak{G}$. This object differs from the *gauge field* used by physicists by numerical constants (with units). Conversely we have the following proposition:

Proposition 561 *Given $\omega \in \sec T^*U \otimes \mathfrak{G}$ and a differentiable section of $\pi^{-1}(U) \subset P$, $U \subset M$, there exists one and only one connection $\boldsymbol{\omega}$ on $\pi^{-1}(U)$ such that $s^*\boldsymbol{\omega} = \omega$.*

Consider now

$$\begin{aligned} \omega \in T^*U \otimes \mathfrak{G}, \quad \omega = (\Phi^{-1}(x,e))^*\boldsymbol{\omega} = s^*\boldsymbol{\omega}, \quad s(x) = \Phi^{-1}(x,e) \ , \\ \omega' \in T^*U' \otimes \mathfrak{G}, \quad \omega' = (\Phi'^{-1}(x,e))^*\boldsymbol{\omega} = s'^*\boldsymbol{\omega}, \quad s'(x) = \Phi'^{-1}(x,e). \end{aligned} \tag{A.33}$$

Then we can write, for each $p \in P$ ($\pi(p) = x$), parameterized by the local trivializations Φ and Φ' respectively as (x, g) and (x, g') with $x \in U \cap U'$, that

$$\omega_p = g^{-1}dg + g^{-1}\omega_x g = g'^{-1}dg' + g'^{-1}\omega'_x g' \ . \tag{A.34}$$

Now, if

$$g' = hg \ , \tag{A.35}$$

we immediately get from (A.34) that

$$\bar{\omega}'_x = hdh^{-1} + h\bar{\omega}_x h^{-1} \ , \tag{A.36}$$

which can be called the *transformation law* for the gauge fields.

A.4 Exterior Covariant Derivatives

Let $\bigwedge^k(P,\mathfrak{G}) = \bigwedge^k T^*P \otimes \mathfrak{G}, 0 \leq k \leq n$, be the set of all k-form fields over P with values in the Lie algebra $\mathfrak{G}$ of the gauge group G (and, of course, the connection $\boldsymbol{\omega} \in \sec \bigwedge^1(P,\mathfrak{G})$).

Definition 562 *For each $\boldsymbol{\varphi} \in \sec \bigwedge^k(P,\mathfrak{G})$ we define the so-called* horizontal form $\boldsymbol{\varphi}^h \in \sec \bigwedge^k(P,\mathfrak{G})$ *by*

$$\boldsymbol{\varphi}_p^h(\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_k) = \boldsymbol{\varphi}(\mathbf{X}_1^h,\mathbf{X}_2^h,...,\mathbf{X}_k^h)\ , \tag{A.37}$$

where $\mathbf{X}_i \in T_pP$, $i = 1,2,..,k$.

Notice that $\boldsymbol{\varphi}_p^h(\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_k) = 0$ if one (or more) of the $\mathbf{X}_i \in T_pP$ are vertical.

Definition 563 $\boldsymbol{\varphi} \in \sec \bigwedge^k T^*P \otimes \mathbf{V}$ *(where $\mathbf{V}$ is a vector space) is said to be horizontal if $\boldsymbol{\varphi}_p(\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_k) = 0$, implies that at least one of the $\mathbf{X}_i \in T_pP$, $i = 1,2,\ldots,k$ is vertical.*

Definition 564 $\boldsymbol{\varphi} \in \sec \bigwedge^k T^*P \otimes \mathbf{V}$ *is said to be of type $(\rho,\mathbf{V})$ if $\forall g \in G$ we have*

$$R_g^*\boldsymbol{\varphi} = \rho(g^{-1})\boldsymbol{\varphi}\ . \tag{A.38}$$

Definition 565 *Let $\boldsymbol{\varphi} \in \sec \bigwedge^k T^*P \otimes \mathbf{V}$ be horizontal. Then, $\boldsymbol{\varphi}$ is said to be tensorial of type $(\rho,\mathbf{V})$.*

Definition 566 *The exterior covariant derivative of $\boldsymbol{\varphi} \in \sec \bigwedge^k(P,\mathfrak{G})$ in relation to the connection ω is*

$$D^{\boldsymbol{\omega}}\boldsymbol{\varphi} = (d\boldsymbol{\varphi})^h \in \sec \bigwedge^{k+1}(P,\mathfrak{G})\ , \tag{A.39}$$

where $D^{\boldsymbol{\omega}}\boldsymbol{\varphi}_p(\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_k,\mathbf{X}_{k+1}) = d\boldsymbol{\varphi}_p(\mathbf{X}_1^h,\mathbf{X}_2^h,...,\mathbf{X}_k^h,\mathbf{X}_{k+1}^h)$. Notice that $d\boldsymbol{\varphi} = d\boldsymbol{\varphi}^a \otimes \mathcal{G}_a$ where $\boldsymbol{\varphi}^a \in \sec \bigwedge^k(T^*P)$, $a = 1,2,...,n$.

Definition 567 *The commutator of $\boldsymbol{\varphi} \in \sec \bigwedge^i(P,\mathfrak{G})$ and $\boldsymbol{\psi} \in \sec \bigwedge^j(P,\mathfrak{G})$, $0 \leq i,j \leq n$, denoted by $[\boldsymbol{\varphi},\boldsymbol{\psi}] \in \sec \bigwedge^{i+j}(P,\mathfrak{G})$ such that if $\mathbf{X}_1,...,\mathbf{X}_{i+j} \in \sec TP$, then*

$$\begin{aligned}&[\boldsymbol{\varphi},\boldsymbol{\psi}](\mathbf{X}_1,...,\mathbf{X}_{i+j})\\ &= \frac{1}{i!j!}\sum_{\sigma\in\mathcal{S}_n}(-1)^\sigma[\boldsymbol{\varphi}(\mathbf{X}_{\iota(1)},...,\mathbf{X}_{\iota(i)}),\boldsymbol{\psi}(\mathbf{X}_{\iota(i+1)},...,\mathbf{X}_{\iota(i+j)})],\end{aligned} \tag{A.40}$$

where $\mathcal{S}_n$ is the permutation group of n elements and $(-1)^\sigma = \pm 1$ is the sign of the permutation. The brackets [,] *in the second member of* (A.40) *are the Lie brackets in $\mathfrak{G}$.*

Writing

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^a \otimes \mathcal{G}_a, \quad \boldsymbol{\psi} = \boldsymbol{\psi}^a \otimes \mathcal{G}_a, \quad \boldsymbol{\varphi}^a \in \sec \bigwedge^i T^*P, \quad \boldsymbol{\psi}^a \in \sec \bigwedge^j T^*P\ , \tag{A.41}$$

we have[3]

$$\begin{aligned} [\boldsymbol{\varphi}, \boldsymbol{\psi}] &= \boldsymbol{\varphi}^a \wedge \boldsymbol{\psi}^b \otimes [\mathcal{G}_a, \mathcal{G}_b] \\ &= f_{ab}^c (\boldsymbol{\varphi}^a \wedge \boldsymbol{\psi}^b) \otimes \mathcal{G}_c, \end{aligned} \tag{A.42}$$

where f_{ab}^c are the structure constants of the Lie algebra.

With (A.42) we can prove *easily* the following important properties involving commutators:

$$[\boldsymbol{\varphi}, \boldsymbol{\psi}] = (-1)^{1+ij} [\boldsymbol{\psi}, \boldsymbol{\varphi}]\ , \tag{A.43}$$

$$(-1)^{ik} [[\boldsymbol{\varphi}, \boldsymbol{\psi}], \boldsymbol{\tau}] + (-1)^{ji} [[\boldsymbol{\psi}, \boldsymbol{\tau}], \boldsymbol{\varphi}] + (-1)^{kj} [[\boldsymbol{\tau}, \boldsymbol{\varphi}], \boldsymbol{\psi}] = 0\ , \tag{A.44}$$

$$d[\boldsymbol{\varphi}, \boldsymbol{\psi}] = [d\boldsymbol{\varphi}, \boldsymbol{\psi}] + (-1)^i [\boldsymbol{\varphi}, d\boldsymbol{\psi}], \tag{A.45}$$

for $\boldsymbol{\varphi} \in \sec \bigwedge^i (P, \mathfrak{G})$, $\boldsymbol{\psi} \in \sec \bigwedge^j (P, \mathfrak{G})$, $\boldsymbol{\tau} \in \sec \bigwedge^k (P, \mathfrak{G})$.

We shall also need the following identity

$$[\boldsymbol{\omega}, \boldsymbol{\omega}](\mathbf{X_1}, \mathbf{X_2}) = 2[\boldsymbol{\omega}(\mathbf{X}_1), \boldsymbol{\omega}(\mathbf{X}_2)]\ . \tag{A.46}$$

The proof of (A.46) is as follows:

(i) Recall that

$$[\boldsymbol{\omega}, \boldsymbol{\omega}] = (\boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b) \otimes [\mathcal{G}_a, \mathcal{G}_b]\ . \tag{A.47}$$

(ii) Let $\mathbf{X}_1, \mathbf{X}_2 \in \sec TP$ (i.e. $\mathbf{X}_1$ and $\mathbf{X}_2$ are vector fields on P). Then,

$$\begin{aligned} [\boldsymbol{\omega}, \boldsymbol{\omega}](\mathbf{X_1}, \mathbf{X_2}) &= (\boldsymbol{\omega}^a(\mathbf{X}_1)\boldsymbol{\omega}^b(\mathbf{X_2}) - \boldsymbol{\omega}^a(\mathbf{X_2})\boldsymbol{\omega}^b(\mathbf{X_1}))[\mathcal{G}_a, \mathcal{G}_b] \\ &= 2[\boldsymbol{\omega}(\mathbf{X}_1), \boldsymbol{\omega}(\mathbf{X}_2)]\ . \end{aligned} \tag{A.48}$$

Definition 568 *The curvature form of the connection $\boldsymbol{\omega} \in \sec \bigwedge^1 (P, \mathfrak{G})$ is $\boldsymbol{\Omega}^{\boldsymbol{\omega}} \in \sec \bigwedge^2 (P, \mathfrak{G})$ defined by*

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}} \boldsymbol{\omega}. \tag{A.49}$$

Definition 569 *The connection $\boldsymbol{\omega}$ is said to be flat if $\boldsymbol{\Omega}^{\boldsymbol{\omega}} = 0$.*

[3] In this Appendix in order to obtain formulas that can be easily compared with the ones appearing in standard texts we use the exterior product as defined in Remark 22. This, we hope, generates no confusion.

Proposition 570

$$D^{\omega}\boldsymbol{\omega}(\mathbf{X_1},\mathbf{X_2}) = d\boldsymbol{\omega}(\mathbf{X}_1,\mathbf{X}_2) + [\boldsymbol{\omega}(\mathbf{X}_1),\boldsymbol{\omega}(\mathbf{X}_2)] \ . \tag{A.50}$$

(A.50) *can be written using* (A.48) (*and recalling that* $\omega(\mathbf{X}) = \omega^a(\mathbf{X})\mathcal{G}_a$) *as*

$$\boldsymbol{\Omega}^{\omega} = D^{\omega}\boldsymbol{\omega} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}] \ . \tag{A.51}$$

Proof. See [1]. □

Proposition 571 (Bianchi identities):

$$D\boldsymbol{\Omega}^{\omega} = 0 \ . \tag{A.52}$$

Proof. (i) Let us calculate $d\boldsymbol{\Omega}^{\omega}$. We have,

$$d\boldsymbol{\Omega}^{\omega} = d\left(d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}]\right) \ . \tag{A.53}$$

We now take into account that $d^2\boldsymbol{\omega} = 0$ and that from the properties of the commutators given by (A.43), (A.44), (A.45) above, we have

$$\begin{aligned} d[\boldsymbol{\omega},\boldsymbol{\omega})] &= [d\boldsymbol{\omega},\boldsymbol{\omega}] - [\boldsymbol{\omega},d\boldsymbol{\omega}], \\ [d\boldsymbol{\omega},\boldsymbol{\omega}] &= -[\boldsymbol{\omega},d\boldsymbol{\omega}] \ , \\ [[\boldsymbol{\omega},\boldsymbol{\omega}],\boldsymbol{\omega}] &= 0 \ . \end{aligned} \tag{A.54}$$

By using (A.54) in (A.53) gives

$$d\boldsymbol{\Omega}^{\omega} = [d\boldsymbol{\omega},\boldsymbol{\omega}] \ . \tag{A.55}$$

(ii) In (A.55) use (A.51) and the last equation in (A.54) to obtain

$$d\boldsymbol{\Omega}^{\omega} = [\boldsymbol{\Omega}^{\omega},\boldsymbol{\omega}] \ . \tag{A.56}$$

(iii) Use now the definition of the exterior covariant derivative [(A.39)] together with the fact that $\boldsymbol{\omega}(\mathbf{X^h}) = 0$, for all $\mathbf{X} \in T_pP$ to obtain

$$D^{\omega}\boldsymbol{\Omega}^{\omega} = 0 \ ,$$

that proves the proposition. □

We can then write the very important formula (known as the Bianchi identity),

$$D^{\omega}\boldsymbol{\Omega}^{\omega} = d\boldsymbol{\Omega}^{\omega} + [\boldsymbol{\omega},\boldsymbol{\Omega}^{\omega}] = 0 \ . \tag{A.57}$$

A.4.1 Local curvature in the Base Manifold M

Let (U, Φ) be a local trivialization of $\pi^{-1}(U)$ and s the associated cross section as defined above. Then, $s^*\boldsymbol{\Omega}^{\boldsymbol{\omega}} := \Omega^\omega$ (the pullback of $\boldsymbol{\Omega}^{\boldsymbol{\omega}}$) is a well defined 2-form field on U which takes values in the Lie algebra $\mathfrak{G}$. Let $\omega = s^*\boldsymbol{\omega}$ (see (A.33)). If we recall that the differential operator d commutes with the pullback, we immediately get

$$\Omega^\omega = s^* D^{\boldsymbol{\omega}}\boldsymbol{\omega} = d\omega + \frac{1}{2}[\omega,\omega] \ . \tag{A.58}$$

It is convenient to define the symbols

$$\mathbf{D}\omega := s^* D^{\boldsymbol{\omega}}\boldsymbol{\omega} \ , \tag{A.59}$$

$$\mathbf{D}\Omega^\omega := s^* D^{\boldsymbol{\omega}}\boldsymbol{\Omega}^{\boldsymbol{\omega}}, \tag{A.60}$$

and to write

$$\begin{aligned} \mathbf{D}\Omega^\omega &= 0 \ , \\ \mathbf{D}\Omega^\omega &= d\Omega^\omega + [\omega,\Omega^\omega] = 0. \end{aligned} \tag{A.61}$$

Equation (A.61) is *also* known as Bianchi identity.

Remark 572 *In gauge theories (Yang-Mills theories) Ω^ω is (except for numerical factors with physical units) called a* field strength *in the gauge Φ.*

Remark 573 *When G is a matrix group, as is the case in the presentation of gauge theories by physicists,* Definition 567 *of the commutator $[\boldsymbol{\varphi}, \boldsymbol{\psi}] \in$* sec $\bigwedge^{i+j}(P, \mathfrak{G})$ *($\boldsymbol{\varphi} \in$* sec $\bigwedge^{i}(P, \mathfrak{G})$, *$\boldsymbol{\psi} \in$* sec $\bigwedge^{j}(P, \mathfrak{G})$*) gives*

$$[\boldsymbol{\varphi}, \boldsymbol{\psi}] = \boldsymbol{\varphi} \wedge \boldsymbol{\psi} - (-1)^{ij}\boldsymbol{\psi} \wedge \boldsymbol{\varphi} \ , \tag{A.62}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are considered as matrices of forms with values in $\mathbb{R}$ and $\boldsymbol{\varphi} \wedge \boldsymbol{\psi}$ stands for the usual matrix multiplication, with entries multiplied by the exterior product. Then, when G is a matrix group, we can write (A.51) *and* (A.58) *as*

$$\boldsymbol{\Omega}^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}}\boldsymbol{\omega} = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}, \tag{A.63}$$

$$\Omega^\omega := \mathbf{D}\omega = d\omega + \omega\wedge\omega. \tag{A.64}$$

A.4.2 Transformation of the Field Strengths Under a Change of Gauge

Consider two local trivializations (U, Φ) and (U', Φ') of P such that $p \in \pi^{-1}(U \cap U')$ has (x, g) and (x, g') as images in $(U \cap U') \times G$, where $x \in U \cap U'$. Let s, s' be the associated cross sections to Φ and Φ' respectively. By writing

$s'^* \boldsymbol{\Omega}^{\boldsymbol{\omega}} = \Omega^{\omega\prime}$, we have the following relation for the local curvature in the two different gauges such that $g' = hg$

$$\Omega^{\omega\prime} = h\Omega^{\omega} h^{-1}, \quad \forall x \in U \cap U' \ . \tag{A.65}$$

We now give the *coordinate expressions* for the potential and field strengths in the trivialization Φ. Let $\langle x^\mu \rangle$ be a local chart for $U \subset M$ and let $\{\partial_\mu = \frac{\partial}{\partial x^\mu}\}$ and $\{dx^\mu\}$, $\mu = 0, 1, 2, 3$, be (dual) bases of TU and T^*U respectively. Then,

$$\omega = \omega^a \otimes \mathcal{G}_a = \omega^a_\mu dx^\mu \otimes \mathcal{G}_a \ , \tag{A.66}$$

$$\Omega^\omega = (\Omega^\omega)^a \otimes \mathcal{G}_a = \frac{1}{2}\Omega^a_{\mu\nu} dx^\mu \wedge dx^\nu \otimes \mathcal{G}_a \ . \tag{A.67}$$

where ω^a_μ, $\Omega^a_{\mu\nu} : M \supset U \to \mathbb{R}$ (or $\mathbb{C}$) and we get

$$\Omega^a_{\mu\nu} = \partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + f^a_{bc}\omega^b_\mu \omega^c_\nu \ . \tag{A.68}$$

The following objects appear frequently in the presentation of gauge theories by physicists,

$$(\Omega^\omega)^a = \frac{1}{2}\Omega^a_{\mu\nu} dx^\mu \wedge dx^\nu = d\omega^a + \frac{1}{2} f^a_{bc}\omega^b \wedge \omega^c \ , \tag{A.69}$$

$$\Omega^\omega_{\mu\nu} = \Omega^a_{\mu\nu}\mathcal{G}_a = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \ , \tag{A.70}$$

$$\omega_\mu = \omega^a_\mu \mathcal{G}_a \ . \tag{A.71}$$

We now give the local expression of Bianchi identities. Using (A.61) and (A.69) we have

$$\mathbf{D}\Omega^{\omega} := \frac{1}{2}(\mathbf{D}\Omega^{\omega})_{\rho\mu\nu} dx^\rho \wedge dx^\mu \wedge dx^\nu = 0 \ . \tag{A.72}$$

By putting

$$(\mathbf{D}\Omega^{\omega})_{\rho\mu\nu} := \mathbf{D}_\rho \Omega^{\omega}_{\mu\nu} \ , \tag{A.73}$$

we have

$$\mathbf{D}_\rho \Omega^{\omega}_{\mu\nu} = \partial_\rho \Omega^{\omega}_{\mu\nu} + [\omega_\rho, \Omega^{\omega}_{\mu\nu}], \tag{A.74}$$

and

$$\mathbf{D}_\rho \Omega^{\omega}_{\mu\nu} + \mathbf{D}_\mu \Omega^{\omega}_{\nu\rho} + \mathbf{D}_\nu \Omega^{\omega}_{\rho\mu} = 0 \ . \tag{A.75}$$

Physicists call the operator

$$\mathbf{D}_\rho := \partial_\rho + [\omega_\rho,] \ . \tag{A.76}$$

the *covariant derivative.*

A.4.3 Induced Connections

Let (P_1, M_1, π_1, G_1) and (P_2, M_2, π_2, G_2) be two principal bundles and let $\mathcal{F} : P_1 \to P_2$ be a bundle homomorphism, i.e. $\mathcal{F}$ is fiber preserving, it induces a diffeomorphism $f : M_1 \to M_2$ and there exists a homomorphism $\lambda : G_1 \to G_2$ such that for $g_1 \in G_1$, $p_1 \in P_1$ we have

$$\mathcal{F}(p_1 g_1) = R_{\lambda(g_1)} \mathcal{F}(p_1) \ . \tag{A.77}$$

Proposition 574 *Let $\mathcal{F} : P_1 \to P_2$ be a bundle homomorphism. Then a connection $\boldsymbol{\omega}_1$ on P_1 determines a unique connection on P_2.*

Remark 575 *Let $(P, M, \pi', \mathrm{O}_{p,q}) = \mathbf{P}_{\mathrm{O}p,q}(M)$ be the orthonormal frame bundle, which is as explained above reduction of the frame bundle $F(M)$. Then, a connection on $\mathbf{P}_{\mathrm{O}p,q}(M)$ determines a unique connection on $F(M)$. This is a very important result that has been used implicitly in* Sect. 4.7.8 *and the solution of* Exercise 310.

Proposition 576 *Let $F(M)$ be the frame bundle of a paracompact manifold M. Then, $F(M)$ can be reduced to a principal bundle with structure group $\mathrm{O}_{p,q}$, and to each reduction there corresponds a Riemannian metric field on M.*

Remark 577 *If M has dimension 4, and we substitute $\mathrm{O}_{p,q} \mapsto \mathrm{SO}^e_{1,3}$ then with each reduction of $F(M)$ there corresponds a Lorentzian metric field on M.*

A.4.4 Linear Connections on a Manifold M

Definition 578 *A linear connection on a smooth manifold M is a connection $\boldsymbol{\omega} \in \sec T^*F(M) \otimes gl(n, \mathbb{R})$.*

Remark 579 *Given a Riemannian (Lorentzian) manifold $(M, \mathbf{g})$ a connection on $F(M)$ which is determined by a connection on the orthonormal frame bundle $\mathbf{P}_{\mathrm{O}p,q}(M)$ $(\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M))$ is called a metric connection. After introducing the concept of covariant derivatives on vector bundles, we can show that the covariant derivative of the metric tensor with respect to a metric connection is null.*

Consider the mapping $f|_p : T_x(M) \to \mathbb{R}^n$ (with $p = (x, \Sigma_x)$ in a given trivialization) which sends $\mathbf{v} \in T_x(M)$ into its components relative to the frame $\Sigma_x = \{ e_1|_x, ..., e_n|_x\}$. Let $\{ \theta^j\big|_x\}$ be the dual basis of $\{ e_i|_x\}$. We write

$$f|_p(\mathbf{v}) = (\theta^j\big|_x(\mathbf{v})) \ . \tag{A.78}$$

Definition 580 *The canonical soldering form of* M *is the* 1*-form* $\boldsymbol{\theta} \in \sec T^*F(M) \otimes \mathbb{R}^n$ *such that for any* $\mathbf{v} \in \sec T_pF(M)$ *such that* $\mathbf{v} = \pi_* v$ *we have*

$$(\boldsymbol{\theta}(v)) := \boldsymbol{\theta}^a\big|_p (v)\boldsymbol{e}_a \\ = \theta^a\big|_x (\mathbf{v})\boldsymbol{e}_a, \tag{A.79}$$

where $\{\mathbf{e}_a\}$ *is the canonical basis of* $\mathbb{R}^n$ *and* $\{\boldsymbol{\theta}^a\}$ *is a basis of* $T^*F(M)$*, with* $\boldsymbol{\theta}^a = \pi^*\theta^a$*,* $\boldsymbol{\theta}^a\big|_p (v) = \theta^a\big|_x (\mathbf{v})$*.*

Definition 581 *The torsion form of a linear connection* $\boldsymbol{\omega} \in \sec T^*F(M) \otimes gl(n, \mathbb{R})$ *is the* 2*-form* $D\boldsymbol{\theta} = \boldsymbol{\Theta} \in \sec \bigwedge^2 T^*F(M) \otimes \mathbb{R}^n$ *.*

As it is easy to verify, the soldering form $\boldsymbol{\theta}$ and the torsion 2-form $\boldsymbol{\Theta}$ are tensorial of type $(\rho, \mathbb{R}^n)$, where $\rho(u) = u$, $u \in \mathrm{Gl}(n, \mathbb{R})$.

Using the same techniques employed in the calculation of $D^{\boldsymbol{\omega}}\boldsymbol{\omega}(\mathbf{X_1}, \mathbf{X_2})$ (A.50) it can be shown that

$$\boldsymbol{\Theta} = d\boldsymbol{\theta} + [\boldsymbol{\omega}, \boldsymbol{\theta}], \tag{A.80}$$

where [,] is the commutator product in the Lie algebra of the *affine* group $A(n, \mathbb{R}) = \mathrm{Gl}(n, \mathbb{R}) \boxtimes \mathbb{R}^n$, where $\boxtimes$ means the *semi-direct* product. Suppose that $(\mathbf{E}_a^b, \boldsymbol{e}_c)$ is the canonical basis of $a(n, \mathbb{R})$, the Lie algebra of $A(n, \mathbb{R})$. Recalling that

$$\boldsymbol{\omega}(v) = \boldsymbol{\omega}_b^a(v)\mathbf{E}_a^b\ , \tag{A.81}$$

$$\boldsymbol{\theta}(v) = \boldsymbol{\theta}^a(v)\boldsymbol{e}_a\ , \tag{A.82}$$

we can show without difficulties that

$$D^{\boldsymbol{\omega}}\boldsymbol{\Theta} = [\boldsymbol{\Omega}, \boldsymbol{\theta}]. \tag{A.83}$$

A.4.5 Torsion and Curvature on *M*

Let $\{x^i\}$ be the coordinate functions associated with a local chart (U, φ) of the maximal atlas of M. Let $\Sigma \in \sec F(U)$ with $e_i = F_i^j \frac{\partial}{\partial x^j}$ and $\boldsymbol{\theta} = \boldsymbol{\theta}^a \boldsymbol{e}_a$. Take $\pi_* v = \mathbf{v}$. Then

$$(\boldsymbol{\theta}_p(v)) = f\big|_p (\mathbf{v}) = f\big|_p (dx^j(\mathbf{v})\partial_j) = f\big|_p (dx^j(\mathbf{v})(F_j^k)^{-1} e_k) \\ = ((F_j^k)^{-1} dx^j(\pi_* v))\ . \tag{A.84}$$

With this result it is quite obvious that given any $\boldsymbol{w} \in \mathbb{R}^n$, $\boldsymbol{\theta}$ determines a horizontal field $v_{\mathbf{w}} \in secTF(M)$ by

$$(\boldsymbol{\theta}(v_{\mathbf{w}}(p))) = \boldsymbol{w}\ . \tag{A.85}$$

With these preliminaries we have the following proposition:

Proposition 582 *There is a bijective correspondence between sections of $T^*M \otimes T^r_s M$ and sections of $T^*F(M) \otimes \mathbb{R}^{n_q}$, the space of tensorial forms of the type $(\rho, \mathbb{R}^{n_q})$ in $F(M)$, with ρ and q being determined by $T^r_s M$.*

Using the above proposition and recalling that the soldering form is tensorial of type $(\rho(u), \mathbb{R}^n)$, $\rho(u) = u$, we see that it determines on M a vector valued differential 1-form[4] $\theta = e_a \otimes \theta^a \in \sec TM \otimes \bigwedge^1 T^*M$. Also, the torsion $\boldsymbol{\Theta}$ is tensorial of type $(\rho(u), \mathbb{R}^n)$, $\rho(u) = u$ and thus define a vector valued 2-form on M, $\Theta = e_a \otimes \Theta^a \in \sec TM \otimes \bigwedge^2 T^*M$. We can show from (A.80) that given $u, w \in T_pF(M)$,

$$\Theta^a(\pi_* u, \pi_* w) = d\theta^a(\pi_* u, \pi_* w) + \omega^a_b(\pi_* u)\theta^b(\pi_* w) - \omega^a_b(\pi_* w)\theta^b(\pi_* u) \,. \quad \text{(A.86)}$$

On the basis manifold this equation is often written:

$$\begin{aligned} \Theta &= \mathbf{D}\theta = e_a \otimes (\mathbf{D}\theta^a) \\ &= e_a \otimes (d\theta^a + \omega^a_b \wedge \theta^b) \,, \end{aligned} \quad \text{(A.87)}$$

where we recognize $\mathbf{D}\theta^a$ as the exterior covariant derivative of index forms introduced in Sect. 3.3.4.

Also, the curvature $\boldsymbol{\Omega}^{\boldsymbol{\omega}}$ is tensorial of type $(\mathrm{Ad}, \mathbb{R}^{n^2})$. It then defines $\Omega = e_a \otimes \theta^b \otimes \mathcal{R}^a_b \in \sec T^1_1 M \otimes \bigwedge^2 T^*M$ which we easily find to be given by

$$\begin{aligned} \Omega &= e_a \otimes \theta^b \otimes \mathcal{R}^a_b \\ &= e_a \otimes \theta^b \otimes (d\omega^a_b + \omega^a_c \wedge \omega^c_b) \,, \end{aligned} \quad \text{(A.88)}$$

where the $\mathcal{R}^a_b \in \sec \bigwedge^2 T^*M$ are the curvature 2-forms introduced in Chap. 2, explicitly given by

$$\mathcal{R}^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b \,. \quad \text{(A.89)}$$

Note that sometimes the symbol $\mathbf{D}\omega^a_b$ such that $\mathcal{R}^a_b := \mathbf{D}\omega^a_b$ is introduced in some texts. Of course, the symbol $\mathbf{D}$ cannot be interpreted in this case as the exterior covariant derivative of index forms[5]. This is expected since $\boldsymbol{\omega} \in \sec \bigwedge^1 T^*P \otimes gl(n, \mathbb{R})$ is *not* tensorial.

A.5 Covariant Derivatives on Vector Bundles

Consider a vector bundle $(E, M, \pi_1, G, \boldsymbol{V})$[6] associated with a PFB bundle (P, M, π, G) by the linear representation ρ of G in the vector space $\boldsymbol{V}$ over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Also, let $\dim_{\mathbb{F}} \boldsymbol{V} = m$. Consider again the trivializations of P and E given by (A.7)–(A.9). Then, we have the definition:

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[4] θ is clearly the identity operator in the space of vector fields.

[5] See Sect. A.3.

[6] Also denoted $E = P \times_\rho \mathbf{V}$.

Definition 583 *The* parallel transport *of* $\mathbf{\Psi_0} \in E$, $\pi_1(\mathbf{\Psi_0}) = x_0$, *along the curve* $\sigma : \mathbb{R} \ni I \to M$, $t \mapsto \sigma(t)$ *from* $x_0 = \sigma(0) \in M$ *to* $x = \sigma(t)$ *is the element* $\mathbf{\Psi}_{\|t} \in E$ *such that:*

(i) $\pi_1(\mathbf{\Psi}_{\|t}) = x$,

(ii) $\chi_i(\mathbf{\Psi}_{\|t}) = \rho(\varphi_i(p_{\|t}) \circ \varphi_i^{-1}(p_0))\chi_i(\mathbf{\Psi_0})$.

(iii) $p_{\|t} \in P$ *is the parallel transport of* $p_0 \in P$ *along* σ *from* x_0 *to* x *as defined in* (A.24) *above.*

Definition 584 *Let* Y *be a vector at* x_0 *tangent to the curve* σ (*as defined above*). *The covariant derivative of* $\mathbf{\Psi} \in \sec E$ *in the direction of* Y *is denoted* $(D_Y^E \mathbf{\Psi})_{x_0} \in \sec E$ *and*

$$(D_Y^E \mathbf{\Psi})(x_0) \equiv (D_Y^E \mathbf{\Psi})_{x_0} = \lim_{t \to 0} \frac{1}{t}(\mathbf{\Psi}^0_{\|t} - \mathbf{\Psi_0}) \, , \tag{A.90}$$

where $\mathbf{\Psi}^0_{\|t}$ *is the "vector"* $\mathbf{\Psi}_t \equiv \mathbf{\Psi}(\sigma(t))$ *of a section* $\mathbf{\Psi} \in \sec E$ *parallel transported along* σ *from* $\sigma(t)$ *to* x_0, *the only requirement on* σ *being*

$$\left. \frac{d}{dt}\sigma(t) \right|_{t=0} = Y \, . \tag{A.91}$$

In the local trivialization (U_i, Ξ_i) of E (see (A.7)–(A.9)) if Ψ_t is the element in $\mathbf{V}$ representing $\mathbf{\Psi}_t$, we have

$$\chi_i(\Psi^0_{\|t}) = \rho(g_0 g_t^{-1})\chi_{i|\sigma(t)}(\Psi_t) \, . \tag{A.92}$$

By choosing p_0 such that $g_0 = e$ we can compute (A.90):

$$\begin{aligned} (D_Y^E \mathbf{\Psi})_{x_0} &= \left. \frac{d}{dt}\rho(g^{-1}(t)\Psi_t) \right|_{t=0} \\ &= \left. \frac{d\Psi_t}{dt} \right|_{t=0} - \left(\rho'(e) \left. \frac{dg(t)}{dt} \right|_{t=0} \right) (\Psi_0) \, . \end{aligned} \tag{A.93}$$

This formula is trivially generalized for the covariant derivative in the direction of an arbitrary vector field $Y \in \sec TM$.

With the aid of (A.93) we can calculate, e.g. the covariant derivative of $\mathbf{\Psi} \in \sec E$ in the direction of the vector field $Y = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu$. This covariant derivative is denoted $D^E_{\partial_\mu}\mathbf{\Psi}$.

We need now to calculate $\left.\frac{dg(t)}{dt}\right|_{t=0}$. In order to do that, recall that if $\frac{d}{dt}$ is a tangent to the curve σ in M, then $s_*\left(\frac{d}{dt}\right)$ is a tangent to $\hat{\sigma}$ the horizontal lift of σ, i.e. $s_*\left(\frac{d}{dt}\right) \in HP \subset TP$. As defined before $s = \Phi_i^{-1}(x, e)$ is the cross section associated with the trivialization Φ_i of P (see (A.6). Then, as g is a mapping $U \to G$ we can write

$$\left[s_*(\frac{d}{dt}) \right](g) = \frac{d}{dt}(g \circ \sigma) \, . \tag{A.94}$$

To simplify the notation, introduce local coordinates $\{x^\mu, g\}$ in $\pi^{-1}(U)$ and write $\sigma(t) = (x^\mu(t))$ and $\hat{\sigma}(t) = (x^\mu(t), g(t))$. Then,

$$s_*\left(\frac{d}{dt}\right) = \dot{x}^\mu(t)\frac{\partial}{\partial x^\mu} + \dot{g}(t)\frac{\partial}{\partial g}\,, \tag{A.95}$$

in the local coordinate basis of $T(\pi^{-1}(U))$. An expression like the second member of (A.95) defines in general a vector tangent to P but, according to its definition, $s_*\left(\frac{d}{dt}\right)$ is in fact horizontal. We must then impose that

$$s_*\left(\frac{d}{dt}\right) = \dot{x}^\mu(t)\frac{\partial}{\partial x^\mu} + \dot{g}(t)\frac{\partial}{\partial g} = \alpha^\mu\left(\frac{\partial}{\partial x^\mu} + \omega^a_\mu \mathcal{G}_a g \frac{\partial}{\partial g}\right)\,, \tag{A.96}$$

for some α^μ.

We used the fact that $\frac{\partial}{\partial x^\mu} + \omega^a_\mu \mathcal{G}_a g \frac{\partial}{\partial g}$ is a basis for HP, as can easily be verified from the condition that $\boldsymbol{\omega}(Y^h) = 0$, for all $Y \in HP$. We immediately get that

$$\alpha^\mu = \dot{x}^\mu(t)\,, \tag{A.97}$$

and

$$\frac{dg(t)}{dt} = \dot{g}(t) = -\dot{x}^\mu(t)\omega^a_\mu \mathcal{G}_a g\,, \tag{A.98}$$

$$\left.\frac{dg(t)}{dt}\right|_{t=0} = -\dot{x}^\mu(0)\omega^a_\mu \mathcal{G}_a\,. \tag{A.99}$$

With this result we can rewrite (A.93) as

$$(D^E_Y \boldsymbol{\Psi})_{x_0} = \left.\frac{d\Psi_t}{dt}\right|_{t=0} - \rho'(e)\omega(Y)(\Psi_0), \qquad Y = \left.\frac{d\sigma}{dt}\right|_{t=0}\,. \tag{A.100}$$

which generalizes trivially for the covariant derivative along a vector field $Y \in \sec TM$.

Remark 585 *Many texts introduce the covariant derivative operator D^E_Y acting on sections of the vector bundle E as follows.*

Definition 586 *A connection D^E on M is a mapping*

$$\begin{aligned} D^E &: \sec TM \times \sec E \to \sec E\,, \\ (\mathbf{X}, \boldsymbol{\Psi}) &\mapsto D^E_{\mathbf{X}}\boldsymbol{\Psi}. \end{aligned} \tag{A.101}$$

such that $D^E_{\mathbf{X}} : \sec E \to \sec E$ satisfies the following properties:

$$\begin{aligned} &\text{(i)}\ \ D^E_{\mathbf{X}}(a\boldsymbol{\Psi}) = aD^E_{\mathbf{X}}\boldsymbol{\Psi}, \\ &\text{(ii)}\ \ D^E_{\mathbf{X}}(\boldsymbol{\Psi} + \boldsymbol{\Phi}) = D^E_{\mathbf{X}}\boldsymbol{\Psi} + D^E_X\boldsymbol{\Phi}\,, \\ &\text{(iii)}\ \ D^E_{\mathbf{X}}(f\boldsymbol{\Psi}) = \mathbf{X}(f) + fD^E_{\mathbf{X}}\boldsymbol{\Psi}\,, \\ &\text{(iv)}\ \ D^E_{\mathbf{X}+\mathbf{Y}}\boldsymbol{\Psi} = D^E_{\mathbf{X}}\boldsymbol{\Psi} + D^E_{\mathbf{Y}}\boldsymbol{\Psi}\,, \\ &\text{(v)}\ \ D^E_{f\mathbf{X}}\boldsymbol{\Psi} = fD^E_{\mathbf{X}}\boldsymbol{\Psi}. \end{aligned} \tag{A.102}$$

$\forall\, \mathbf{X}, \mathbf{Y} \in \sec TM$*,* $\mathbf{\Psi}, \mathbf{\Phi} \in \sec E$*,* $\forall a \in \mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ *(the field of scalars entering the definition of the vector space* $\boldsymbol{V}$ *of* E*)* $\forall f \in C^{\infty}(M)$*, where* $C^{\infty}(M)$ *is the set of smooth functions with values in* $\mathbb{F}$*.*

Of course, all properties in (A.102) follows directly from (A.100). However, the point of view encoded in Definition 586 may be appealing to physicists. To see, first recall that $E = P \times_{\rho} \boldsymbol{V}$. Recall that ϱ stands for the representation of G in the vector space $\boldsymbol{V}$.

Definition 587 *The dual bundle of* E *is the bundle* $E^* = P \times_{\rho^*} \boldsymbol{V}^*$*, where* $\boldsymbol{V}^*$ *is the dual space of* $\boldsymbol{V}$ *and* ϱ^* *is the representation of* G *in the vector space* $\boldsymbol{V}^*$*.*

Example 588 *As examples we have the tangent bundle which is* $TM = F(M) \times_{\rho} \mathbb{R}^n$ *where* $\rho : \mathrm{Gl}(n, \mathbb{R}) \to \mathrm{Gl}(n, \mathbb{R})$ *denotes the standard representation and* $T^*M = F(M) \times_{\rho^*} (\mathbb{R}^n)^*$ *where the dual representation* ρ^* *satisfies* $\rho^*\,(g) = \rho(g^{-1})^t$*. Also, the tensor bundle of tensors of type* (r, s)*, the bundle of homogeneous* k*-vectors and the bundle of homogeneous* k*-forms are:*

$$\begin{aligned} T_s^r M &= \bigotimes\nolimits_s^r TM = F(M) \times_{\otimes_s^r} \left(\bigotimes\nolimits_s^r \mathbb{R}^n\right), \\ \bigwedge\nolimits^k TM &= F(M) \times_{\bigwedge_\rho^k} \bigwedge\nolimits^k \mathbb{R}^n, \\ \bigwedge\nolimits^k T^*M &= F(M) \times_{\bigwedge_{\rho^*}^k} \bigwedge\nolimits^k \mathbb{R}^n, \end{aligned} \tag{A.103}$$

where $\bigotimes_s^r$*,* $\bigwedge_\rho^k$ *and* $\bigwedge_{\rho^*}^k$ *are the induced tensor product and exterior powers representations.*

Definition 589 *The bundle* $E \otimes E^*$ *is called the bundle of endomorphisms of* E *and will be denoted by* $\mathrm{End}\, E$*.*

Definition 590 *A connection* D^{E^*} *acting on* E^* *is defined by*

$$(D_{\mathbf{X}}^{E^*} \mathbf{\Xi}^*)(\mathbf{\Psi}) = \mathbf{X}\left(\mathbf{\Xi}^*(\mathbf{\Psi})\right) - \mathbf{\Xi}^*(D_{\mathbf{X}}^{E} \mathbf{\Psi}), \tag{A.104}$$

for $\forall \mathbf{\Xi}^* \in \sec E^*$*,* $\forall \mathbf{\Psi} \in \sec E$ *and* $\forall \mathbf{X} \in \sec TM$*.*

Definition 591 *A connection* $D^{E \otimes E^*}$ *acting on sections of* $E \otimes E^*$ *is defined for* $\forall \mathbf{\Xi}^* \in \sec E^*$*,* $\forall \mathbf{\Psi} \in \sec E$ *and* $\forall \mathbf{X} \in \sec TM$ *by*

$$D_{\mathbf{X}}^{E \otimes E^*} \mathbf{\Xi}^* \otimes \mathbf{\Psi} = D_{\mathbf{X}}^{E^*} \mathbf{\Xi}^* \otimes \mathbf{\Psi} + \mathbf{\Xi}^* \otimes D_{\mathbf{X}}^{E} \mathbf{\Psi}. \tag{A.105}$$

We shall abbreviate $D^{E \otimes E^*}$ by $D^{\mathrm{End} E}$. Eq(A.105) may be generalized in an obvious way in order to define a connection on arbitrary tensor products of bundles $E \otimes E' \otimes ... E'^{\cdots\prime}$. Finally, we recall for completeness that given two bundles, say E and E' and given connections D^E and $D^{E'}$ there is an obvious connection $D^{E \oplus E'}$ defined in the Whitney bundle $E \oplus E'$ (recall Definition 551). It is given by

$$D_{\mathbf{X}}^{E \oplus E'}(\mathbf{\Psi} \oplus \mathbf{\Psi}') = D_{\mathbf{X}}^{E} \mathbf{\Psi} \oplus D_{\mathbf{X}}^{E'} \mathbf{\Psi}', \tag{A.106}$$

for $\forall \mathbf{\Psi} \in \sec E$, $\forall \mathbf{\Psi}' \in \sec E'$ and $\forall \mathbf{X} \in \sec TM$.

A.5.1 Connections on E Over a Lorentzian Manifold

In what follows we suppose that $(M, \mathbf{g})$ is a Lorentzian manifold (Definition 229). We recall that the manifold M in a Lorentzian structure is supposed paracompact. Then, according to Proposition 543 the bundles E, E^*, $T_s^r M$ and $\operatorname{End} E$ admit global cross sections.

We then write for the covariant derivative of $\boldsymbol{\Psi} \in \sec E$ and $\mathbf{X} \in \sec TM$,

$$D_{\mathbf{X}}^{E}\boldsymbol{\Psi} = D_{\mathbf{X}}^{0E}\boldsymbol{\Psi} + \mathcal{W}(\mathbf{X})\boldsymbol{\Psi}, \tag{A.107}$$

where $\mathcal{W} \in \sec \operatorname{End} E \otimes T^*M$ will be called *connection* 1-form (or *potential*) for D_X^E and D_X^{0E} is a well defined connection on E, that we are going to determine.

Consider then a open set $U \subset M$ and a trivialization of E in U. Such a trivialization is said to be a *choice of a gauge.*

Let $\{\mathbf{e}_i\}$ be the canonical basis of $\mathbf{V}$. Let $\boldsymbol{\Psi}|_U \in \sec\ E|_U = \pi^{-1}(U)$. Consider the trivialization $\Xi : \pi^{-1}(U) \to U \times \mathbf{V}$, $\Xi(\boldsymbol{\Psi}) = (\pi(\boldsymbol{\Psi}), \chi(\boldsymbol{\Psi})) = (x, \chi(\boldsymbol{\Psi}))$. In this trivialization we write

$$\boldsymbol{\Psi}|_U := (x, \Psi(x)) \ , \tag{A.108}$$

$\Psi(x) \in \mathbf{V}$, $\forall x \in U$, with $\Psi : U \to \boldsymbol{V}$ a smooth function. Let $\{s_i\} \in \sec\ E|_U$, $s_i = \chi^{-1}(\mathbf{e}_i)$ $i = 1, 2, ..., m$ be a basis of sections of $E|_U$ and $\{e_\mu\} \in \sec F(U), \mu = 0, 1, 2, 3$ a basis for TU. Let also $\{\theta^\nu\}$, $\theta^\nu \in \sec T^*U$, be the dual basis of $\{e_\mu\}$ and $\{s^{*i}\} \in \sec\ E^*|_U$, be a basis of sections of $E^*|_U$ dual to the basis $\{s_i\}$.

We define the connection coefficients in the chosen gauge by

$$D_{e_\mu}^{E} s_i = \mathcal{W}_{\mu i}^{j} s_j \ . \tag{A.109}$$

Then, if $\Psi = \Psi^i s_i$ and $\mathbf{X} = \mathbf{X}^\mu e_\mu$

$$\begin{aligned} D_{\mathbf{X}}^{E}\Psi &= X^\mu D_{e_\mu}^{E}(\Psi^i s_i) \\ &= X^\mu \left[e_\mu(\Psi^i) + \mathcal{W}_{\mu j}^{i}\Psi^j \right] s_i \ . \end{aligned} \tag{A.110}$$

Now, let us concentrate on the term $X^\mu \mathcal{W}_{\mu j}^{i}\Psi^j s_i$. It is, of course a new section $F := (x, X^\mu \mathcal{W}_{\mu j}^{i}\Psi^j s_i)$ of $E|_U$ and $X^\mu \mathcal{W}_{\mu j}^{i}\Psi^j s_i$ is linear in both X and Ψ.

This observation shows that $\mathcal{W}^U \in \sec(\operatorname{End} E|_U \otimes T^*U)$, such that in the trivialization introduced above is given by

$$\mathcal{W}^U = \mathcal{W}_{\mu j}^{i} s_j \otimes s^{*i} \otimes \theta^\mu \ , \tag{A.111}$$

is the representative of $\mathcal{W}$ in the chosen gauge.

Note that if $\mathbf{X} \in \sec TU$ and $\boldsymbol{\Psi} := (x, \Psi(x)) \in \sec\ E|_U$ we have

$$\begin{aligned} \omega^U(\mathbf{X}) &:= \omega_{\mathbf{X}}^{U} = X^\mu \mathcal{W}_{\mu j}^{i} s_j \otimes s^{*i}, \\ \omega_{\mathbf{X}}^{U}(\Psi) &= X^\mu \mathcal{W}_{\mu j}^{i}\Psi^i s_j \ . \end{aligned} \tag{A.112}$$

We can then write

$$D_{\mathbf{X}}^{E}\Psi = \mathbf{X}(\Psi) + \omega_{\mathbf{X}}^{U}(\Psi) \ , \tag{A.113}$$

thereby identifying $D^{0E}_{\mathbf{X}}\Psi = \mathbf{X}(\Psi)$. In this case $D^{0E}_{\mathbf{X}}$ is called the standard *flat* connection.

Now, we can state a very important result which has been used in Chap. 2 to write the different decompositions of Riemann-Cartan connections.

Proposition 592 *Let D^{0E} and D^{E} be arbitrary connections on E then there exists $\bar{\mathcal{W}} \in \sec \mathrm{End} E \otimes T^*M$ such that for any $\mathbf{\Psi} \in \sec E$ and $\mathbf{X} \in \sec TM$,*

$$D^{E}_{\mathbf{X}}\mathbf{\Psi} = D^{0E}_{\mathbf{X}}\mathbf{\Psi} + \bar{\mathcal{W}}(\mathbf{X})\mathbf{\Psi}. \tag{A.114}$$

A.5.2 Gauge Covariant Connections

Definition 593 *A connection D^E on E is said to be a G-connection if for any $u \in G$ and any $\mathbf{\Psi} \in \sec E$ there exists a connection D'^E on E such that for any $\mathbf{X} \in \sec TM$*

$$D'^{E}_{\mathbf{X}}\left(\rho(u)\mathbf{\Psi}\right) = \rho(u) D^{E}_{\mathbf{X}}\mathbf{\Psi}. \tag{d11}$$

Proposition 594 *If $D^{E}_{\mathbf{X}}\mathbf{\Psi} = D^{0E}_{\mathbf{X}}\mathbf{\Psi} + \bar{\mathcal{W}}(\mathbf{X})\mathbf{\Psi}$ for $\mathbf{\Psi} \in \sec E$ and $\mathbf{X} \in \sec TM$, then $D'^{E}_{\mathbf{X}}\mathbf{\Psi} = D^{0E}_{\mathbf{X}}\mathbf{\Psi} + \bar{\mathcal{W}}'(\mathbf{X})\mathbf{\Psi}$ with*

$$\bar{\mathcal{W}}'(\mathbf{X}) = u\bar{\mathcal{W}}(\mathbf{X})u^{-1} + udu^{-1}\ . \tag{A.115}$$

Suppose that the vector bundle E has the same structural group as the orthonormal frame bundle $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$, which as we know is a reduction of the frame bundle $F(M)$. In this case we give the definition:

Definition 595 *A connection D^E on E is said to be a generalized G-connection if for any $u \in G$ and any $\mathbf{\Psi} \in \sec E$ there exists a connection D'^E on E such that for any $\mathbf{X} \in \sec TM$, $TM = \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M) \times_{\rho^{TM}} \mathbb{R}^4$*

$$D'^{E}_{\mathbf{X}'}\left(\rho(u)\mathbf{\Psi}\right) = \rho(u) D^{E}_{\mathbf{X}}\mathbf{\Psi}, \tag{A.116}$$

where $\mathbf{X}' = \rho^{TM}\mathbf{X} \in \sec TM$.

A.5.3 Curvature Again

Definition 596 *Let D^E be a G-connection on E. The curvature operator $\mathbf{R}^E \in \sec \bigwedge^2 T^*M \otimes \mathrm{End}E$ of D^E is the mapping*

$$\mathbf{R}^E \colon \sec TM \otimes TM \otimes E \to E, \tag{A.117}$$

$$\mathbf{R}^E(\mathbf{X},\mathbf{Y})\mathbf{\Psi} = D^E_{\mathbf{X}} D^E_{\mathbf{Y}}\mathbf{\Psi} - D^E_{\mathbf{Y}} D^E_{\mathbf{X}}\mathbf{\Psi} - D^E_{[\mathbf{X},\mathbf{Y}]}\mathbf{\Psi}$$

$$\mathbf{R}^E(\mathbf{X},\mathbf{Y}) = D^E_{\mathbf{X}} D^E_{\mathbf{Y}} - D^E_{\mathbf{Y}} D^E_{\mathbf{X}} - D^E_{[\mathbf{X},\mathbf{Y}]}, \tag{d14}$$

for any $\mathbf{\Psi} \in \sec E$ and $\mathbf{X},\mathbf{Y} \in \sec TM$.

If $\mathbf{X} = \partial_\mu, \mathbf{Y} = \partial_\nu \in \sec TU$ are coordinate basis vectors associated with the coordinate functions $\{x^\mu\}$ we have

$$\mathbf{R}^E(\partial_\mu,\partial_\nu) := \mathbf{R}^E_{\mu\nu} = \left[D^E_{\partial_\mu}, D^E_{\partial_\nu}\right]\ . \tag{A.118}$$

In a local basis $\{s_i \otimes s^{*j}\}$ of End$E$ we have under the local trivialization used above

$$
\begin{aligned}
\mathbf{R}^{E}_{\mu\nu} &= \mathbf{R}_{\mu\ \nu b}^{\ a} s_a \otimes s^{*b} \ ,\\
\mathbf{R}_{\mu\ \nu b}^{\ a} &= \partial_\mu \mathcal{W}^a_{\nu b} - \partial_\nu \mathcal{W}^a_{\mu b} + \mathcal{W}^a_{\mu c}\mathcal{W}^c_{\nu b} - \mathcal{W}^a_{\nu c}\mathcal{W}^c_{\mu b} \ .
\end{aligned}
\tag{A.119}
$$

(A.119) can also be written

$$
\mathbf{R}^{E}_{\mu\nu} = \partial_\mu \mathcal{W}_\nu - \partial_\nu \mathcal{W}_\mu + [\mathcal{W}_\mu, \mathcal{W}_\nu] \ . \tag{A.120}
$$

A.5.4 Exterior Covariant Derivative Again

Definition 597 *Consider* $\Psi \otimes A_r \in \sec E \otimes \bigwedge^r T^*M$ *and* $B_s \in \bigwedge^s T^*M$ *. We define* $(\Psi \otimes A_r) \otimes_\wedge B_s$ *by*

$$
(\Psi \otimes A_r) \otimes_\wedge B_s = \Psi \otimes (A_r \wedge B_s) \ . \tag{A.121}
$$

Definition 598 *Let* $\Psi \otimes A_r \in \sec E \otimes \bigwedge^r T^*M$ *and* $\mathbf{\Pi} \otimes B_s \in \sec \mathrm{End} E \otimes \bigwedge^s T^*M$. *We define* $(\Pi \otimes B_s) \otimes_\wedge (\Psi \otimes A_r)$ *by*

$$
(\Pi \otimes B_s) \otimes_\wedge (\Psi \otimes A_r) = \Pi(\Psi) \otimes (B_s \wedge A_r) \ . \tag{A.122}
$$

Definition 599 *Given a connection* D^E *acting on* E, *the exterior covariant derivative* $\boldsymbol{d}^{D^E}$ *acting on sections of* $E \otimes \bigwedge^r T^*M$ *and the exterior covariant derivative* $\boldsymbol{d}^{D^{\mathrm{End}\,E}}$ *acting on sections of* End$E \otimes \bigwedge^s T^*M$ $(r, s = 0, 1, 2, 3, 4)$ *is given by:*

(i) *if* $\mathbf{\Psi} \in \sec E$ *then for any* $\mathbf{X} \in \sec TM$

$$
\mathbf{d}^{D^E} \Psi(\mathbf{X}) = D^E_{\mathbf{X}} \Psi, \tag{A.123}
$$

(ii) *For any* $\Psi \otimes A_r \in \sec E \otimes \bigwedge^r T^*M$

$$
\mathbf{d}^{D^E} (\Psi \otimes A_r) = \mathbf{d}^{D^E} \Psi \otimes \wedge A_r + \Psi \otimes dA_r \ , \tag{A.124}
$$

(iii) *For any* $\mathbf{\Pi} \otimes B_s \in \sec \mathrm{End}\, E \otimes \bigwedge^s T^*M$

$$
\mathbf{d}^{D^{\mathrm{End}\,E}} (\Pi \otimes B_s) = \mathbf{d}^{D^{\mathrm{End}\,E}} \Pi \otimes_\wedge B_s + \Pi \otimes dB_s \ . \tag{A.125}
$$

Proposition 600 *Consider the bundle product* $\mathfrak{E} = (\mathrm{End}\, E \otimes \bigwedge^s T^*M) \otimes_\wedge (E \otimes \bigwedge^r T^*M)$. *Let* $\mathbf{\Pi} = \Pi \otimes B_s \in \sec \mathrm{End}\, E \otimes \bigwedge^s T^*M$ *and* $\mathbf{\Psi} = \Psi \otimes A_r \in \sec E \otimes \bigwedge^r T^*M$ *. Then the exterior covariant derivative* $\mathbf{d}^{D^{\mathfrak{E}}}$ *acting on sections of* $\mathfrak{E}$ *satisfies*

$$
\mathbf{d}^{D^{\mathfrak{E}}} (\mathbf{\Pi} \otimes_\wedge \mathbf{\Psi}) = (\mathbf{d}^{D^{\mathrm{EndE}}} \mathbf{\Pi}) \otimes_\wedge \mathbf{\Psi} + (-1)^s \mathbf{\Pi} \otimes_\wedge \mathbf{d}^{D^E} \mathbf{\Psi}. \tag{A.126}
$$

Exercise 601 *The reader can now show several interesting results, which make contact with results obtained earlier when we analyzed the connections and curvatures on principal bundles and which allowed us sometimes the use of sloppy notations in the main text:*

(i) *Suppose that the bundle admits a flat connection* D^{0E}. *We put* $\mathbf{d}^{D^{0E}} = d$. *Then, if* $\chi \in \sec E \otimes \bigwedge^r T^*M$ *we have*

$$\mathbf{d}^{D^{0E}} \chi = \mathbf{d}\chi + \mathcal{W} \otimes_\wedge \chi \ .$$

(ii) *If* $\chi \in \sec E \otimes \bigwedge^r T^*M$ *we have*

$$(\mathbf{d}^{D^E})^2 \chi = \mathbf{R}^E \otimes_\wedge \chi \ . \tag{A.127}$$

(iii) *If* $\chi \in \sec E \otimes \bigwedge^r T^*M$ *we have*

$$(\mathbf{d}^{D^E})^3 \chi = \mathbf{R}^E \otimes_\wedge \mathbf{d}^{D^E} \chi \ . \tag{A.128}$$

(iii) *Suppose that the bundle admits a flat connection* D^{0E}. *We put* $\mathbf{d}^{D^{0E}} = d$. *Then, if*

(iv) $\mathbf{\Pi} \in \sec \operatorname{End} E \otimes \bigwedge^s T^*M$ *we have*

$$\mathbf{d}^{D^{\operatorname{End} E}} \mathbf{\Pi} = \mathbf{d}\mathbf{\Pi} + [\mathcal{W}, \mathbf{\Pi}] \ . \tag{A.129}$$

(v)

$$\mathbf{d}^{D^{\operatorname{End} E}} \mathbf{R}^E = 0 \ . \tag{A.130}$$

(vi)

$$\mathbf{R}^E = d\mathcal{W} + \mathcal{W} \otimes_\wedge \mathcal{W} \ . \tag{A.131}$$

Remark 602 *Note that* $\mathbf{R}^E \neq \mathbf{d}^{D^{\operatorname{End} E}} \mathcal{W}$.

We end here this long Appendix, hopping that the material presented be enough to permit our reader to follow the more difficult parts of the text and in particular to see the reason for our use of many eventual sloppy notations.

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Acronyms and Abbreviations

ADM	Arnowitt-Deser-Misner
CCSP	Classical Charged Spinning Particles
CDS	Covariant Dirac Spinor
DE$\mathcal{C}\ell^{\ell}$	Dirac Equation for a DHSF
DHE	Dirac-Hestenes Equation
DHS	Dirac-Hestenes Spinor
DHSF	Dirac-Hestenes Spinor Field
EP	Equivalence Principle
ELE	Euler-Lagrange Equation
EMFS	Even Multiform Fields
FECD	Fake Exterior Covariant Differential
GR	General Relativity
GRT	General Relativity Theory
IMT$_{\gamma}$	Inertial Moving Tetrad Along γ
IRF	Inertial Reference Frame
LIASF	Left Ideal Algebraic Spinor Field
LLC	Local Lorentz Coordinates
LLCC	Local Lorentz Coordinates Chart
LLF	Local Lorentz Frame
LLRF$_{\gamma}$	Local Lorentz Reference Frame Associated with γ
MDE	Maxwell-Dirac Equivalence
ME	Maxwell Equations
MTW	Misner-Thorne-Wheeler
NLDHE	Non-Linear Dirac-Hestenes Equation
PWS	Plane Wave Solution
PFB	Principal Fiber Bundle
PLLI	Principle of Local Lorentz Invariance
PIRF	Pseudo Inertial Reference Frame
RCST	Riemann-Cartan Spacetime

RCWS	Riemann-Cartan-Weyl Space
RIASF	Right Ideal Algebraic Spinor Field
SRT	Special Relativity Theory
SAP	Stationary Action Principle
i.e.	Id Est
e.g.	Exempli Gratia

List of Symbols

$\boldsymbol{V}$	Vector Space	19
$\boldsymbol{V}^*$	Dual Space of $\boldsymbol{V}$	19
$\oplus$	Direct Sum	19
dim	Dimension	19
$\mathbb{N}$	Set of Natural Numbers	19
$\mathbb{R}$	Real Field	19
$T_k(\boldsymbol{V})$	Space of k-Cotensors	19
$\bigoplus_{k=0}^{\infty}$	See Definition 3	19
$T(\boldsymbol{V}^*)$	Space of Multicotensors	19
$\langle\rangle_k$	k-Part Operator	20
$\otimes$	Tensor Product	20
$T_s^r(\boldsymbol{V})$	Space of r-Contravariant, s-Covariant Tensors	21
$\boldsymbol{\mathcal{T}}(\boldsymbol{V})$	Tensor Algebra of $\boldsymbol{V}$	21
$\mathrm{T}(\mathbf{V})$	Space of Multitensors	21
$\hat{}$	Main Involution	21
$\tilde{}$	Reversion	21
$\bar{}$	Conjugation	22
$\boldsymbol{g}$	Metric Tensor of $\boldsymbol{V}$	22
g	Metric Tensor of $\boldsymbol{V}^*$	22
$\underset{g}{\cdot}$	Scalar Produc Induced by g	22
$\cdot$	Scalar Product	22
$\{\boldsymbol{e}_k\}$	Basis of $\boldsymbol{V}$	22
$\{\varepsilon^k\}$	Dual Basis of $\{\boldsymbol{e}_k\}$	22
$\{\boldsymbol{e}^k\}$	Reciprocal Basis $\{\boldsymbol{e}_k\}$	22
$\{\varepsilon_k\}$	Reciprocal Basis of $\{\varepsilon^k\}$	23
$\bigwedge \boldsymbol{V}^*$	Exterior Algebra	23
$\wedge$	Exterior Product	23
A	Antisymmetrization Operator	24
$\epsilon_{i_1\ldots i_k}$	Permutation Symbol of Order k	25

$\mathring{\mathbf{g}}$	Fiducial Metric Tensor of $\boldsymbol{V}$	25
$\mathring{g}$	Fiducial Metric Tensor of $\boldsymbol{V}^*$	25
$(\boldsymbol{V}^*, \mathring{g})$	Metric Vector Space	25
$\star$	Hodge Operator	25
$\underset{\mathring{g}}{\star}$	Hodge Operator Associated with $\underset{\mathring{g}}{\star}$	26
$\llcorner$	Left Contraction	27
$\lrcorner$	Right Contration	27
$\underset{\mathring{g}}{\llcorner}$	Left Contraction Associated with $\mathring{g}$	27
$\underset{\mathring{g}}{\lrcorner}$	Right Contraction Associated with $\mathring{g}$	27
$(\bigwedge \boldsymbol{V}^*, \mathring{g})$	Grassmann Algebra	27
$\mathcal{C}\ell(\boldsymbol{V}^*, \mathring{g})$	Clifford Algebra of $(\boldsymbol{V}^*, \mathring{g})$	28
AB	Clifford Product of A and B	28
$A \hookrightarrow B$	A is Embedded in B and $A \subseteq B$	31
$ext\text{-}(\boldsymbol{V}^*)$	Space of Extensors	33
$ext(\bigwedge^p \boldsymbol{V}^*, \bigwedge^q \boldsymbol{V}^*)$	Space of (p,q) Extensors	33
$\dagger$	Adjoint Operator	34
$\underline{}$	Exterior Power Extension Operator	35
tr	Trace Operator	36
$\mathfrak{det}$	Determinant Operator of a Extensor	36
$bif(t)$	Biform of t	37
g	Endomorphism Associated with g	39
$\bullet$	Scalar Product	40
$S_{[ab]}$	Skew-Symmetric Part of S_{ab}	44
$\mathbb{R}^{1,3}$	Minkowski Vector Space	46
$\|\|\mathbf{v}\|\|$	Norm of $\mathbf{v}$	47
$\uparrow$	Time Orientation	47
$\mathbf{L}$	Lorentz Transformation	48
$\mathrm{O}_{1,3}$	Lorentz Group	48
$\mathrm{SO}_{1,3}$	Proper Lorentz Group	48
$\mathrm{SO}^e_{1,3} \equiv \mathcal{L}^{\uparrow}_{+}$	Proper Orthochronous Lorentz Group	48
$\mathbf{P}$	Poincaré Transformation	49
$\boxtimes$	Semi-Direct Product	49
$\mathcal{P}$	Poincaré Group	49
$\circledast$	Any one of the Products $\wedge$, $\cdot$, $\llcorner$, $\lrcorner$ or Clifford Product	52
$\partial_Y \circledast F(Y)$	One of the Derivative Operators Acting on $F(Y)$	53
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