Appendix

In the introduction of these notes, we claimed that groups as well as buildings can be viewed as distinguished classes of schemes. The purpose of this appendix is to prove these claims. Let us start with groups.

Let Θ be a group. For each $\theta \in \Theta$, we define

$$\tilde{\theta} := \{(\zeta, \eta) \in \Theta \times \Theta \mid \zeta\theta = \eta\}.$$

We set $\tilde{\Theta} := \{\tilde{\theta} \mid \theta \in \Theta\}$ and

$$\mathcal{T}(\Theta) := (\Theta, \tilde{\Theta}).$$

We set

$$\mathcal{G}(X, G) := \{\{g\} \mid g \in G\}.$$

Theorem A (i) *Assume that (X, G) is thin. Then $\mathcal{G}(X, G)$ is a group with respect to the complex multiplication and with $\{1\}$ as identity element.*

(ii) *Let Θ be a group. Then $\mathcal{T}(\Theta)$ is a thin scheme.*

(iii) *If (X, G) is thin, $\mathcal{T}(\mathcal{G}(X, G)) \cong (X, G)$.*

(iv) *For each group Θ, $\mathcal{G}(\mathcal{T}(\Theta)) \cong \Theta$.*

Proof. (i) From Theorem 1.2.7 we conclude that the complex product of two elements of $\mathcal{G}(X, G)$ is again an element of $\mathcal{G}(X, G)$. Thus, by Lemma 1.2.1(ii), $\mathcal{G}(X, G)$ is a semigroup.

From Lemma 1.1.1(i) we obtain that $\{1\}$ is the identity element of $\mathcal{G}(X, G)$. Finally, as (X, G) is assumed to be thin, we deduce from Lemma 1.2.6 that, for each $g \in G$, $\{g^*\}\{g\} = \{1\} = \{g\}\{g^*\}$.

(ii)[1] First of all, it is clear that $\tilde{1} = 1_\Theta$ and that, for each $\theta \in \Theta$,

$$\tilde{\theta}^* = \widetilde{\theta^{-1}}.$$

But also the regularity condition is easily verified for the pair $(\Theta, \tilde{\Theta})$. Let β, γ, ϵ, ζ, $\eta \in \Theta$ be given, and assume that $(\beta, \gamma) \in \tilde{\eta}$. Then, $\beta\tilde{\epsilon} \cap \gamma\tilde{\zeta}^* \neq \emptyset$ if

[1] We repeat the argument given in the introduction of these notes.

and only if $\beta\epsilon = \gamma\zeta^{-1}$ if and only if $\beta\epsilon\zeta = \gamma$ if and only if $\epsilon\zeta = \eta$. Thus, $|\beta\tilde{\epsilon} \cap \gamma\tilde{\zeta}^*| = \delta_{\epsilon\zeta,\eta}$.[2]

(iii) Set $\Theta := \mathcal{G}(X, G)$, and fix $v \in X$. For each $x \in X$, we denote by $x\phi$ the uniquely determined element of Θ such that $x \in v(x\phi)$. For each $g \in G$, we define

$$g\phi := \widetilde{\{g\}}.$$

Then

$$\phi : X \cup G \to \Theta \cup \tilde{\Theta}$$

is a well-defined bijective map with $X\phi \subseteq \Theta$ and $G\phi \subseteq \tilde{\Theta}$.

Let y, $z \in X$, and let $g \in G$ such that $(y, z) \in g$ be given. By Lemma 1.7.3(iii), we shall be done if we succeed in showing that $(y\phi, z\phi) \in g\phi$.

Let $e \in G$ be such that $(v, y) \in e$, and let $f \in G$ be such that $(v, z) \in f$. Then, by Lemma 1.2.4, $f \in \{e\}\{g\}$. From this we conclude that $\{f\} = \{e\}\{g\}$; see Theorem 1.2.7, and recall that (X, G) is assumed to be thin. Thus, by definition,

$$(\{e\}, \{f\}) \in \widetilde{\{g\}}.$$

On the other hand, as $(v, y) \in e$, $\{e\} = y\phi$. Similarly, as $(v, z) \in f$, $\{f\} = z\phi$. Therefore, we conclude that $(y\phi, z\phi) \in g\phi$.

(iv) For each $\theta \in \Theta$, we define

$$\theta\phi := \{\tilde{\theta}\}.$$

Then

$$\phi : \Theta \to \mathcal{G}(\mathcal{T}(\Theta))$$

is a well-defined bijective map.

Let ζ, $\eta \in \Theta$ be given, and set $\theta := \zeta\eta$. We shall be done if we succeed in showing that $\theta\phi = \zeta\phi\eta\phi$.

First of all, since $\zeta\eta = \theta$, $(\zeta, \theta) \in \tilde{\eta}$. Moreover, we also have $(1, \zeta) \in \tilde{\zeta}$ and $(1, \theta) \in \tilde{\theta}$. Thus, by Lemma 1.2.4, $\tilde{\theta} \in \tilde{\zeta}\tilde{\eta}$. But, by (ii), $n_{\tilde{\zeta}} = 1 = n_{\tilde{\eta}}$. Thus, by Theorem 1.2.7,

$$\{\tilde{\theta}\} = \{\tilde{\zeta}\}\{\tilde{\eta}\}.$$

It follows that $\theta\phi = \zeta\phi\eta\phi$. □

Let us now see how buildings can be viewed as a distinguished class of schemes.

Let C be a set, and let $r \subseteq C \times C$ be given.

We shall say that r is *regular* if, for all a, $b \in C$, $|ar| = |br|$.

Assume that r is regular, and let $c \in C$ be given. Then $|cr|$ is called the *valency* of r.

For each $s \subseteq C \times C$, we define

[2] Here δ is the Kronecker delta.

$$r \circ s := \{(a,b) \in C \times C \mid ar \cap bs^* \neq \emptyset\}.$$

The element r is called an *incidence relation* of C if

$$1_C \cap r = \emptyset \neq r = r^*$$

and

$$r \circ r \subseteq 1_C \cup r.$$

We shall denote by

$$\mathrm{Inc}(C)$$

the set of the incidence relations of C.[3]

Lemma B *Let C be a set, and let R be a partition of $C \times C$ such that $\emptyset \notin R$. Let $l \in \mathrm{Inc}(C)$ be given, assume that l is regular, and let n denote the valency of l.*

Let s, $t \in R$ be such that $s \circ l = t \neq s$. Let a, $b \in C$, and let $r \in R$ be such that $(a,b) \in r$. Then $|as \cap bl| = \delta_{rt}$ and

$$|at \cap bl| = \begin{cases} n & \text{if } r = s, \\ n-1 & \text{if } r = t, \\ 0 & \text{if } r \notin \{s,t\}. \end{cases}$$

Proof. Clearly, if $r = t$, $as \cap bl \neq \emptyset$. Conversely, assume that $as \cap bl \neq \emptyset$. Then

$$(a,b) \in r \cap (s \circ l) = r \cap t,$$

whence $r = t$.

Assume that $2 \leq |as \cap bl|$. Let d, $e \in as \cap bl$ be such that $d \neq e$. Then, since $l = l^*$ and $l \circ l \subseteq 1_C \cup l$, $(d,e) \in l$. Thus,

$$(a,e) \in s \cap (s \circ l) = s \cap t.$$

It follows that $s = t$, contrary to our hypothesis.

Thus we have shown that $|as \cap bl| = \delta_{rt}$. Let us now compute $|at \cap bl|$.

If $r = s$, $bl \subseteq a(s \circ l) = at$, whence $|at \cap bl| = n$.

Now assume that $at \cap bl \neq \emptyset$ and that $r \neq s$. Let $d \in at \cap bl$ be given. Since $t \subseteq s \circ l$, $as \cap dl \neq \emptyset$. Let $e \in as \cap dl$ be given. Then $el \subseteq a(s \circ l) = at$. On the other hand, as $(e,b) \in l \circ l \subseteq 1_C \cup l$ and $s \neq r$, $b \in el$. Thus, $r = t$ and $bl \setminus \{e\} \subseteq el \subseteq at$. It follows that $|at \cap bl| = n - 1$. □

[3] There is an obvious bijection between $\mathrm{Inc}(C)$ and the set of the non-identity equivalence relations of C. For each $r \in \mathrm{Inc}(C)$, $1_C \neq 1_C \cup r$ and $1_C \cup r$ is an equivalence relation of C. Conversely, for each equivalence relation r of C such that $1_C \neq r$, $r \setminus 1_C \in \mathrm{Inc}(C)$.

Let C be a set, and let $L \subseteq \mathrm{Inc}(C)$ be given. Then the pair (C, L) will be called a *chamber system*.[4] It is called *regular* if each element of L is regular.

Let (C, L) be a chamber system.

Recall that we denote by $\mathbf{F}(L)$ the free monoid constructed on L. By $*$ we denote the multiplication of $\mathbf{F}(L)$. The identity element of $\mathbf{F}(L)$ is denoted by $\mathbf{1}$.

Let $n \in \mathbb{N} \setminus \{0\}$, let $c_0, \ldots, c_n \in C$, and let $l_1, \ldots, l_n \in L$ be given. Set $\mathbf{f} := l_1 * \cdots * l_n$. Then $(c_0, \ldots, c_n)$ is called a *gallery of type* $\mathbf{f}$ *(from* c_0 *to* c_n*)* if, for each $i \in \{1, \ldots, n\}$, $(c_{i-1}, c_i) \in l_i$.[5]

For each $c \in C$, (c) will be called a *gallery of type* $\mathbf{1}$.

Recall that we denote by $\mathbf{R}(G)$ the monoid of all non-empty subsets of G with respect to the complex multiplication. Moreover, for each $L \subseteq \mathrm{Inv}(G)$, ρ_L denotes the uniquely determined monoid homomorphism from $\mathbf{F}(L)$ to $\mathbf{R}(G)$ such that, for each $l \in L$, $l\rho_L = \{l\}$.

Lemma C *Let* $L \subseteq \mathrm{Inv}(G)$ *be given. Then we have*

(i) (X, L) *is a regular chamber system.*

(ii) *Let* $y, z \in X$, *and let* $g \in G$ *be such that* $(y, z) \in g$. *Let* $\mathbf{f} \in \mathbf{F}(L)$ *be given. Set* $\rho := \rho_L$. *Then* $g \in \mathbf{f}\rho$ *if and only if* (X, L) *possesses a gallery of type* $\mathbf{f}$ *from* y *to* z.

Proof. (i) From Lemma 1.4.5(i), (ii) we obtain that $\mathrm{Inv}(G) \subseteq \mathrm{Inc}(X)$. This proves (i).

(ii) Clearly, we have $g \in \mathbf{1}\rho$ if and only if $g = 1$ if and only if $y = z$. Therefore, the claim is obvious if $\mathbf{1} = \mathbf{f}$.

Asume now that $\mathbf{1} \neq \mathbf{f}$. Then there exist $n \in \mathbb{N} \setminus \{0\}$ and $l_1, \ldots, l_n \in L$ such that $\mathbf{f} = l_1 * \cdots * l_n$.

If $g \in \mathbf{f}\rho$, $g \in l_1 \cdots l_n$. Thus, by Lemma 1.2.4, there exist $x_0, \ldots, x_n \in X$ such that $x_0 = y$, $x_n = z$, and, for each $i \in \{1, \ldots, n\}$, $(x_{i-1}, x_i) \in l_i$. This means that there exists a gallery of type $\mathbf{f}$ from y to z.

Conversely, assume that there exists a gallery of type $\mathbf{f}$ from y to z. Then there exist $x_0, \ldots, x_n \in X$ such that $x_0 = y$, $x_n = z$, and, for each $i \in \{1, \ldots, n\}$, $(x_{i-1}, x_i) \in l_i$. Thus, by Lemma 1.2.4, $g \in l_1 \cdots l_n$. (Recall that $(y, z) \in g$.) It follows that $g \in \mathbf{f}\rho$. □

Let (C, L) be a chamber system, and let m be a Coxeter map of L.

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[4] Formally, our definition of chamber systems differs slightly from the usual one; see, e.g., [28]. Usually, the pair (C, L) is called a chamber system if L is a family of partitions on C. But, as mentioned earlier, non-identity partitions on C and incidence relations of C correspond to each other uniquely. Thus, our chamber systems correspond to those chamber systems in the sense of [28] which neither contain the identity partition nor have repeated partitions.

[5] What we call a "gallery" here is the same as a "simple gallery" in [28].

Recall that $\mathbf{F}_m(L)$ denotes the set of the m-reduced elements of $\mathbf{F}(L)$.

We shall denote by $\Gamma_m(L)$ the Coxeter group defined by m. By ϕ_m we shall denote the natural monoid homomorphism from $\mathbf{F}(L)$ to $\Gamma_m(L)$.

The pair (C, L) is called a *building of type* m if (C, L) is regular and if there exists a map δ from $C \times C$ to $\Gamma_m(L)$ such that, for all a, $b \in C$ and, for each $\mathbf{f} \in \mathbf{F}_m(L)$, $\delta(a, b) = \mathbf{f}\phi_m$ if and only if (C, L) possesses a gallery of type $\mathbf{f}$ from a to b.

The pair (C, L) is called a *building* if there exists a Coxeter map m of L such that (C, L) is a building of type m.[6]

Assume now that (C, L) is a building. Then, by definition, there exist a Coxeter map m of L and a map δ from $C \times C$ to $\Gamma_m(L)$ as above. For each $\gamma \in \Gamma_m(L)$, we define

$$r_\gamma := \delta^{-1}(\gamma).$$

For each $\gamma \in \Gamma_m(L)$, we shall denote by $\gamma\ell$ the length of γ as an element of the Coxeter group $\Gamma_m(L)$.

Lemma D *Let m be a Coxeter map, and let (C, L) be a building of type m. Let γ, $\lambda \in \Gamma_m(L)$ be such that $\lambda\ell = 1$ and $\gamma\ell + 1 = (\gamma\lambda)\ell$. Then $r_\gamma \circ r_\lambda = r_{\gamma\lambda}$.*

Proof. Let us abbreviate

$$\phi := \phi_m.$$

Let $\mathbf{f} \in \mathbf{F}_m(L)$ be such that

$$\mathbf{f}\phi = \gamma,$$

and let $l \in L$ be such that

$$l\phi = \lambda.$$

Then the hypothesis $\gamma\ell + 1 = (\gamma\lambda)\ell$ implies that $\mathbf{f} * l \in \mathbf{F}_m(L)$.

Now let a, $b \in C$ be given, and let us assume first that $(a, b) \in r_{\gamma\lambda}$. Then, by definition,

$$\delta(a, b) = \gamma\lambda = \mathbf{f}\phi l\phi = (\mathbf{f} * l)\phi.$$

Thus, as $\mathbf{f} * l \in \mathbf{F}_m(L)$, (C, L) possesses a gallery $(c_0, \dots, c_n)$, say, of type $\mathbf{f} * l$ from a to b. Set $c := c_{n-1}$. Then, $(c_0, \dots, c_{n-1})$ is a gallery of type $\mathbf{f}$ from a to c, and (c_{n-1}, c_n) is a gallery of type l from c to b. Therefore, by definition, $\delta(a, c) = \mathbf{f}\phi = \gamma$ and $\delta(c, b) = l\phi = \lambda$. It follows that $(a, c) \in r_\gamma$ and $(c, b) \in r_\lambda$. Thus, $(a, b) \in r_\gamma \circ r_\lambda$.

Let us now assume that $(a, b) \in r_\gamma \circ r_\lambda$. Then there exists $c \in C$ such that $(a, c) \in r_\gamma$ and $(c, b) \in r_\lambda$. It follows that $\delta(a, c) = \gamma = \mathbf{f}\phi$ and $\delta(c, b) = \lambda = l\phi$.

[6] As indicated in the introduction of these notes, our definition of buildings is midway between the two definitions given in [27] and [28].

Thus, by definition, (C, L) possesses a gallery $(c_0, \ldots, c_n)$, say, of type $\mathbf{f}$ from a to c and a gallery of type l from c to b. Now $(c_0, \ldots, c_n, b)$ is a gallery of type $\mathbf{f} * l$ from a to b. Thus, as $\mathbf{f} * l \in \mathbf{F}_m(L)$,

$$\delta(a, b) = (\mathbf{f} * l)\phi = \mathbf{f}\phi l\phi = \gamma\lambda.$$

It follows that $(a, b) \in r_{\gamma\lambda}$. □

For each building (C, L), we set $R(C, L) := \{r_\gamma \mid \gamma \in \Gamma_m(L)\}$ and

$$\mathcal{A}(C, L) := (C, R(C, L)).$$

For each $L \subseteq \mathrm{Inv}(G)$, we define

$$\mathcal{B}_L(X, G) := (X, L).$$

The second part of the following theorem uses Theorem 3.1.5(ii).

Theorem E (i) *Let $L \subseteq \mathrm{Inv}(G)$ be given, and assume that (X, G) is a Coxeter scheme with respect to L. Then $\mathcal{B}_L(X, G)$ is a building of type m_L.*[7]

(ii) *Let m be a Coxeter map, and let (C, L) be a building of type m. Then $\mathcal{A}(C, L)$ is a Coxeter scheme with respect to L. Moreover, $m = m_L$.*

(iii) *Let $L \subseteq \mathrm{Inv}(G)$ be given, and assume that (X, G) is a Coxeter scheme with respect to L. Then $\mathcal{A}(\mathcal{B}_L(X, G)) = (X, G)$.*

(iv) *For each building (C, L), $\mathcal{B}_L(\mathcal{A}(C, L)) = (C, L)$.*

Proof. (i) From Lemma C(i) we know already that $\mathcal{B}_L(X, G)$ is a regular chamber system.

Let us abbreviate $m := m_L$ and $\rho := \rho_L$.

Let y, $z \in X$, and let $g \in G$ be such that $(y, z) \in g$. Then, by Proposition 5.1.3(ii), there exists $\mathbf{f} \in \mathbf{F}_m(L)$ such that $g \in \mathbf{f}\rho$. We define

$$\delta(y, z) := \mathbf{f}\phi_m.$$

Let us first convince ourselves that δ is a well-defined map from $X \times X$ to $\Gamma_m(L)$. Let y, $z \in X$, let $g \in G$ such that $(y, z) \in g$, and let $\mathbf{d}, \mathbf{e} \in \mathbf{F}_m(L)$ such that $g \in \mathbf{d}\rho \cap \mathbf{e}\rho$ be given.

Assuming that (X, G) is a Coxeter scheme with respect to L, we have, in particular, that (X, G) is L-constrained. Therefore, as $\mathbf{d}\rho \cap \mathbf{e}\rho \neq \emptyset$, we obtain that $\mathbf{d}\rho = \mathbf{e}\rho$. This gives $\mathbf{d} \sim_m \mathbf{e}$. (We use once again that (X, G) is a Coxeter scheme with respect to L.) It follows that $\mathbf{d}\phi_m = \mathbf{e}\phi_m$.

[7] Recall that m_L is a Coxeter map which was introduced in the beginning of Section 5.1.

Let y, $z \in X$, and let $\mathbf{f} \in \mathbf{F}_m(L)$ be given. We have to prove that $\delta(y, z) = \mathbf{f}\phi_m$ if and only if $\mathcal{B}_L(X, G)$ possesses a gallery of type $\mathbf{f}$ from y to z.

Let $g \in G$ be such that $(y, z) \in g$.

Assume first that $\delta(y, z) = \mathbf{f}\phi_m$. Then the definition of δ yields $g \in \mathbf{f}\rho$. Thus, by Lemma C(ii), $\mathcal{B}_L(X, G)$ possesses a gallery of type $\mathbf{f}$ from y to z.

Conversely, assume that $\mathcal{B}_L(X, G)$ possesses a gallery of type $\mathbf{f}$ from y to z. Then, by Lemma C(ii), $g \in \mathbf{f}\rho$. Now, as $(y, z) \in g$, we have $\delta(y, z) = \mathbf{f}\phi_m$.

(ii) Since (C, L) is assumed to be a building of type m, there exists a map δ from $C \times C$ to $\Gamma_m(L)$ such that, for all a, $b \in C$ and, for each $\mathbf{f} \in \mathbf{F}_m(L)$, $\delta(a, b) = \mathbf{f}\phi_m$ if and only if (C, L) possesses a gallery of type $\mathbf{f}$ from a to b.

Let us abbreviate

$$R := R(C, L).$$

Clearly, δ is surjective. Therefore, $\emptyset \notin R$. Moreover, $1_C = r_1 \in R$ and, for each $\gamma \in \Gamma_m(L)$, $(r_\gamma)^* = r_{\gamma^{-1}} \in R$. Thus, in order to prove that $\mathcal{A}(C, L)$ is a scheme, we only have to verify the regularity condition for $\mathcal{A}(C, L)$.

Let r, g, $l \in R$ be given. Let a, $b \in C$ be such that $(a, b) \in r$. In order to prove that $\mathcal{A}(C, L)$ satisfies the regularity condition, we have to show that the cardinality of $ag \cap bl^*$ does not depend on the choice of $(a, b) \in r$.

Let $\gamma \in \Gamma_m(L)$ be such that

$$g = r_\gamma,$$

and let $\lambda \in \Gamma_m(L)$ be such that

$$l = r_\lambda.$$

There is nothing to prove if $1_C = l$. Thus, we may assume that $1_C \neq l$. Moreover, by induction on $\lambda\ell$, we may assume that $\lambda\ell = 1$.

Assume first that $\gamma\ell + 1 = (\gamma\lambda)\ell$. Set $t := r_{\gamma\lambda}$. Then, by Lemma D,

$$g \circ l = r_\gamma \circ r_\lambda = r_{\gamma\lambda} = t.$$

Thus, by Lemma B, $|ag \cap bl| = \delta_{rt}$. (Set $s := g$.)

Assume now that $(\gamma\lambda)\ell + 1 = \gamma\ell$. Set $s := r_{\gamma\lambda}$. Then, by Lemma D,

$$s \circ l = r_{\gamma\lambda} \circ r_\lambda = r_\gamma = g.$$

Thus, by Lemma B, $|ag \cap bl|$ does not depend on the choice of $(a, b) \in r$. (Set $t := g$, and use the hypothesis that l is regular.)

We now have proved that $\mathcal{A}(C, L)$ is a scheme, and we still have to prove that $\mathcal{A}(C, L)$ is a Coxeter scheme with respect to L. (Note that $L \subseteq \mathrm{Inv}(R)$.)

Let $\mathbf{R}(R)$ denote the monoid of all non-empty subsets of R with respect to the complex multiplication (in R). Let us denote by ρ' the uniquely determined monoid homomorphism from $\mathbf{F}(L)$ to $\mathbf{R}(R)$ such that, for each $l \in L$, $l\rho' = \{l\}$.

We first shall prove that $\mathcal{A}(C, L)$ is L-constrained, in other words that, for each $\mathbf{f} \in \mathbf{F}_m(L)$, $|\mathbf{f}\rho'| = 1$. (It is clear that $\langle L \rangle = R$.)

Let $\mathbf{f} \in \mathbf{F}_m(L)$ be given. Then $\mathbf{f}\phi_m \in \Gamma_m(L)$. Set $r := \delta^{-1}(\mathbf{f}\phi_m)$. Then $r \in R$.

Let a, $b \in C$ be such that $(a,b) \in r$. Then, by definition, $\delta(a,b) = \mathbf{f}\phi_m$. Thus, by definition, (C, L) possesses a gallery of type $\mathbf{f}$ from a to b. Thus, by Lemma C(ii), $r \in \mathbf{f}\rho'$.

Let $s \in \mathbf{f}\rho'$ be given. Then $s \in R$. Therefore, there exists $\mathbf{e} \in \mathbf{F}_m(L)$ such that $s = \delta^{-1}(\mathbf{e}\phi_m)$. Let a, $b \in C$ be such that $(a,b) \in s$. Then, by definition, $\delta(a,b) = \mathbf{e}\phi_m$. Thus, (C,L) possesses a gallery of type $\mathbf{e}$ from a to b. Thus, by Lemma C(ii), $s \in \mathbf{e}\rho'$. But, as $s \in \mathbf{f}\rho'$, (C, L) possesses a gallery of type $\mathbf{f}$ from a to b; use Lemma C(ii) once again. Thus, $\mathbf{e}\phi_m = \mathbf{f}\phi_m$, whence $r = s$.

Since $s \in \mathbf{f}\rho'$ has been chosen arbitrarily, we have shown that $\{r\} = \mathbf{f}\rho'$. Since $\mathbf{f} \in \mathbf{F}_m(L)$ has been chosen arbitrarily, we have shown that $\mathcal{A}(C, L)$ is L-constrained.

Let $\mathbf{d}$, $\mathbf{e} \in \mathbf{F}_m(L)$ be such that $\mathbf{d}\rho' = \mathbf{e}\rho'$. Let $r \in \mathbf{d}\rho'$, and let a, $b \in C$ be such that $(a,b) \in r$. Then, by Lemma C(ii), (C, L) possesses a gallery of type $\mathbf{d}$ from a to b as well as a gallery of type $\mathbf{e}$ from a to b. It follows that $\mathbf{d}\phi_m = \mathbf{e}\phi_m$. Therefore, $\mathbf{d} \approx_m \mathbf{e}$. Thus, by Theorem 3.1.5(ii), $\mathbf{d} \sim_m \mathbf{e}$.

Since $\mathbf{d}$, $\mathbf{e} \in \mathbf{F}_m(L)$ have been chosen arbitrarily, we now have proved that $\mathcal{A}(C, L)$ is a Coxeter scheme with respect to m.

(iii) Let us abbreviate

$$R := R(X, L).$$

Then we just have to prove that $G = R$.

Let $\mathbf{R}(R)$ denote the monoid of all non-empty subsets of R with respect to the complex multiplication. Let us denote by ρ' the uniquely determined monoid homomorphism from $\mathbf{F}(L)$ to $\mathbf{R}(R)$ such that, for each $l \in L$, $l\rho' = \{l\}$.

Let $g \in G$, and let $r \in R$ be such that $g \cap r \neq \emptyset$. We shall prove that $g = r$.

Let y, $z \in X$ be such that $(y, z) \in g \cap r$. By Proposition 5.1.3(ii), there exists $\mathbf{f} \in \mathbf{F}_m(L)$ such that

$$\{g\} = \mathbf{f}\rho.$$

Thus, (X, L) possesses a gallery of type $\mathbf{f}$ from y to z; see Lemma C(ii). From Lemma C(ii) we now deduce that $r \in \mathbf{f}\rho'$. Thus, as (X, R) is L-constrained, we obtain that

$$\{r\} = \mathbf{f}\rho'.$$

Let v, $w \in X$ be such that $(v, w) \in g$, and let $s \in R$ be such that $(v, w) \in s$. Since $(v, w) \in g$, (X, L) possesses a gallery of type $\mathbf{f}$ from v to w; see Lemma C(ii). Thus, using Lemma C(ii) once again, we conclude that $s \in \mathbf{f}\rho' = \{r\}$. It follows that $r = s$. In particular, $(v, w) \in r$.

Since v, $w \in X$ have been chosen arbitrarily such that $(v, w) \in g$, we conclude that $g \subseteq r$. Similarly, we obtain that $r \subseteq g$. It follows that $g = r$.

Since $g \in G$ and $r \in R$ have been chosen arbitrarily, we have shown that $G = R$.

(iv) Since (C, L) is assumed to be a building, $\mathcal{A}(C, L)$ is a Coxeter scheme with respect to L; see (ii). Thus, by definition, $\mathcal{B}_L(\mathcal{A}(C, L)) = (C, L)$. □

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