Appendix

The use of Mathematica®

We are going to illustrate the use of Mathematica® with a concrete example of the construction of a 4–dimensional AB–group. We will execute all the computations needed in case number 3 in the table of section 7.2.

A.1 Choose a crystallographic group Q

The group E to be constructed fits in a short exact sequence

$$1 \to \mathbb{Z} \to E \to Q \to 1$$

where Q is the 3–dimensional crystallographic group listed in [10] on page 62 as follows (the numbering of the lines is added):

```
1   FAMILY II: ...

2     CRYSTAL SYSTEM 2: ...

3       Q-CLASS 2/1: ORDER 2; ISOM TYPE 2.1; 2 Z-CLASSES ...

4                    REL: A2=I

5       Z-CLASS 2/1/1: Z(P2); ...

6           GEN:       A -1  0  0
7                         0  1  0
8                         0  0 -1

9           SPGR: 01 A [0,0,0]     IT 3; OBT 1
10            FF  02 A [0,1,0]/2   IT 4; OBT 1
```

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```
11      Z-CLASS ...
```

We extract the information we need in the following way:

- On line 3 we see that the holonomy group F of Q is of order 2 and so F is isomorphic to $\mathbb{Z}_2$ (= Isomorphism Type 2.1 in [10, Table 6B]).
- Line 4 describes the holonomy group F. There is one generator A and one relation $A^2 = 1$. We will use α (or `alfa`) to denote this generator in the sequel.
- The group Q has an affine representation (seen in $Gl(4,\mathbb{R})$) as follows:
  First there are the three translations, which are always the same and which we denote by a, b, c:

  $$a = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

  Further, the affine transformation corresponding to α is indicated in lines 6,7,8 (the rotational part) and in line 9 (the translational part). So

  $$\alpha = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- A presentation for Q can be written down now very easily following section 5.2. One should keep in mind that the action of F on $\mathbb{Z}^3$ is given by the rotational part of α. For this group, we can see, since the translational part of α equals zero, that $Q = \mathbb{Z}^3 \rtimes \mathbb{Z}_2$. For other groups, one will have to use the matrix representation of Q to complete its presentation. Conclusion:

$$Q = < a, b, c, \alpha \parallel \begin{array}{ll} [b,a] = 1 & [c,a] = 1 \\ [c,b] = 1 & \alpha^2 = 1 \\ \alpha a = a^{-1}\alpha & \alpha b = b\alpha \\ \alpha c = c^{-1}\alpha & \end{array} > .$$

- Line 10 denotes the next crystallographic group (number 4) and the symbol **FF** in front of it, indicates that this group is torsion free. (Fixedpoint Free space group).

Continuing as in section 5.2 we now describe any extension of Q by $\mathbb{Z}$ compatible with the trivial action of Q on $\mathbb{Z}$ (which factors through $\mathbb{Z}_2$). (Since $\alpha^2 = 1$, we cannot hope that extensions compatible with the non trivial action, will be torsion free). Such a group E has a presentation as follows:

$$E : < a, b, c, d, \alpha \parallel \begin{array}{ll} [b,a] = d^{l_1} & [d,a] = 1 \\ [c,a] = d^{l_2} & [d,b] = 1 \\ [c,b] = d^{l_3} & [d,c] = 1 \\ \alpha a = a^{-1}\alpha d^{l_4} & \alpha^2 = d^{l_7} \\ \alpha b = b\alpha d^{l_5} & \alpha d = d\alpha \\ \alpha c = c^{-1}\alpha d^{l_6} & \end{array} > .$$

for some integers $l_1, l_2, \ldots, l_7$. We use Mathematica® to find the computational consistent ones, which is described in the following section.

A.2 Determination of computational consistent presentations

The following little program contains a function "matrixmacht" which computes formal powers of a unitriangular matrix, based on lemma 4.4.5. This program is runned automatically each time we start a Mathematica® session.

```
(* Personal Commands Used in the Infra-nil computations *)
(* ---------------------------------------------------- *)

(* Remark: All variables are written with 3 identical symbols to *)
(*         prevent confusion with variables used within a        *)
(*         Mathematica session.                                  *)


(* Definition of the binomial coefficients *)
(* xxx : is a formal parameter *)
(* nnn : is an integer *)

bin[xxx_,nnn_ ]:=(xxx-nnn+1)/nnn bin[xxx,nnn-1]
bin[xxx_,0 ]:=1
```

```
(* matrixmacht : Power of an uppertringular matrix "aaa" with *)
(*               formal power "xxx".                            *)
(*               Based on the formula:                          *)
(*               $A^x=((A-I)+I)^x=\sum_x \bn{x}{l} (A-I)^x$   *)

matrixmacht[aaa_ , xxx_ ]:=Sum[ (bin[xxx , lll]*
                      MatrixPower[aaa-IdentityMatrix[ Length[aaa]],lll]),
                      {lll,0,Length[aaa]}]


(* Some convenient abbreviations *)

mat[aaa_]:=TableForm[Expand[aaa]]
com[aaa_,bbb_]:=Expand[Inverse[aaa].Inverse[bbb].aaa.bbb]
```

Now we are ready to start the program used do determine the computational consistent presentations for E. The program is stored in a file with the name `groep.3` and looks like

```
(* Determining the computational consistent presentations and affine *)
(* representations for this class of AC-groups.                       *)

(* A general canonical type representation built up from the data    *)
(* in the book of Neubueser e.a.                                      *)

a={{1,A1,A2,A3,A4},
   {0,1,0,0,1},
   {0,0,1,0,0},
   {0,0,0,1,0},
   {0,0,0,0,1}}

b={{1,B1,B2,B3,B4},
   {0,1,0,0,0},
   {0,0,1,0,1},
   {0,0,0,1,0},
   {0,0,0,0,1}}

c={{1,C1,C2,C3,C4},
   {0,1,0,0,0},
   {0,0,1,0,0},
   {0,0,0,1,1},
   {0,0,0,0,1}}

d={{1,0,0,0,1},
   {0,1,0,0,0},
   {0,0,1,0,0},
```

```
   {0,0,0,1,0},
   {0,0,0,0,1}}

alfa={{1,alf1,alf2,alf3,alf4},(* The "1" on this line indicates that we*)
      {0,-1, 0, 0, 0},        (* are dealing with a trivial action of  *)
      {0, 0, 1, 0, 0},        (* alfa on d                             *)
      {0, 0, 0,-1, 0},
      {0, 0, 0, 0, 1}}

(*The conditions, which should be satisfied by the unknowns A1,...,alf4*)
(*And the conditions on l1,l2,...,l7 for computational consistency.    *)
(*All matrices printed should be zero.                                 *)

Print[ Expand[ com[b,a]-matrixmacht[d,l1] ]]
Print[ Expand[ com[c,a]-matrixmacht[d,l2] ]]
Print[ Expand[ com[c,b]-matrixmacht[d,l3] ]]
Print[ Expand[ alfa.a-Inverse[a].alfa.matrixmacht[d,l4] ]]
Print[ Expand[ alfa.b-b.alfa.matrixmacht[d,l5] ]]
Print[ Expand[ alfa.c-Inverse[c].alfa.matrixmacht[d,l6] ]]
Print[ Expand[ alfa.alfa-matrixmacht[d,l7] ]]
```

We now list a recorded version of a Mathematica® session to show how we use this program:

```
euler% math
Mathematica 2.1 for SPARC
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In[1]:= <<groep3.1
{{0, 0, 0, 0, -A2 + B1 - l1}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, -A3 + C1 - l2}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, -B3 + C2 - l3}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 0, 2 A2, 0,alf1 - A1 + 2 A4 - l4}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 2 B1, 0, 2 B3, alf2 - l5}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 0, 2 C2, 0,alf3 - C3 + 2 C4 - l6}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
```

```
{{0, 0, 2 alf2, 0, 2 alf4 - 17}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}

In[2]:= A2=B1-l1;A3=C1-l2;B3=C2-l3;alf2=l5;alf4=17/2;

In[3]:= <<groep3.1
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 2 B1 - 2 l1, 0, alf1 - A1 + 2 A4 - l4}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 2 B1, 0, 2 C2 - 2 l3, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 0, 2 C2, 0,alf3 - C3 + 2 C4 - l6}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 0, 2 l5, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}

In[4]:= B1=0;l1=0;C2=0;l3=0;l5=0;alf1=A1-2 A4 + l4;

In[5]:= <<groep3.1
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0,alf3 - C3 + 2 C4 - l6}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
```

```
>   {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}

In[6]:= alf3=C3 -2 C4 + 16;

In[7]:= <<groep3.1
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}
{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},

>   {0, 0, 0, 0, 0}}

In[8]:= Print[l1," ",l2," ",l3," ",l4," ",l5," ",l6," ",l7]
0 l2 0 l4 0 l6 l7

In[9]:= l2=k1;l4=k2;l6=k3;l7=k4;

In[10]:= mat[a]

Out[10]//TableForm= 1   A1   0   C1 - k1   A4

                    0   1    0   0         1

                    0   0    1   0         0

                    0   0    0   1         0

                    0   0    0   0         1

In[11]:= A1=0;C1=k1/2;A4=0;
```

```
In[12]:= mat[b]

Out[12]//TableForm= 1    0    B2    0    B4

                    0    1    0     0    0

                    0    0    1     0    1

                    0    0    0     1    0

                    0    0    0     0    1

In[13]:= B2=0;B4=0;

In[14]:= mat[c]

Out[14]//TableForm=      k1
                         --
                    1    2     0    C3    C4

                    0    1     0    0     0

                    0    0     1    0     0

                    0    0     0    1     1

                    0    0     0    0     1

In[15]:= C3=0;C4=0;

In[16]:= mat[alfa]

Out[16]//TableForm=                       k4
                                          --
                    1    k2    0    k3    2

                    0    -1    0    0     0

                    0    0     1    0     0

                    0    0     0    -1    0

                    0    0     0    0     1

In[17]:= mat[a]

Out[17]//TableForm=                -k1
                                   ---
                    1    0    0     2     0
```

```
                  0   1   0   0      1

                  0   0   1   0      0

                  0   0   0   1      0

                  0   0   0   0      1

In[18]:= mat[b]

Out[18]//TableForm= 1   0   0   0   0

                  0   1   0   0   0

                  0   0   1   0   1

                  0   0   0   1   0

                  0   0   0   0   1

In[19]:= mat[c]

Out[19]//TableForm=     k1
                        --
                  1     2    0   0   0

                  0     1    0   0   0

                  0     0    1   0   0

                  0     0    0   1   1

                  0     0    0   0   1

In[20]:= Save["representatie3.1",a,b,c,d,alfa]

In[21]:= Quit
euler%
```

The program specifies the computational consistent groups E, which can all be presented by means of the four parameters k_1, k_2, k_3 and k_4:

$$E : < a, b, c, d, \alpha \parallel \begin{array}{ll} [b,a] = 1 & [d,a] = 1 \\ [c,a] = d^{k_1} & [d,b] = 1 \\ [c,b] = 1 & [d,c] = 1 \\ \alpha a = a^{-1}\alpha d^{k_2} & \alpha^2 = d^{k_4} \\ \alpha b = b\alpha & \alpha d = d\alpha \\ \alpha c = c^{-1}\alpha d^{k_3} & \end{array} > .$$

A.3 Computation of $H^2(Q, Z)$

The previous section also shows that the set of standard cocycles

$$SZ^2_\varphi(Q, \mathbb{Z}) \cong \mathbb{Z}^4$$

and that a canonical representation for each group E is saved in a file with the name `representatie 3.1`.

The following step in the process is to determine which standard cocycles are cohomologous to zero. This is done with a program having the name `cocyk3.1`:

```
(* Which standard cocycles are cohomologous to zero?   *)
(* By theorem 5.2.2 we may suppose that                *)
k1=0
<<representatie3.1

(* The action of alfa on d can be read of as the first entry of the *)
(* matrix which represents alfa                                     *)
actalfa=alfa[[1,1]]


For[kk=0,kk<=1,kk++,
  For[mm=0,mm<=1,mm++,
     {product=matrixmacht[a,x1].matrixmacht[b,x2].
              matrixmacht[c,x3].MatrixPower[alfa,kk].
              matrixmacht[a,y1].matrixmacht[b,y2].
              matrixmacht[c,y3].MatrixPower[alfa,mm],
              (* Product is a general product of two elements.  *)
              (* Product can be written in the form:            *)
              (* a^acoef b^bcoef c^ccoef d^f(x,y) alfa^alfacoef *)
              (* The following commands determine all these     *)
              (* coefficients.                                  *)
        alfacoef=Mod[kk+mm,2], (* Computation in Z-modulo 2 *)
        product=product.MatrixPower[alfa,-alfacoef],
        acoef=product[[2,5]], (* This is a direct consequence *)
```

```
        bcoef=product[[3,5]], (* of the canonical type          *)
        ccoef=product[[4,5]], (* representation                 *)
product=matrixmacht[c,-ccoef].matrixmacht[b,-bcoef].
                matrixmacht[a,-acoef].product,
        (* The power of d equals the cocycle *)
        f=Simplify[product[[1,5]]],
        (* A check for exactness of the program: *)
product[[1,5]]=0,
If[product==IdentityMatrix[5],Print[],Print["Something's wrong!"]],
      Print["Cocykel:"],
Print[f],
term=(actalfa)^(kk), (* term= the action of x on d *)
        (* The general form of delta gamma for x and y *)
deltagamma=( (term y1 + x1 -acoef) gammaa +
             (term y2 + x2 -bcoef) gammab +
             (term y3 + x3 -ccoef) gammac +
             (term mm + kk -alfacoef )gammaalfa ),
difference=Expand[f-deltagamma],
Print[],
Print["f(x,y)-deltagamma(x,y):"],
Print[difference],
Print["______________________________________",
              "_____________________________________"],
} ]]
Print["====================================",
     "====================================="]
```

The recorded Mathematica® session is as follows:

```
euler% math
Mathematica 2.1 for SPARC
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In[1]:= <<cocyk3.1
Cocykel:
0
f(x,y)-deltagamma(x,y):
0
-------------------------------------------------------------------------
Cocykel:
0
f(x,y)-deltagamma(x,y):
0
-------------------------------------------------------------------------
Cocykel:
k2 y1 + k3 y3
f(x,y)-deltagamma(x,y):
```

```
-2 gammaa y1 + k2 y1 - 2 gammac y3 + k3 y3
----------------------------------------------------------------------
Cocykel:
k4 + k2 y1 + k3 y3
f(x,y)-deltagamma(x,y):
-2 gammaalfa + k4 - 2 gammaa y1 + k2 y1 - 2 gammac y3 + k3 y3
----------------------------------------------------------------------
======================================================================

In[2]:= gammaa=k2/2;gammac=k3/2;gammaalfa=k4/2;

In[3]:= <<cocyk3.1
Cocykel:
0
f(x,y)-deltagamma(x,y):
0
----------------------------------------------------------------------
Cocykel:
0
f(x,y)-deltagamma(x,y):
0
----------------------------------------------------------------------
Cocykel:
k2 y1 + k3 y3
f(x,y)-deltagamma(x,y):
0
----------------------------------------------------------------------
Cocykel:
k4 + k2 y1 + k3 y3
f(x,y)-deltagamma(x,y):
0
----------------------------------------------------------------------
======================================================================

In[4]:= Quit
euler%
```

This shows that a cocycle f depending on the four variables k_1, k_2, k_3 and k_4 is cohomologous to 0 if and only if

$$k_1 = 0,\ k_2, k_3, k_4 \in 2\mathbb{Z}$$

This allows us to conclude that

$$H^2_\varphi(Q, \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}_2)^3$$

A.4 Investigation of the torsion

The following problem is to discover in which groups E there is torsion. To achieve this, we first look for all torsion elements in the crystallographic group Q. Since the order of the holonomy group is 2, the order of a torsion element will also be equal to two. The program used to find torsion in Q is called `torsion` and is the following:

```
(* Searching the torsion elements of the crystallographic group Q *)

(* Loading the representation for E and so for Q *)
<<representatie3.1

(* order= The order of a possible torsion element *)
order=2

Print[" "]
(* A general element having a possible finite order *)
test=matrixmacht[a,m1].matrixmacht[b,m2].matrixmacht[c,m3].alfa

(* testelement to the power=order *)
testpower=MatrixPower[test,order]

(* Looking at the element in Q rather than in E *)
testcutoff=IdentityMatrix[4]
For[i=1,i<=4,i++,
 For[j=1,j<=4,j++,
  testcutoff[[i,j]]=testpower[[i+1,j+1]] ]]

(* What are the possible torsions? *)
Print[mat[testcutoff]]
Print[" "]
Print[Solve[testcutoff==IdentityMatrix[4],{m1,m2,m3}] ]
Print[" "]
```

The result of this program is:

```
euler% math
Mathematica 2.1 for SPARC
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In[1]:= <<torsion

1   0   0   0

0   1   0   2 m2
```

```
0    0    1    0

0    0    0    1

{{m2 -> 0}}


In[2]:= Quit
euler%
```

Before we continue, we have to notice that the file `representatie3.1` is in a dangerous form, since it contains a lot of redundancy, which may cause errors. Therefore we do the following:

```
euler% cat representatie3.1
a = {{1, A1, 0, C1 - l2, A4}, {0, 1, 0, 0, 1}, {0, 0, 1, 0, 0},
   {0, 0, 0, 1, 0}, {0, 0, 0, 0, 1}}

A1 = 0

C1 = k1/2

l2 = k1

A4 = 0

b = {{1, 0, B2, 0,B4}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 1}, {0, 0, 0, 1, 0},
   {0, 0, 0, 0, 1}}

B2 = 0

B4 = 0

c = {{1,C1, 0,C3, C4}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 1},
   {0, 0, 0, 0, 1}}

C3 = 0

C4 = 0

d = {{1, 0, 0, 0, 1}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0},
   {0, 0, 0, 0, 1}}

alfa = {{1, A1 - 2*A4 + l4, 0, C3 - 2*C4 + l6, l7/2}, {0, -1, 0, 0, 0},
   {0, 0, 1, 0, 0}, {0, 0, 0, -1, 0}, {0, 0, 0, 0, 1}}

l4 = k2
```

```
16 = k3

17 = k4
euler% math
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In[1]:= <<representatie3.1

Out[1]= k4

In[2]:= DeleteFile["representatie3.1"]

In[3]:= a>>representatie3.1

In[4]:= b>>>representatie3.1

In[5]:= c>>>representatie3.1

In[6]:= d>>>representatie3.1

In[7]:= alfa>>>representatie3.1

In[8]:= Quit
euler% cat representatie3.1
{{1, 0, 0,-k1/2, 0}, {0, 1, 0, 0, 1}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0},
  {0, 0, 0, 0, 1}}
{{1, 0, 0, 0, 0}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 1}, {0, 0, 0, 1, 0},
  {0, 0, 0, 0, 1}}
{{1, k1/2, 0, 0, 0}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 1},
  {0, 0, 0, 0, 1}}
{{1, 0, 0, 0, 1}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0},
  {0, 0, 0, 0, 1}}
{{1,k2, 0,k3, k4/2}, {0, -1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, -1, 0},
  {0, 0, 0, 0, 1}}
euler% vi representatie3.1
       (--> a little bit of file editing <--)
euler% cat representatie3.1
a={{1, 0, 0,-k1/2, 0}, {0, 1, 0, 0, 1}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0},
  {0, 0, 0, 0, 1}}
b={{1, 0, 0, 0, 0}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 1}, {0, 0, 0, 1, 0},
  {0, 0, 0, 0, 1}}
c={{1, k1/2, 0, 0, 0}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 1},
  {0, 0, 0, 0, 1}}
d={{1, 0, 0, 0, 1}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0},
  {0, 0, 0, 0, 1}}
alfa={{1, k2, 0, k3, k4/2}, {0, -1, 0, 0, 0}, {0, 0, 1, 0, 0},
      {0, 0, 0, -1, 0} , {0, 0, 0, 0, 1}}
euler%
```

Now we are really ready to start the computations concerning torsion. We know from corollary 6.1.2 that we only have to deal with $k_1, k_2, k_3, k_4 \in \{0, 1\}$. Let $q = a^{m_1} b^{m_2} c^{m_3} \alpha$ be a torsion element in Q (of order 2), then for the lift $\tilde{q} = a^{m_1} b^{m_2} c^{m_3} \alpha$ of q in E we have that

$$\tilde{q}^2 = (a^{m_1} b^{m_2} c^{m_3} \alpha)^2 = d^{P(m_1, m_2, m_3)}$$

where $P(m_1, m_2, m_3)$ is some polynomial function in the variables m_1, m_2 and m_3. If q acts non trivially on d, we know that $\tilde{q}$ is indeed a torsion element. So suppose that the action of q on d is trivial. Now it's easy to see that there exists a lift $\tilde{q}$ of q which is a torsion element of order 2 if and only if

$$P(m_1, m_2, m_3) \equiv 0 \bmod 2.$$

To investigate this, the following lemma is very helpful:

Lemma A.4.1 *Let $P(x)$ be an integer valued polynomial function (i.e. $P(z) \in \mathbb{Z}$, $\forall z \in \mathbb{Z}$) of total degree $\leq M$. Then*

$$\forall z, k, n \in \mathbb{Z} : \; P(z) \equiv P(z + kn(M!)) \bmod n.$$

Proof: Every integer valued polynomial in the variable x of degree $\leq M$, is a $\mathbb{Z}$–linear combination of polynomials of the form

$$\binom{x}{i}, \; (0 \leq i \leq M).$$

We remark that $\binom{x}{0}$ denotes the constant 1. For $z, k, n \in \mathbb{Z}$, $1 \leq i \leq M$:

$$\begin{aligned} \binom{z + kn(M!)}{i} &= \frac{(z + M!kn)((z-1) + M!kn)\ldots((z-i+1) + M!kn)}{1.2.3\ldots i} \\ &\equiv \frac{z(z-1)(z-2)\ldots(z-i+1)}{1.2.3\ldots i} \bmod n. \end{aligned}$$

This finishes the proof.

■

The lemma tells us that in checking if $P(m_1, m_2, m_3) \equiv 0 \bmod 2$, we may restrict ourselves to $m_i \in \{0, 1, \ldots, 2(\text{degree of } P \text{ in } m_i)! - 1\}$, for $i = 1, 2, 3$. We used all of this in writing the following program (`torsion3.1`) to find the AB–groups:

```
(* maxki= the maximum value of ki < the order of the torsion we are *)
(*        looking for.                                              *)
maxk1=1
maxk2=1
maxk3=1
maxk4=1
order=2
   For[k1=0,k1<=maxk1,k1++,
    For[k2=0,k2<=maxk2,k2++,
     For[k3=0,k3<=maxk3,k3++,
      For[k4=0,k4<=maxk4,k4++,
       {<<representatie3.1,
       (* A general torsionelement in Q, lifted to E *)
       torsionelement[m1_,m2_,m3_]:=matrixmacht[a,m1]. (* Fill in the  *)
                                    matrixmacht[b, 0]. (* solutions     *)
                                    matrixmacht[c,m3]. (* found with    *)
                                    alfa,              (* program torsion *)
       (* If a torsion element acts non trvially on Z, *)
       (* then any lift to E has torsion              *)
       If[(torsionelement[m1,m2,m3])[[1,1]]==-1,
           Print["Ignore the rest, group has torsion!"] ],
       Print["group: (k1=",k1,",k2=",k2,",k3=",k3,",k4=",k4,")"],
       (* We compute the power of d in torsionelement^order *)
       f[m1_,m2_,m3_]:=(
          Expand[(MatrixPower[torsionelement[m1,m2,m3],order])[[1,5]] ]),
       test=f[x,y,z], (* test is a polynomial function in x,y,z *)
       maxm1=(Exponent[test,x]), (* The degree of f in m1 *)
       maxm2=(Exponent[test,y]), (* The degree of f in m2 *)
       maxm3=(Exponent[test,z]), (* The degree of f in m3 *)
       If[maxm1<=0,maxm1=1,maxm1=order (maxm1!)],
       If[maxm2<=0,maxm2=1,maxm2=order (maxm2!)],
       If[maxm3<=0,maxm3=1,maxm3=order (maxm3!)],
       torsion=0,
       For[ll1=0,ll1<maxm1,ll1++,
         For[ll2=0,ll2<maxm2,ll2++,
           For[ll3=0,ll3<maxm3,ll3++,{
              If[Mod[f[ll1,ll2,ll3],order]==0,torsion=torsion+1]} ]]],
       If[torsion>0,Print["torsion"],
              Print["These kind of elements have infinite order!"] ],
       Print[" "],
       m1=., (* Cleaning up *)
       m2=.,
       m3=.,
       f=.,
       a=.,
       b=.,
       c=.,
       d=.,
       alfa=.}
```

```
]]]]
```

The output with Mathematica® looks like:

```
euler% math
Mathematica 2.1 for SPARC
Copyright 1988-92 Wolfram Research, Inc.

In[1]:= <<torsion3.1
group: (k1=0,k2=0,k3=0,k4=0)
torsion

group: (k1=0,k2=0,k3=0,k4=1)
These kind of elements have infinite order!

group: (k1=0,k2=0,k3=1,k4=0)
torsion

group: (k1=0,k2=0,k3=1,k4=1)
torsion

group: (k1=0,k2=1,k3=0,k4=0)
torsion

group: (k1=0,k2=1,k3=0,k4=1)
torsion

group: (k1=0,k2=1,k3=1,k4=0)
torsion

group: (k1=0,k2=1,k3=1,k4=1)
torsion

group: (k1=1,k2=0,k3=0,k4=0)
torsion

group: (k1=1,k2=0,k3=0,k4=1)
torsion

group: (k1=1,k2=0,k3=1,k4=0)
torsion

group: (k1=1,k2=0,k3=1,k4=1)
torsion

group: (k1=1,k2=1,k3=0,k4=0)
torsion

group: (k1=1,k2=1,k3=0,k4=1)
torsion
```

```
group: (k1=1,k2=1,k3=1,k4=0)
torsion

group: (k1=1,k2=1,k3=1,k4=1)
torsion


In[2]:= Quit
euler%
```

We have to interpret this as follows. A group E determined by a 4–tuple (k_1, k_2, k_3, k_4) is an AB–group if and only if

$$\left\{ \begin{array}{l} k_1 \equiv 0 \bmod 2 \\ k_2 \equiv 0 \bmod 2 \\ k_3 \equiv 0 \bmod 2 \\ k_4 \equiv 1 \bmod 2. \end{array} \right.$$

We denote (the cohomology class of) the group E (which is determined by k_1, k_2, k_3, k_4) as $< (k_1, k_2, k_3, k_4) >$. Since $< (k_1, k_2, k_3, k_4) > \cong < (-k_1, -k_2, -k_3, -k_4) >$, we may always suppose that k_1 is positive. (We exclude $k_1 = 0$, since in this case Fitt(E) would be abelian). Of course, the relations on the cohomology–level show that we may restrict ourselves to $k_2, k_3, k_4 \in \{0, 1\}$. All this allows us to conclude that for a fixed k_1 (i.e. a fixed nilmanifold), there is at most one AB–group of this kind, namely if k_1 is even, we may take $E =< (k_1, 0, 0, 1) >$.

A.5 Summary

The previous sections show that any extension of Q by Z has a presentation of the form

$$E : < a, b, c, d, \alpha \mid\mid \begin{array}{ll} [b,a] = 1 & [d,a] = 1 \\ [c,a] = d^{k_1} & [d,b] = 1 \\ [c,b] = 1 & [d,c] = 1 \\ \alpha a = a^{-1}\alpha d^{k_2} & \alpha^2 = d^{k_4} \\ \alpha b = b\alpha & \alpha d = d\alpha \\ \alpha c = c^{-1}\alpha d^{k_3} & \end{array} > .$$

A canonical type affine representation $\lambda : E \to \mathrm{Aff}(\mathbb{R}^4)$ for such an E is given by

$$\lambda(a)=\begin{pmatrix} 1 & 0 & 0 & \frac{-k_1}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \lambda(b)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \lambda(c)=\begin{pmatrix} 1 & \frac{k_1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\lambda(\alpha) = \begin{pmatrix} 1 & k_2 & 0 & k_3 & \frac{k_4}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also showed that the group $H^2(Q,\mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}_2)^3$.
Finally, we indicated that for a fixed nilpotent group, there is maximal one AB–group of this kind, containing this nilpotent group as its maximal nilpotent normal subgroup. More precise, for $k_1 > 0$ even, there is one AB-group, determined by the parameters $(k_1, k_2, k_3, k_4) = (k_1, 0, 0, 1)$, containing

$$N :< a, b, c, d \,||\, [c, a] = d^{k_1} >$$

as its maximal nilpotent normal subgroup.

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