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APPENDIX

THEOREMS OF ALGEBRA, GEOMETRY, AND CALCULUS

The purpose of this appendix is to summarize some basic results from algebra, geometry, and calculus for the purpose of making the monograph more readily accessible to those less familiar with the basic mathematical concepts used in the book. The symbols below have the following meaning:

$$\begin{array}{rll} \forall & \textit{meaning} & \textit{"for every"} \\ A \to B & \textit{meaning} & \textit{"mapping of } A \textit{ into } B\textit{"} \\ \to & \textit{meaning} & \textit{"implies"} \\ x & \textit{meaning} & \textit{"cartesian product of sets"} \\ \times & \textit{meaning} & \textit{"tensor product"} \\ \epsilon & \textit{meaning} & \textit{"belongs to"} \\ \cup & \textit{meaning} & \textit{"union"} \\ \subset & \textit{meaning} & \textit{"is included"} \end{array}$$

1 Concepts from Abstract Algebra

Necessity and Sufficiency. A statement that P is true *if and only if* a condition Q holds is equivalent to the statement that a necessary and sufficient condition for P to be true is that Q holds. The proof must proceed as follows:

Sufficiency (if) (Q is sufficient for P) - Assume Q holds; then show that this implies that P is true.

Necessity (only if) (Q is necessary for P) - Assume P holds; then show that Q follows as a consequence.

Functions. A *function* f from a set A into a set B is an ordered triple of sets (f, A, B) denoted by f: $A \to B$ (f maps A into B), where:

1. $f \subseteq AxB$ (f is a *proper subset* of AxB).

2. $\forall\ x \in A$ there exists a $y \in B$ such that $(x, y) \in f$.

3. $\forall\ x \in A$ and $y_1, y_2 \in B$, if $(x, y_1) \in f$ *and* $(x, y_2) \in f$, then $y_1 = y_2$.

When $(x, y) \in f$, we write

$$y = f(x) \tag{A.1}$$

1. *Surjective (Onto) Functions.* A function $f : A \to B$ is *surjective*, or from A *onto* B, if and only if every $b \in B$ is the image of some element of A.

2. *Injective (One-to-One) Functions.* A function $f : A \to B$ is *injective*, or *one-to-one*, if for every b belonging to the range of f there is exactly one $a \in A$ such that $b = f(a)$.

3. *Bijective (One-to-One and Onto) Functions.* A function $f : A \to B$ is *bijective*, or *one-to-one and onto*, if and only if it is both injective and surjective; *i.e.*, if and only if every $b \in B$ is the unique image of some $a \in A$.

A function $f : A \to B$ is invertible if and only if it is bijective.

Homomorphism. Let $@ = \{A, *\}$ and $\# = \{B, o\}$ denote two systems, @ consisting of a set A and a binary operation $*$ defined on A, and # consisting of a set B and a binary operation o defined on B. A *homomorphism* of @ into # is a mapping $H : @ \to \#$ such that for each $a, b \in A$

$$H(a \; * \; b) = H(a) \; o \; H(b) \tag{A.2}$$

Note that there need not be a one-to-one correspondence between the elements of @ and their images in #.

Isomorphism. Let @ and # be two systems as above. The systems @ and # are *isomorphic* if and only if the following hold:

1. There exists a bijective map $F : A \to B$.

2. The operations are preserved by the mapping F in the sense that if $a, b \in A$, then $F(a \; * \; b) = F(a) \; o \; F(b)$.

The mapping is then referred to as an *isomorphism* or an *isomorphic mapping* of @ into #. Isomorphism is a special case of homomorphism in which the mapping is bijective.

Group. Group is a system consisting of a set G and a binary operation $*$ on G such that the following axioms must hold:

1. *Closure.* If $a, b \in G$, then $a * b \in G$, *i.e.* a group is *closed* under the operation $*$.

2. *Associative Law.* For every $a, b, c \in G$, $a * (b * c) = (a * b) * c$.

3. *Identity Element.* There exists an element $e \in G$ such that $a * e = e * a = a$ for every $a \in G$.

4. *Inverse.* For each $a \in G$, there exists an inverse element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

If $@ = \{G, *\}$ is a group, any group $@_1 = \{G_1, *\}$ such that $G_1 \subset G$ and $e \in G_1$ is a *subgroup* of $@$, denoted by $@_1 \subset @$.

2 Concepts from Linear Vector Spaces

The real linear vector space under consideration in the book is Euclidean with a cartesian coordinate system. The elements of this space are scalars usually indicated by the light-faced italics $a, b, \phi, A, B, \Phi, \ldots$, vectors indicated by bold-faced miniscule letters $\mathbf{a}$,$\mathbf{b}$, $\boldsymbol{\xi}, \ldots$, and tensors indicated by bold-faced letters $\mathbf{A}$,$\mathbf{B}$,$\mathbf{T}$,$\boldsymbol{\Phi}, \ldots$. The standard basis of the space is defined as follows:

$$\mathbf{e}_1 = (1,0,0), \quad \mathbf{e}_2 = (0,1,0), \quad \mathbf{e}_3 = (0,0,1) \tag{A.3}$$

such that if $\mathbf{x}$ is any vector, then

$$\mathbf{x} = x_k \mathbf{e}_k \tag{A.4}$$

where x_k are unique scalars or *components* of vector $\mathbf{x}$. The subscript k is the *tensor index* and repeated implies a summation from 1 to 3. The tensorial indices always occur as subscripts and are denoted by the italic light-faced miniscules $i, j, k, \ldots$ The basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is *orthonormal*, since

$$\begin{aligned} \mathbf{e}_q \cdot \mathbf{e}_\ell = \delta_{q\ell} &= 1 \ \ if \ \ q = \ell \\ &= 0 \ \ if \ \ q \neq \ell \end{aligned} \tag{A.5}$$

where $\delta_{q\ell}$ is the Kronecker delta, with $\delta_{qq} = 3$. The magnitude $|\mathbf{x}|$ of the vector $\mathbf{x}$ is defined with an *inner product*, denoted by a dot, *i.e.*

$$|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x} \tag{A.6}$$

The two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if

$$\mathbf{x} \cdot \mathbf{y} = 0 \tag{A.7}$$

The second order tensor $\mathbf{A}$ is a linear transformation; a mapping $\mathbf{A}$ of the vector space $\$$ into the vector space $\$'$, *i.e.*

$$\mathbf{A}(a\mathbf{x} + b\mathbf{y}) = a\,\mathbf{A}(\mathbf{x}) + b\,\mathbf{A}(\mathbf{y}) = a\,\mathbf{A}\mathbf{x} + b\,\mathbf{A}\mathbf{y} \tag{A.8}$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$ and scalars a and b. The linear transformation $\mathbf{A}$ can have an *inverse* $\mathbf{A}^{-1}$. If $dim\$ = dim\$'$, then any of the following statements is a necessary and sufficient condition that $\mathbf{A}$ has an inverse: $\mathbf{A}$ is one-to-one, $\mathbf{A}$ maps $\$$ onto $\$'$, the nullspace of $\mathbf{A}$ contains only vector $\mathbf{0}$.

If $\mathbf{A}^{-1}$ exists, it is a linear mapping, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$. The unit linear transformation **I** is a *unit tensor*, satisfying for all **x**

$$\mathbf{I}\mathbf{x} = \mathbf{x} \tag{A.9}$$

With $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ defining a basis of the vector space, the condition

$$\mathbf{A}\mathbf{e}_i = A_{ij}\mathbf{e}_j \tag{A.10}$$

defines the *components* A_{ij} of **A** relative to the basis. The components of **A** form a matrix $\| A_{ij} \|$, with the row index i and column index j. The *determinant* of the tensor is the determinant of the matrix $\| A_{ij} \|$, and **A** is invertible if $det\mathbf{A} \neq 0$. If **A** and **B** are tensors, their composition **AB** is the *product* of **A** and **B**. The components of **AB** are $A_{ij}B_{jk}$, and

$$det(\mathbf{AB}) = det\mathbf{A}\, det\mathbf{B} = det(\mathbf{BA}) \tag{A.11}$$

If $det\mathbf{A} = \pm 1$, then **A** is *unimodular*.

The *tensor product* of vectors **a** and **b** is the tensor $\mathbf{a} \times \mathbf{b}$, such that

$$(\mathbf{a} \times \mathbf{b})\mathbf{u} = (\mathbf{u} \cdot \mathbf{b})\mathbf{a} \quad \vee \quad \mathbf{u} \in \$ \tag{A.12}$$

The set of tensors $\mathbf{e}_i \times \mathbf{e}_j$ forms a basis for the space of tensors, *i.e.*

$$\begin{aligned} \mathbf{A} &= A_{ij}\, \mathbf{e}_i \times \mathbf{e}_j \\ \mathbf{I} &= \delta_{ij}\, \mathbf{e}_i \times \mathbf{e}_j \end{aligned} \tag{A.13}$$

Similarly, the products $\mathbf{e}_i \times \mathbf{e}_j \times \mathbf{e}_k$ form a basis for a third order tensor **U**,

$$\mathbf{U} = U_{ijk}\, \mathbf{e}_i \times \mathbf{e}_j \times \mathbf{e}_k \tag{A.14}$$

with components U_{ijk}.

The *transpose* $\mathbf{A}^T$ of the tensor **A** satisfies

$$(A^T)_{ij} = A_{ji} \tag{A.15}$$

and

$$\begin{gathered} (\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T, \;\; (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T, \;\; (\mathbf{A}^T)^T = \mathbf{A} \\ (\mathbf{a} \times \mathbf{b})^T = \mathbf{b} \times \mathbf{a} \end{gathered} \tag{A.16}$$

and if **A** is invertible, then so is $\mathbf{A}^T$, and

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T \tag{A.17}$$

A tensor **S** is *symmetric*, or **W** *skew-symmetric* if

$$\mathbf{S} = \mathbf{S}^T, \;\; S_{ij} = S_{ji}; \quad \mathbf{W} = -\mathbf{W}^T, \;\; W_{ij} = -W_{ji} \tag{A.18}$$

and any second order tensor **A** has a unique representation as a sum of symmetric and skew-symmetric tensors,

$$\mathbf{A} = \mathbf{S} + \mathbf{W} \tag{A.19}$$

where

$$\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T), \quad \mathbf{W} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) \tag{A.20}$$

or in terms of brackets $(\ldots)$ indicating symmetrization and $[\ldots]$ indicating skew-symmetrization

$$A_{ij} = A_{(ij)} + A_{[ij]}; \quad A_{(ij)} = S_{ij}, \quad A_{[ij]} = W_{ij} \tag{A.21}$$

The symmetry and skew-symmetry can also be defined for a third-order tensor **U**, *i.e.*

$$U_{ijk} = U_{(ijk)} + U_{[ijk]} + \frac{2}{3}(U_{[ij]k} + U_{[kj]i} + U_{(ij)k} - U_{k(ij)}) \tag{A.22}$$

where

$$\begin{aligned} U_{(ijk)} &= \frac{1}{6}(U_{ijk} + U_{jki} + U_{kij} + U_{jik} + U_{kji} + U_{ikj}) \\ U_{[ijk]} &= \frac{1}{6}(U_{ijk} + U_{jki} + U_{kij} - U_{jik} - U_{kji} - U_{ikj}) \\ U_{(ij)k} &= \frac{1}{2}(U_{ijk} + U_{jik}) \\ U_{[ij]k} &= \frac{1}{2}(U_{ijk} - U_{jik}) \end{aligned} \tag{A.23}$$

The third order *alternating tensor* is defined as

$$\epsilon_{ijk}\, \mathbf{e}_i \times \mathbf{e}_j \times \mathbf{e}_k \tag{A.24}$$

$$\epsilon_{ijk} = \left| \begin{array}{c} +1 \text{ if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 \text{ if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 \text{ if two or more indices } i,j,k \text{ are equal} \end{array} \right|$$

In particular, the following results are useful

$$\begin{aligned} \epsilon_{pqs}\epsilon_{snr} &= \delta_{np}\delta_{rq} - \delta_{nq}\delta_{rp} \\ \epsilon_{pqs}\epsilon_{sqr} &= \delta_{qp}\delta_{rq} - \delta_{qq}\delta_{rp} = \delta_{pr} - 3\delta_{rp} = -2\delta_{pr} \end{aligned} \tag{A.25}$$

The *contraction* operation of a tensor lowers its index by two. For example, the contraction of a second order tensor **A** leads to a tensor A_{ii} of order zero, or the *trace* of **A**, $tr\mathbf{A}$, *i.e.*

$$A_{ii} = tr\mathbf{A}, \quad \mathbf{a}\cdot\mathbf{b} = tr(\mathbf{a} \times \mathbf{b}) \tag{A.26}$$

The *cross product* $\mathbf{w} = \mathbf{a} \wedge \mathbf{b}$ of vectors **a** and **b** is a pseudovector

$$w_i = \epsilon_{ijk} a_j a_k \tag{A.27}$$

and with any second order skew-symmetric tensor **W** can be associated a vector **w**, such that

$$w_i = \frac{1}{2}\epsilon_{ijk} W_{jk}, \qquad W_{ij} = \epsilon_{ijk} w_k \tag{A.28}$$

or in matrix form

$$\| W_{ij} \| = \begin{vmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{vmatrix} = \begin{vmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{vmatrix} \tag{A.29}$$

A mapping **Q** of an inner-product space is *orthogonal* if it preserves the inner-product:

$$\mathbf{Q}(\mathbf{a})\mathbf{Q}(\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} \tag{A.30}$$

This condition is satisfied *if and only if* **Q** is an invertible tensor such that

$$\mathbf{Q}^{-1} = \mathbf{Q}^T \tag{A.31}$$

and hence

$$det\mathbf{Q} = \pm 1 \tag{A.32}$$

If $det\mathbf{Q} = 1$, the orthogonal tensor is *proper* or a *rotation.*

Two topological spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$, where X and Y are sets and $\mathcal{X}$ and $\mathcal{Y}$ are topologies, are *homeomorphic* (or topologically equivalent) if and only if there exists a map $H : X \rightarrow Y$ such that

1. H is bijective.

2. H is continuous.

3. H^{-1} is continuous.

The map is then a *homeomorphism* from (X, Y) to $(\mathcal{X}, \mathcal{Y})$.

The distance between the points **a** and **b** is the magnitude $|\mathbf{b} - \mathbf{a}|$ or metric that can define a topology of the Euclidean space, with standard procedures defining then the continuity, convergence, limits, compactness, *etc.* When a tensor **A** maps a subspace into itself, that subspace is *invariant* under **A**. Every tensor **A** has invariant subspaces: the whole vector space, the subspace $\{\mathbf{0}\}$, the range of **A**, and the nullspace of **A**. Moreover, if λ is any scalar the nullspace of $\mathbf{A} - \lambda\mathbf{I}$ is also an invariant subspace of **A**. It is called the *proper space* of $\mathbf{A}$ corresponding to λ, and its dimension is the *multiplicity* of λ for $\mathbf{A}$. The *characteristic equation* of $\mathbf{A}$ is

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{A.33}$$

where the *principal invariants* I_k of $\mathbf{A}$ are uniquely determined by the complex numbers a_1, a_2, a_3 such that

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)$$
$$I_1 = a_1 + a_2 + a_3, \quad I_2 = a_2a_3 + a_3a_1 + a_1a_2, \quad I_3 = a_1a_2a_3 \tag{A.34}$$

If $\mathbf{A}$ is symmetric then a_1, a_2, and a_3 are basic invariants in the sense that any invariant of $\mathbf{A}$ can be expressed in terms of them, *i.e.*

$$I_1 = tr\mathbf{A}, \quad I_2 = \frac{1}{2}[(tr\mathbf{A})^2 - tr\mathbf{A}^2], \quad I_3 = det\mathbf{A} \tag{A.35}$$

Another set of invariants of $\mathbf{A}$ are $tr\mathbf{A}$, $tr\mathbf{A}^2$, and $tr\mathbf{A}^3$. The invariants, by definition, are independent of the choice of the coordinate system and play a central role in the theory of constitutive equations as further discussed below in section 4.

3 Basic Identities from Calculus

Let f and g be scalars, $\mathbf{f}$ and $\mathbf{g}$ vectors, $\mathbf{A}$ a second order tensor, and all differentiable. Then,

$$\begin{aligned}
\nabla(fg) &= f\nabla g + g\nabla f, \quad (fg)_{,i} = fg_{,i} + gf_{,i} \\
\nabla(\mathbf{f}\cdot\mathbf{g}) &= (\nabla\mathbf{f})^T\mathbf{g} + (\nabla\mathbf{g})^T\mathbf{f}, \quad (f_ig_i)_{,j} = f_{i,j}\,g_i + f_i\,g_{i,j} \\
\nabla(f\mathbf{g}) &= f\nabla\mathbf{g} + \mathbf{g}\times\nabla f, \quad (fg_i)_{,j} = fg_{i,j} + g_if_{,j} \\
div(\mathbf{f}) &= \nabla\cdot\mathbf{f} = tr(\nabla\mathbf{f}) = f_{i,i} \\
\Delta f &= div(\nabla f) = tr(\nabla^2 f), \quad f_{,ii} = (f_{,i})_{,i} \\
\nabla\cdot f\mathbf{g} &= div(f\mathbf{g}) = \mathbf{g}\cdot\nabla f + f\nabla\cdot\mathbf{g}, \quad (fg_i)_{,i} = g_if_{,i} + fg_{i,i} \\
\nabla\cdot(\mathbf{A}\mathbf{g}) &= (\nabla\cdot\mathbf{A}^T)\cdot\mathbf{g} + tr(\mathbf{A}\nabla\mathbf{g}), \quad (A_{ij}g_j)_{,i} = A_{ij,i}\,g_j + A_{ij}\,g_{j,i} \\
\nabla\cdot(\nabla\mathbf{g})^T &= \nabla(\nabla\cdot\mathbf{g}), \quad g_{i,ji} = g_{i,ij} \\
\nabla\cdot(\nabla\mathbf{g} \pm (\nabla\mathbf{g})^T) &= \Delta\mathbf{g} \pm \nabla(\nabla\cdot\mathbf{g}), \quad (g_{i,j} \pm g_{j,i})_{,j} = g_{i,jj} \pm g_{j,ji}
\end{aligned} \tag{A.36}$$

4 Results from the Theory of Invariants and Tensor Representation Theorems

The invariance of constitutive equations under orthogonal transformation $\mathbf{Q}$ is discussed in chapters 4 and 6. A function

$$f(\mathbf{v}_1, \ldots, \mathbf{v}_P, \mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{W}_1, \ldots, \mathbf{W}_M) \tag{A.37}$$

of P vectors $\mathbf{v}$, N symmetric second order tensors $\mathbf{A}$, and M skew-symmetric second order tensors $\mathbf{W}$ is an *absolute invariant* of the vectors and tensors under the transformation $\mathbf{Q}$ if

$$f(\mathbf{Q}\mathbf{v}_1,\ldots,\mathbf{Q}\mathbf{v}_P,\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T,\ldots,\mathbf{Q}\mathbf{A}_N\mathbf{Q}^T,\mathbf{Q}\mathbf{W}_1\mathbf{Q}^T,\ldots,\mathbf{Q}\mathbf{W}_M\mathbf{Q}^T) = f(\mathbf{v}_1,\ldots,\mathbf{v}_P,\mathbf{A}_1,\ldots,\mathbf{A}_N,\mathbf{W}_1,\ldots,\mathbf{W}_M) \tag{A.38}$$

The central problem in the theory of invariants is to determine a set of basic invariants from which all other can be generated, given a set of variables (vectors and tensors) and a group of transformations. When it is sufficient to consider a function of polynomial invariants, the polynomial invariant is *reducible* if it can be expressed as a polynomial in other invariants. A set of polynomial invariants with the property that any polynomial invariant can be expressed as a polynomial in members of the given set is an *integrity basis.* The main problem is to determine *minimal* integrity bases and polynomial relations between invariants which do not permit any one invariant to be expressed as a polynomial in the remainder.

In this section, a summary of representations of isotropic scalar, vector, and tensor functions will be presented from the works of WANG (1971), SMITH (1971), and corrections introduced by BOEHLER (1977). For polynomial invariants, the article by SPENCER (1971) may also be consulted.

Let f, $\mathbf{g}$, $\mathbf{H}$, and $\mathbf{K}$ be scalar, vector, symmetric second order tensor, and skew-symmetric second order tensor isotropic functions, respectively, of vectors and tensors as in (A.37). The *irreducible set of invariants* of P vectors $\mathbf{v}$, N tensors $\mathbf{A}$, and M tensors $\mathbf{W}$ are as follows:

$$\begin{gathered}\mathbf{v}_m\cdot\mathbf{v}_m,\ \mathbf{v}_n\cdot\mathbf{v}_m,\ tr\mathbf{A}_i,\ tr\mathbf{A}_i^2,\ tr\mathbf{A}_i^3,\ tr\mathbf{A}_i\mathbf{A}_j,\\ tr\mathbf{A}_i^2\mathbf{A}_j,\ tr\mathbf{A}_i\mathbf{A}_j^2,\ tr\mathbf{A}_i^2\mathbf{A}_j^2,\ tr\mathbf{A}_i\mathbf{A}_j\mathbf{A}_k,\ tr\mathbf{W}_p^2,\ tr\mathbf{W}_p\mathbf{W}_q,\\ tr\mathbf{W}_p\mathbf{W}_q\mathbf{W}_r,\ \mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_m,\ \mathbf{v}_m\cdot\mathbf{A}_i^2\mathbf{v}_m,\ \mathbf{v}_m\cdot\mathbf{A}_i\mathbf{A}_j\mathbf{v}_m,\ \mathbf{v}_m\cdot\mathbf{A}_i\mathbf{v}_n,\ \mathbf{v}_m\cdot\mathbf{A}_i^2\mathbf{v}_n,\\ \mathbf{v}_m\cdot(\mathbf{A}_i\mathbf{A}_j-\mathbf{A}_j\mathbf{A}_i)\mathbf{v}_n,\ \mathbf{v}_m\cdot\mathbf{W}_p^2\mathbf{v}_m,\ \mathbf{v}_m\cdot\mathbf{W}_p\mathbf{W}_q\mathbf{v}_m,\ \mathbf{v}_m\cdot\mathbf{W}_p^2\mathbf{W}_q\mathbf{v}_m,\\ \mathbf{v}_m\cdot\mathbf{W}_p\mathbf{W}_q^2\mathbf{v}_m,\ \mathbf{v}_m\cdot\mathbf{W}_p\mathbf{v}_n,\ \mathbf{v}_m\cdot\mathbf{W}_p^2\mathbf{v}_n,\ \mathbf{v}_m\cdot(\mathbf{W}_p\mathbf{W}_q-\mathbf{W}_q\mathbf{W}_p)\mathbf{v}_n,\\ tr\mathbf{A}_i\mathbf{W}_p^2,\ tr\mathbf{A}_i^2\mathbf{W}_p^2,\ tr\mathbf{A}_i^2\mathbf{W}_p^2\mathbf{A}_i\mathbf{W}_p,\ tr\mathbf{A}_i\mathbf{W}_p\mathbf{W}_q,\ tr\mathbf{A}_i\mathbf{W}_p\mathbf{W}_q^2,\\ tr\mathbf{A}_i\mathbf{W}_p^2\mathbf{W}_q,\ tr\mathbf{A}_i\mathbf{A}_j\mathbf{W}_p,\ tr\mathbf{A}_i\mathbf{W}_p^2\mathbf{A}_j\mathbf{W}_p,\ tr\mathbf{A}_i\mathbf{A}_j^2\mathbf{W}_p,\ tr\mathbf{A}_i^2\mathbf{A}_j\mathbf{W}_p,\\ \mathbf{v}_m\cdot\mathbf{A}_i\mathbf{W}_p\mathbf{v}_m,\ \mathbf{v}_m\cdot\mathbf{W}_p\mathbf{A}_i\mathbf{W}_p^2\mathbf{v}_m,\ \mathbf{v}_m\cdot\mathbf{A}_i^2\mathbf{W}_p\mathbf{v}_m,\ \mathbf{v}_m\cdot(\mathbf{A}_i\mathbf{W}_p-\mathbf{W}_p\mathbf{A}_i)\mathbf{v}_n\end{gathered} \tag{A.39}$$

where $i,j,k=1,\ldots,N$; $i<j<k$, where $p,q,r=1,\ldots,M$; $p<q<r$, and where $m,n=1,\ldots,P$; $m<n$.

The representation of a *vector-valued isotropic function* $\mathbf{g}$ is given by

$$\mathbf{g}(\mathbf{v}_m,\mathbf{A}_i,\mathbf{W}_p)=\sum_r \alpha_r\,\mathbf{g}_r(\mathbf{v}_m,\mathbf{A}_i,\mathbf{W}_p) \tag{A.40}$$

where α_r are scalar-valued isotropic functions of the invariants (A.39), whereas $\mathbf{g}_r$ are vector-valued functions given by

$$\begin{gathered}\mathbf{v}_m,\ \mathbf{A}_i\mathbf{v}_m,\ \mathbf{A}_i^2\mathbf{v}_m,\ (\mathbf{A}_i\mathbf{A}_j-\mathbf{A}_j\mathbf{A}_i)\mathbf{v}_m,\ \mathbf{W}_p\mathbf{v}_m,\\ \mathbf{W}_p^2\mathbf{v}_m,\ (\mathbf{W}_p\mathbf{W}_q-\mathbf{W}_q\mathbf{W}_p)\mathbf{v}_m,\ (\mathbf{A}_i\mathbf{W}_p-\mathbf{W}_p\mathbf{A}_i)\mathbf{v}_m\end{gathered}\tag{A.41}$$

where $i,j=1,\ldots,N$; $i<j$, where $p,q=1,\ldots,M$; $p<q$, and where $m=1,\ldots,P$.

A *symmetric tensor-valued isotropic function* $\mathbf{H}$ has the following representation

$$\mathbf{H}(\mathbf{v}_m,\ \mathbf{A}_i,\ \mathbf{W}_p)=\sum_r\beta_r\,\mathbf{H}_r(\mathbf{v}_m,\ \mathbf{A}_i,\ \mathbf{W}_p)\tag{A.42}$$

where β_r are scalar-valued isotropic functions of the invariants (A.39), and where $\mathbf{H}_r$ are symmetric tensor-valued isotropic functions given by

$$\begin{gathered}\mathbf{I},\ \mathbf{A}_i,\ \mathbf{A}_i^2,\ \mathbf{A}_i\mathbf{A}_j+\mathbf{A}_j\mathbf{A}_i,\ \mathbf{A}_i^2\mathbf{A}_j+\mathbf{A}_j\mathbf{A}_i^2,\ \mathbf{A}_i\mathbf{A}_j^2+\mathbf{A}_j^2\mathbf{A}_i,\\ \mathbf{v}_m\times\mathbf{v}_m,\ \mathbf{v}_m\times\mathbf{v}_n+\mathbf{v}_n\times\mathbf{v}_m,\ \mathbf{W}_p^2,\ \mathbf{W}_p\mathbf{W}_q+\mathbf{W}_q\mathbf{W}_p,\ \mathbf{W}_p\mathbf{W}_q^2-\mathbf{W}_q^2\mathbf{W}_p,\\ \mathbf{W}_p^2\mathbf{W}_q-\mathbf{W}_q\mathbf{W}_p^2,\ \mathbf{v}_m\times\mathbf{A}_i\mathbf{v}_m+\mathbf{A}_i\mathbf{v}_m\times\mathbf{v}_m,\ \mathbf{v}_m\times\mathbf{A}_i^2\mathbf{v}_m+\mathbf{A}_i^2\mathbf{v}_m\times\mathbf{v}_m,\\ \mathbf{A}_i(\mathbf{v}_m\times\mathbf{v}_n-\mathbf{v}_n\times\mathbf{v}_m)-(\mathbf{v}_m\times\mathbf{v}_n-\mathbf{v}_n\times\mathbf{v}_m)\mathbf{A}_i,\ \mathbf{A}_i\mathbf{W}_p-\mathbf{W}_p\mathbf{A}_i,\\ \mathbf{W}_p\mathbf{A}_i\mathbf{W}_p,\ \mathbf{A}_i^2\mathbf{W}_p-\mathbf{W}_p\mathbf{A}_i^2,\ \mathbf{W}_p\mathbf{A}_i\mathbf{W}_p^2-\mathbf{W}_p^2\mathbf{A}_i\mathbf{W}_p,\ \mathbf{W}_p\mathbf{v}_m\times\mathbf{W}_p\mathbf{v}_m,\\ \mathbf{v}_m\times\mathbf{W}_p\mathbf{v}_m+\mathbf{W}_p\mathbf{v}_m\times\mathbf{v}_m,\ \mathbf{W}_p\mathbf{v}_m\times\mathbf{W}_p^2\mathbf{v}_m+\mathbf{W}_p^2\mathbf{v}_m\times\mathbf{W}_p\mathbf{v}_m,\\ \mathbf{W}_p(\mathbf{v}_m\times\mathbf{v}_n-\mathbf{v}_n\times\mathbf{v}_m)+(\mathbf{v}_m\times\mathbf{v}_n-\mathbf{v}_n\times\mathbf{v}_m)\mathbf{W}_p\end{gathered}\tag{A.43}$$

where $i,j=1,\ldots,N$; $i<j$, where $p,q=1,\ldots,M$; $p<q$, and where $m,n=1,\ldots,P$; $m<n$.

A *skew-symmetric tensor valued isotropic function* $\mathbf{K}$ has the representation as follows

$$\mathbf{K}(\mathbf{v}_m,\ \mathbf{A}_i,\ \mathbf{W}_p)=\sum_r\gamma_r\,\mathbf{K}_r(\mathbf{v}_m,\ \mathbf{A}_i,\ \mathbf{W}_p)\tag{A.44}$$

where γ_r are scalar-valued isotropic functions of the invariants (A.39), whereas $\mathbf{K}_r$ are skew-symmetric tensor-valued isotropic functions given by

$$\begin{gathered}\mathbf{W}_p,\ \mathbf{W}_p\mathbf{W}_q-\mathbf{W}_q\mathbf{W}_p,\ \mathbf{A}_i\mathbf{A}_j-\mathbf{A}_j\mathbf{A}_i,\\ \mathbf{A}_i^2\mathbf{A}_j-\mathbf{A}_j\mathbf{A}_i^2,\ \mathbf{A}_i^2\mathbf{A}_j^2-\mathbf{A}_j^2\mathbf{A}_i^2,\ \mathbf{A}_i\mathbf{A}_j\mathbf{A}_i^2-\mathbf{A}_i^2\mathbf{A}_j\mathbf{A}_i,\ \mathbf{A}_j\mathbf{A}_i\mathbf{A}_j^2-\mathbf{A}_j^2\mathbf{A}_i\mathbf{A}_j,\\ \mathbf{A}_i\mathbf{A}_j\mathbf{A}_k+\mathbf{A}_j\mathbf{A}_k\mathbf{A}_i+\mathbf{A}_k\mathbf{A}_i\mathbf{A}_j-\mathbf{A}_j\mathbf{A}_i\mathbf{A}_k-\mathbf{A}_i\mathbf{A}_k\mathbf{A}_j-\mathbf{A}_k\mathbf{A}_j\mathbf{A}_i,\\ \mathbf{v}_m\times\mathbf{v}_n-\mathbf{v}_n\times\mathbf{v}_m,\ \mathbf{v}_m\times\mathbf{A}_i\mathbf{v}_m-\mathbf{A}_i\mathbf{v}_m\times\mathbf{v}_m,\ \mathbf{v}_m\times\mathbf{A}_i^2\mathbf{v}_m-\mathbf{A}_i^2\mathbf{v}_m\times\mathbf{v}_m,\end{gathered}$$

$$\begin{gathered}\mathbf{A}_i\mathbf{v}_m \times \mathbf{A}_i^2\mathbf{v}_m - \mathbf{A}_i^2\mathbf{v}_m \times \mathbf{A}_i\mathbf{v}_m,\ \mathbf{A}_i\mathbf{v}_m \times \mathbf{A}_j\mathbf{v}_m \\ -\mathbf{A}_j\mathbf{v}_m \times \mathbf{A}_i\mathbf{v}_m + \mathbf{v}_m \times (\mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i)\mathbf{v}_m - (\mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i)\mathbf{v}_m \times \mathbf{v}_m, \\ \mathbf{A}_i(\mathbf{v}_m \times \mathbf{v}_n - \mathbf{v}_n \times \mathbf{v}_m) + (\mathbf{v}_m \times \mathbf{v}_n - \mathbf{v}_n \times \mathbf{v}_m)\mathbf{A}_i,\ \mathbf{A}_i\mathbf{W}_p + \mathbf{W}_p\mathbf{A}_i, \\ \mathbf{A}_i\mathbf{W}_p^2 - \mathbf{W}_p^2\mathbf{A}_i,\ \mathbf{v}_m \times \mathbf{W}_p\mathbf{v}_m - \mathbf{W}_p\mathbf{v}_m \times \mathbf{v}_m,\ \mathbf{v}_m \times \mathbf{W}_p^2\mathbf{v}_m - \mathbf{W}_p^2\mathbf{v}_m \times \mathbf{v}_m, \\ \mathbf{W}_p(\mathbf{v}_m \times \mathbf{v}_n - \mathbf{v}_n \times \mathbf{v}_m) - (\mathbf{v}_m \times \mathbf{v}_n - \mathbf{v}_n \times \mathbf{v}_m)\mathbf{W}_p\end{gathered} \tag{A.45}$$

where $i,j,k=1,\ldots,N$; $i<j<k$, where $p,q=1,\ldots,M$; $p<q$, and where $m,n=1,\ldots,P$; $m<n$.

The representation of a general second order tensor-valued isotropic function is obtained by adding (A.42) and (A.44). *Form-invariant functionals* may also be constructed in terms of polynomial invariants (SPENCER, 1971).

An occurring problem in the studies of linearized constitutive equations is to obtain representations for *isotropic tensors*. An isotropic tensor of order μ has *components* which are unchanged by an orthogonal transformation of rectangular cartesian coordinates (SPENCER,1971), *i.e.*

$$\alpha^*_{i_1 i_2 \ldots i_\mu} = \alpha_{i_1 i_2 \ldots i_\mu} \tag{A.46}$$

μ is *even*: $\alpha_{i_1 i_2 \ldots i_\mu}$ is the sum of the terms of the type

$$\delta_{i_\alpha i_\beta}\,\delta_{i_\gamma i_\delta}\ldots\delta_{i_\sigma i_\tau}$$

μ is *odd*: $\alpha_{i_1 i_2 \ldots i_\mu}$ is a sum of the terms of the type

$$\delta_{i_\alpha i_\beta}\,\delta_{i_\gamma i_\delta}\ldots\delta_{i_\lambda i_\nu}\,\epsilon_{i_\rho i_\sigma i_\tau}$$

where in each case $\alpha,\beta,\gamma,\delta,\ldots,\rho,\sigma,\tau$ is a permutation of $1,2,\ldots,\mu$. As can be seen, the isotropic tensors of even order are invariant under improper orthogonal transformations, whereas those of odd order change sign under such transformations. For the use in chapters 7 and 8, it can be shown from the above that

$$\begin{gathered}\alpha_{ij} = a\,\delta_{ij}, \qquad \alpha_{ijk} = a\,\epsilon_{ijk} \\ \alpha_{ijpq} = a\,\delta_{ij}\delta_{pq} + b\,\delta_{ip}\delta_{jq} + c\,\delta_{iq}\delta_{jp} \\ \alpha_{ijkmn\ell} = a_1\,\delta_{ij}\delta_{k\ell}\delta_{mn} + a_2\,\delta_{ij}\delta_{km}\delta_{\ell n} + a_3\,\delta_{ij}\delta_{\ell m}\delta_{kn} + a_4\,\delta_{ik}\delta_{j\ell}\delta_{mn} \\ +a_5\,\delta_{ik}\delta_{\ell m}\delta_{jn} + a_6\,\delta_{ik}\delta_{jm}\delta_{\ell n} + a_7\,\delta_{jk}\delta_{i\ell}\delta_{mn} + a_8\,\delta_{jk}\delta_{im}\delta_{\ell n} \\ +a_9\,\delta_{jk}\delta_{in}\delta_{m\ell} + a_{10}\,\delta_{im}\delta_{jn}\delta_{k\ell} + a_{11}\,\delta_{im}\delta_{j\ell}\delta_{kn} + a_{12}\,\delta_{in}\delta_{jm}\delta_{k\ell} \\ +a_{13}\,\delta_{in}\delta_{j\ell}\delta_{km} + a_{14}\,\delta_{i\ell}\delta_{jm}\delta_{kn} + a_{15}\,\delta_{i\ell}\delta_{jn}\delta_{km}\end{gathered} \tag{A.47}$$

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