EPILOGUE

In the last few years we have studied several topics which are related to analyticity and trajectory spaces. In particular we mention our work on Dirac's formalism [EG 4-6], our work on a measure theoretical version of Sobolev's Lemma [EG 7], and on eigenfunction expansions for general self-adjoint operators [EG 8]. Further, in [E 2] Van Eijndhoven developed a functional analytic generalization of the theory of tempered distributions, which is a kind of mirror image of the theory as presented in this book. Also for this theory an extensive list of classical analytic examples is available now. See [EG 1-3], [E 2], [G 5].

In our papers [EK], [EGK] we started a unification and generalization of the theories in [G 1-3] and [E 2], which we find rather exciting. The spaces of this unification not only include most classical distribution spaces and function spaces that we know about, but also the spaces of (extendible) operators on those spaces. In this epilogue we present a brief sketch of the underlying ideas in [EGK].

Let Φ_1 denote the following directed set of functions on $\mathbb{R}$:

$$\Phi_1 = \{\lambda \mapsto e^{-t|\lambda|} \mid t > 0\} .$$

Let A denote a self-adjoint operator in a separable Hilbert space X and define

$$S_{\Phi_1}(X;\mathsf{A}) = \bigcup_{\varphi\in\Phi_1} \varphi(\mathsf{A})X . \qquad (1)$$

This is a linear space and it is natural to endow it with the inductive limit topology σ_{ind} coming from the Hilbert space $\varphi(\mathsf{A})(X)$ with norm $\|\varphi(\mathsf{A})\cdot\|_X$. It will be clear that $S_{\Phi_1}(X;\mathsf{A}) = S_{X,|\mathsf{A}|}$, i.e. the analyticity space of this book with $|\mathsf{A}| = \int |\lambda| dE_\lambda$ if $\mathsf{A} = \int \lambda \, dE_\lambda$.

As a second space, associated with Φ_1, X and A, define

$$T_{\Phi_1}(X;\mathsf{A}) = \bigcap_{\varphi\in\Phi_1} \varphi^{-1}(\mathsf{A})X . \qquad (2)$$

Here $\varphi^{-1}(A)X$ is a completion of X with respect to the norm $\|\varphi(A)\cdot\|_X$. On $T_{\Phi_1}(X;A)$ the projective limit topology τ_{proj} with respect to $\varphi^{-1}(A)X$ is imposed. Comparison with Section I.2 shows that $T_{\Phi_1}(X;A) = T_{X,|A|}$ in every respect.

$S_{\Phi_1}(X;A)$ and $T_{\Phi_1}(X;A)$ can be put in duality by

$$\langle u,F\rangle = (\varphi^{-1}(A)u,\varphi(A)F)_X \tag{3}$$

for suitable $\varphi \in \Phi_1$. This pairing does not depend on the choice of $\varphi \in \Phi_1$. Next, we define a 'compatibility set' $\Phi_1^{\#}$ of Borel functions:

$$\Phi_1^{\#} = \{f \mid f > 0,\ \forall_{\varphi\in\Phi_1} : \sup_{\lambda\in\mathbb{R}} f(\lambda)\varphi(\lambda) < \infty\} .$$

Comparison with Lemma I.1.5 shows that $\Phi_1^{\#}$ consists of the functions

$$f = \begin{cases} f_1(\lambda) & , \quad \lambda \geqq 0 \\ f_2(-\lambda) & , \quad \lambda > 0 \end{cases} \quad \text{with } f_1,f_2 \in \mathbf{B}_+(\mathbb{R}) .$$

Starting from $\Phi_1^{\#}$ instead of Φ_1 the spaces $S_{\Phi_1^{\#}}(X;A)$ and $T_{\Phi_1^{\#}}(X;A)$ can be introduced as in (1) and (2).

From Theorem I.1.6 and Proposition I.2.6 it follows that

$$S_{\Phi_1^{\#}}(X;A) = T_{\Phi_1}(X;A) \quad \text{and} \quad T_{\Phi_1^{\#}}(X;A) = S_{\Phi_1}(X;A) .$$

The fundamental reason for this is Lemma I.1.5, which, rephrased in the language of this epilogue, says that

$$\forall_{\psi\in\Phi_1^{\#\#}}\ \exists_{c>0}\ \exists_{\varphi\in\Phi} : \psi \leqq c\varphi . \tag{4}$$

One of the consequences of property (4) is that $S_{\Phi_1}(X;A)$ and $T_{\Phi_1}(X;A)$ are both inductive limits *and* projective limits of Hilbert spaces. Thence, the nonstrict inductive limits $S_{\Phi_1}(X;A)$ and $T_{\Phi_1}(X;A)$ behave very much like *strict* inductive limits.

In [EGK] the above sketched ideas have been worked out for a general directed set Φ of nonnegative Borel functions on $\mathbb{R}^n$ with the following properties:

A.0. $\forall_{\varphi\in\Phi}$: $\varphi \geqq 0$ and bounded on bounded sets of $\mathbb{R}^n$.

$\forall_{\varphi_1,\varphi_2\in\Phi}\ \exists_{\varphi_3\in\Phi}\ \exists_{c>0}:\ c\varphi_3 \geqq \max(\varphi_1,\varphi_2)$.

A.1. $\forall_{\varphi\in\Phi}$: $\tilde{\varphi}^{-1}(\lambda) = \begin{cases} \varphi(\lambda)^{-1} & \text{if } \varphi(\lambda) > 0 \\ 0 & \text{if } \varphi(\lambda) = 0 \end{cases}$ is bounded on bounded sets.

A.2. The supports of φ, $\varphi \in \Phi$, cover the whole $\mathbb{R}^n$

$$\mathbb{R}^n = \bigcup_{\varphi\in\Phi} \{\lambda \in \mathbb{R} \mid \varphi(\lambda) > 0\} .$$

A.3. $\forall_{\varphi\in\Phi}\ \exists_{\psi\in\Phi}\ \exists_{c>0}\ \forall_{m\in Z^n}$: $(1 + |m|) \sup_{\lambda\in Q_m} \varphi(\lambda) \leqq c \inf_{\lambda\in Q_m} \psi(\lambda)$.

Here $m = (m_1,\ldots,m_n)$ and $Q_m = [m_1-1,m_1] \times \ldots \times [m_n-1,m_n]$.

Associated with Φ a so-called compatibility class $\Phi^\#$ of Borel functions is introduced as follows.

B.1. $\forall_{f\in\Phi^\#}$: $f \geqq 0$ and $\tilde{f}^{-1}$, cf. A.1, is bounded on bounded sets.

B.2. $\forall_{f\in\Phi^\#}\ \forall_{\varphi\in\Phi}$: $\sup_{\lambda\in\mathbb{R}^n} f(\lambda)\varphi(\lambda) < \infty$.

It can be shown, see [EGK], that $\Phi^\#$ satisfies again all properties A.0-A.3. Now, given a set Φ which satisfies A.0-A.3 and a strongly commuting set of self-adjoint operators $A = (A_1,\ldots,A_n)$ in a Hilbert space X, the spaces $S_\Phi(X;A)$, $T_\Phi(X;A)$, $S_{\Phi^\#}(X;A)$ and $T_{\Phi^\#}(X;A)$ are defined according to (1) and (2). For the solutions of technical problems around these definitions, see [EGK].

The space $S_\Phi(X;A)$ has the following properties:

S.1. Endowed with the semi-norms s_f, $s_f(w) = \|f(A)w\|$, $f \in \Phi^\#$, $S_\Phi(X;A)$ is a barreled and bornological locally convex topological vector space.

S.2. $S_\Phi(X;A)$ is an inductive limit of Hilbert spaces which is nonstrict in general.

The space $T_\Phi(X;A)$ has the following properties:

T.1. Endowed with the semi-norms t_φ, $t_\varphi(F) = \|\varphi(A)F\|_X$, $\varphi \in \Phi$, $T_\Phi(X;A)$ is a complete locally convex topological vector space.

T.2. $T_\Phi(X;A)$ is a projective limit of Hilbert spaces.

T.3. Each bounded set B of $T_\Phi(X;A)$ is homeomorphic to a bounded subset of X. The homeomorphism is established by a well chosen operator $g(A)$, $g \in \Phi^\#$.

T.4. $T_\Phi(X;A) = S_{\Phi^\#}(X;A)$ as a set.

The spaces $S_\Phi(X;A)$ and $T_\Phi(X;A)$ can be paired by

$$\langle u,F\rangle = (\tilde{\varphi}(A)^{-1}u,\varphi(A)F)_X \quad , \qquad u \in S_\Phi(X;A),\ F \in T_\Phi(X;A)$$

where $\varphi \in \Phi$ is such that $u \in \varphi(A)X$. The pairing does not depend on the specific choice of $\varphi \in \Phi$. By this pairing $S_\Phi(X;A)$ and $T_\Phi(X;A)$ can be regarded as each others strong duals.

If Φ contains only functions which are bounded on the joint spectrum $\sigma(A)$ of A, there is the Gelfand triple

$$S_\Phi(X;A) \subset X \subset T_\Phi(X;A) \ .$$

On the other hand, if Φ contains only functions which are bounded away from zero on $\sigma(A)$, there is the Gelfand triple

$$T_\Phi(X;A) \subset X \subset S_\Phi(X;A) \ .$$

In general neither $S_\Phi(X;A)$ nor $T_\Phi(X;A)$ is contained in X.
It may happen that Φ satisfies the following additional axiom A.4 which involves the second compatibility class $(\Phi^\#)^\# = \Phi^{\#\#}$. Cf. Lemma I.1.5, Lemma III.2.7.

A.4. $\forall_{\zeta\in\Phi^{\#\#}}\ \exists_{\varphi\in\Phi}\ \exists_{c>0}:\ \zeta < c\varphi$.

This has a series of important consequences, viz.: $S_\Phi(X;A) = S_{\Phi^{\#\#}}(X;A)$, $T_\Phi(X;A) = T_{\Phi^{\#\#}}(X;A)$, $T_\Phi(X;A) = S_{\Phi^\#}(X;A)$, $S_\Phi(X;A) = T_{\Phi^\#}(X;A)$ as topological vector spaces.
It then follows that all properties S.1, S.2, T.1-T.4, are shared by both spaces $S_\Phi(X;A)$, $T_\Phi(X;A)$. They are of exactly the same topological type.

In particular, both spaces are inductive limits of Hilbert spaces *and* projective limits of Hilbert spaces. Research is being done now on topological tensor products, topological tensor products and operator algebras involving $S_\Phi(X;\mathsf{A})$- and $T_\Phi(X;\mathsf{A})$-spaces. We conclude with a short sketch of some examples.

Example 1. $\Phi_\chi = \{\chi_\Delta \mid \Delta \in \mathbf{B}_b(\mathbb{R}^n)\}$

In words: Φ_χ is the set of all characteristic functions of bounded Borel sets. $\Phi_\chi^\#$ consists of *all* Borel functions which are bounded on bounded Borel sets. All axioms A.0-A.4 are satisfied. If n = 1, $X = \ell_2$, $\mathsf{A} = \mathsf{N} = \mathrm{diag}(1,2,3,\dots)$, then

$S_{\Phi_\chi}(\ell_2;\mathsf{N})$ is the space of finite sequences, and

$T_{\Phi_\chi}(\ell_2;\mathsf{N})$ is the space of all sequences.

If n = 1, $X = \mathbf{L}_2(\mathbb{R})$, $\mathsf{A} = \mathsf{Q}$, $(\mathsf{Q}u)(x) = xf(x)$, $x \in \mathbb{R}$, then

$S_{\Phi_\chi}(\mathbf{L}_2(\mathbb{R});\mathsf{Q})$ is the space of $\mathbf{L}_2(\mathbb{R})$-functions with bounded support,

$T_{\Phi_\chi}(\mathbf{L}_2(\mathbb{R});\mathsf{Q}) = \mathbf{L}_{2,\mathrm{loc}}(\mathbb{R})$.

Example 2. $\Phi_G = \Phi_1 = \{\lambda \mapsto e^{-t|\lambda|} \mid t > 0\}$

This set obviously satisfies A.0-A.3. A.4 is satisfied because of Lemma I.1.5 of this book. We have

$$S_{\Phi_G}(X;\mathsf{A}) = S_{X,|\mathsf{A}|} \subset X \subset T_{\Phi_G}(X;\mathsf{A}) = T_{X,|\mathsf{A}|} ,$$

the spaces of this book. For concrete examples we refer to Chapter 2.

Example 3. $\Phi_E = \{\lambda \mapsto e^{t|\lambda|} \mid t > 0\}$

This satisfies A.0-A.4. The proof of A.4 runs similarly to the proof of Lemma I.1.5. See [EK]. We have

$$T_{\Phi_E}(X;\mathsf{A}) = \tau(X,|\mathsf{A}|) \subset X \subset \sigma(X,|\mathsf{A}|) = S_{\Phi_E}(X;\mathsf{A}) .$$

Here τ and σ are the spaces in [E 2]. Remark that

$$\tau(X,|A|) = \bigcap_{n=1}^{\infty} \mathcal{D}(e^{n|A|}) .$$

All spaces of Chapter II have their counterparts as τ- and σ-spaces. See [E2], [EG 1-3].

The so-called Korevaar pansions, [Ko], can be regarded as elements of a very special $S_{\Phi_E}(X;A)$-space.

Example 4. $\Phi_P = \{\lambda \mapsto (1+\lambda^2)^n \mid n = 0,1,2,\ldots\}$

Φ_P satisfies A.0-A.4. The property A.4 follows by replacing in Example 3 the variable λ by $\log|\lambda|$.

With $X = \mathbf{L}_2(\mathbb{R})$ and the operator $H = \frac{1}{2}(x^2 - \frac{d^2}{dx^2})$

$$S_{\Phi_P}(\mathbf{L}_2(\mathbb{R});H) = T_{\Phi_P^\#}(\mathbf{L}_2(\mathbb{R});H) = S'(\mathbb{R})$$

$$S_{\Phi_P^\#}(\mathbf{L}_2(\mathbb{R});H) = T_{\Phi_P}(\mathbf{L}_2(\mathbb{R});H) = S(\mathbb{R}) .$$

Here $S(\mathbb{R})$ is the Schwarz space of test functions of rapid decrease and $S'(\mathbb{R})$ is the space of tempered distributions.

With $X = \mathbf{L}_2(\mathbb{R})$ and the operator $P = i\frac{d}{dx}$

$$S_{\Phi_P^\#}(\mathbf{L}_2(\mathbb{R});P) = \bigcap_{n=0}^{\infty} \mathbf{H}^n(\mathbb{R}) = \mathbf{H}^{\infty}(\mathbb{R})$$

$$T_{\Phi_P^\#}(\mathbf{L}_2(\mathbb{R});P) = \bigcup_{n=0}^{\infty} \mathbf{H}^{-n}(\mathbb{R}) = \mathbf{H}^{-\infty}(\mathbb{R}) ,$$

where $\mathbf{H}^n(\mathbb{R})$ denotes the Sobolev space of order n.

Example 5. $\Phi_A = \{\lambda \mapsto |\lambda|\ e^{-\frac{1}{2}|\lambda|}\ e^{t|\lambda|} \mid t > 0\}$

With Example 3 the validity of A.0-A.4 easily follows. With

$$X = F = \{f \mid f \text{ entire}, \int_{\mathbb{C}} |f(z)|^2\, e^{-|z|^2}\, dxdy < \infty,\ z = x+iy\} ,$$

the Bargman space, cf. [Ba 1], and $\widetilde{H} = z\frac{d}{dz}$ we get

$$S_{\Phi_A}(F;\widetilde{H}) = \{g \mid g \text{ entire}, \exists_{K,L>0} \; |g(z)| \leq Ke^{L|z|}\} ,$$

i.e. the space of entire functions of exponential growth.

$$T_{\Phi_A}(F;\widetilde{H}) = \{\text{all entire analytic functions}\} .$$

The latter space is described in [AnVa] as a Partial Inner Product Space.

Example 6. $\Phi_B = \{\lambda \mapsto |\lambda|^{\frac{1}{2}|\lambda|} \, e^{-t|\lambda|} \mid t > 0\}$

With Example 2 the validity of A.0-A.4 easily follows.
With $X = F$ and $\widetilde{H} = z\frac{d}{dz}$ of Example 5 we get

$$T_{\Phi_B^{\#}}(F;\widetilde{H}) = \{f \mid f(z) \text{ is analytic on a nbh of the origin}\} ,$$

i.e. the space of germs of analytic functions at 0.

$$S_{\Phi_B^{\#}}(F;\widetilde{H}) = \{g \mid g \text{ entire}, \forall_{\varepsilon>0} \, \exists_{K>0}: \; |g(z)| \leq Ke^{\varepsilon|z|}\} ,$$

i.e. the space of entire functions of sub-exponential growth. Combination of this with Example 5 yields the following quintuple of spaces of entire functions:

$$S_{\Phi_B^{\#}}(F;\widetilde{H}) \subset S_{\Phi_A}(F;\widetilde{H}) \subset F \subset T_{\Phi_A}(F;\widetilde{H}) \subset T_{\Phi_B^{\#}}(F;\widetilde{H}) .$$

From the general theory in [EGK] it follows that these spaces are all inductive limits and projective limits of Hilbert spaces.

Example 7. $\Phi_O = \{(\lambda,\mu) \mapsto f(\lambda)\varphi(\mu) \mid f \in \Phi_G^{\#}, \varphi \in \Phi_G\}$

The axioms A.0-A.3 are obviously satisfied. For two strongly commuting operators A_1 and A_2 in a Hilbert space X we have

$$S_{\Phi_O}(X;A_1,A_2) = ST_{X;A_1,A_2} ,$$

$$T_{\Phi_O}(X;A_1,A_2) = TS_{X;A_1,A_2} ,$$

the spaces of Chapter III. Axiom A.4 is *not* satisfied in this case. In Chapter III it has been shown that

$$\eta: (\lambda,\mu) \mapsto \exp\left[-\mu\left(\frac{\mu}{\lambda+\mu}\right)^2\right]$$

belongs to $\Phi^{\#\#}$, while for all $\varphi \in \Phi_0$ the function $\eta\varphi^{-1}$ is unbounded. Starting from this it has been shown in Chapter III that

$$ST_{X;A_1,A_2} = S_{\Phi_0}(X;A_1,A_2)$$

is not a complete space in general and does not have the usual properties of strict inductive limit spaces. Cf. Theorem III.2.13 and Theorem III.2.14.

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LIST OF SYMBOLS