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APPENDIX 1

MULTI-INDEXED MATRICES AND GENERALIZED MATRIX THEORY

Let I denote a contiguous subset of integers, let $\mathbb{R}$ be the set of real numbers. We define a m-indexed matrix ${}^{m}A$ as a map

$$ {}^{m}A : {}^{m}I \to \mathbb{R} \tag{A.1} $$

where ${}^{m}I = I \times \ldots \times I$ (m-copies). According to (A.1), the index-set ${}^{m}I$ can be called the domain of the m-indexed matrix ${}^{m}A$. This domain will be denoted as $D\,{}^{m}A$. Let us introduce the index-sets

$$ L_n^o \triangleq \{j\}_o^n = \{0,1,\ldots,n\} \tag{A.2} $$

$$ {}^{m}L_n^o \triangleq \underbrace{L_n^o \times \ldots \times L_n^o}_{m} = \{(j_1,\ldots,j_m) : j_k \in L_n^o \,,\ k=1,\ldots,m\} \tag{A.3} $$

A m-indexed matrix ${}^{m}\underline{A}_n$ will be called the n-th order matrix if

$$ D\,{}^{m}\underline{A}_n = {}^{m}L_n^o \tag{A.4a} $$

This matrix can be equivalently expressed in terms of its m-indexed entries as follows

$$ {}^{m}\underline{A}_n = [\, a_{j_1 \ldots j_m} \,]_{(j_1,\ldots,j_m) \in {}^{m}L_n^o} \tag{A.4b} $$

We will consider here some properties of multi-indexed matrices, and introduce operations on those matrices. We will usually drop, for simplicity, the order and domain of the matrices, assuming that all matrices are of the type (A.4), unless otherwise stated.

Symmetric matrix

A m-indexed matrix will be called symmetric if for any permutation $\pi_1,\dots,\pi_m$ of integers $1,\dots,m$ we shall have

$$\underline{a}_{j_{\pi_1}\dots j_{\pi_m}} = \underline{a}_{j_1\dots j_m} \qquad \text{(A.5a)}$$

Consequently, it is sufficient to consider $\binom{m+n}{m}$ 'different' entries of the symmetric matrix instead of n^m entries of non-symmetric matrix. Now let $\gamma_{j_1\dots j_m}$ denote the number of equal elements of the symmetric matrix, corresponding to the sequence $(j_1,\dots,j_m)$. We shall denote the 'symmetric part' of a m-indexed matrix ${}^m\underline{A}_n$ by ${}^mA$, where

$$^mA_n = [\, a_{j_1\dots j_m} \,]_{(j_1,\dots,j_m) \in \mathrm{sym}\,{}^mL_n^o} \qquad \text{(A.5b)}$$

with $\mathrm{sym}\,{}^mL_n^o$ denoting the 'symmetric part' of the m-variate index-set, obtained according to lexicographic or anti-lexicographic ordering. The entries $a_{j_1\dots j_m}$ of ${}^mA_n$ can then be expressed in terms of the entries $\underline{a}_{j_1\dots j_m}$ of the symmetric matrix ${}^m\underline{A}_n$ as

$$a_{j_1\dots j_m} = \gamma_{j_1\dots j_m}\, \underline{a}_{j_1\dots j_m} \qquad \text{(A.5c)}$$

Transpose matrix

Let π be a permutation of the index-set $\{(j_1,\dots,j_m) \in {}^mL_n^o\}$. π may be represented by a map

$$\binom{1\,,\,2\,,\dots,\,m}{\pi_1,\,\pi_2,\dots,\,\pi_m} \qquad \text{(A.6a)}$$

where $(\pi_1, \pi_2, \ldots, \pi_m)$ is a permutation of $\{1,2,\ldots,m\}$. Then, a m-indexed matrix ${}^mA'_\pi$ will be called the transpose matrix due to the permutation π if

$$({}^mA'_\pi)_{j_1 \ldots j_m} = ({}^mA)_{\pi (j_1,\ldots,j_m)} \tag{A.6b}$$

Zero-matrix

A m-indexed matrix will be called a zero-matrix if for each sequence of indices $(j_1,\ldots,j_m) \in D^mA$ we have $a_{j_1 \ldots j_m} = 0$. This matrix will be denoted by

$${}^mO_n = [0_{j_1 \ldots j_m}] \quad (j_1,\ldots,j_m) \in D^mO_n \tag{A.7}$$

where $0_{j_1 \ldots j_m}$ will be the zero-entry with 'coordinates' $(j_1,\ldots,j_m)$.

Unit-matrix

A 2m-indexed matrix will be called the unit-matrix if for each $(j_1,\ldots,j_m)$ and $(k_1,\ldots,k_m) \in D^{2m}A_n$ we have

$$a_{j_1 \ldots j_m k_1 \ldots k_m} = \delta_{j_1 \ldots j_m ; k_1 \ldots k_m} \tag{A.8a}$$

where

$$\delta_{j_1 \ldots j_m ; k_1 \ldots k_m} = \begin{cases} 1 & \text{if } j_1 = k_1 , \ldots, j_m = k_m \\ 0 & \text{otherwise} \end{cases} \tag{A.8b}$$

This matrix will be denoted as

$$^{2m}\mathbf{1}_n = [\delta_{j_1\ldots j_m;k_1\ldots k_m}]_{(j_1,\ldots,j_m,k_1,\ldots,k_m)\in D^{2m}\mathbf{1}_n} \tag{A.8c}$$

Block-matrices

A block-matrix whose block-entries are m-indexed, n-th order matrices

$$^{\{M\}}A_n = [\,^mA_n\,]_{m=1,\ldots,M} \tag{A.9a}$$

will be called a M-block (row), m-indexed, n-th order matrix. Its block-row domain $D\,^{\{M\}}A_n$ will be a vector of simple domains

$$D\,^{\{M\}}A_n = [D\,^mA_n]_{m=1,\ldots,M} \tag{A.9b}$$

Similarly, a block-matrix

$$^{\{M\}}B_n = \text{col}\,[\,^mB_n\,]_{m=1,\ldots,M} \tag{A.10a}$$

will be called a M-block (column), m-indexed, n-th order matrix with the block-column domain

$$D\,^{\{M\}}B_n = \text{col}\,[\,D\,^mB_n\,]_{m=1,\ldots,M} \tag{A.10b}$$

Finally, a block-matrix

$$^{\{M\times M\}}H_n = [\,^{m\oplus u}H_n\,]_{m,u=1,\ldots,M} \tag{A.11a}$$

whose block-entries are (m+u)-indexed matrices

$$^{m\oplus u}H_n = [h_{j_1 \dots j_m k_1 \dots k_u}]_{(j_1,\dots,j_m,k_1,\dots,k_u) \in D^{m\oplus u}H_n} \qquad \text{(A.11b)}$$

will be called a $(M\times M)$-block (square), $(m+u)$-indexed, n-th order matrix. Its block-square domain $D^{\{M\times M\}}H_n$ is given by

$$D^{\{M\times M\}}H_n = [D^{m\oplus u}H_n]_{m,u=1,\dots,M} \qquad \text{(A.11c)}$$

Let us observe that the matrix (A.11) can be described in a generalized 'block-column' form. To do that, let us suppose that $D^{m\oplus u}H_n = D^mH_n \times D^uH_n$, where $(j_1,\dots,j_m) \in D^mH_n$ and $(k_1,\dots,k_u) \in D^uH_n$. Then we can write

$$^{\{M\times M\}}H_n = [^{\{M\}\times u}H_n]_{u=1,\dots,M} \qquad \text{(A.12a)}$$

where

$$^{\{M\}\times u}H_n = \text{col}\,[^{m\oplus u}H_n]_{m=1,\dots,M} \qquad \text{(A.12b)}$$

or, equivalently,

$$^{\{M\}\times u}H_n = [^{\{M\}}H_{n;k_1\dots k_u}]_{(k_1,\dots,k_u) \in D^uH_n} \qquad \text{(A.12c)}$$

with

$$^{\{M\}}H_{n;k_1\dots k_u} = [h_{j_1\dots j_m k_1\dots k_u}]_{(j_1,\dots,j_m) \in D^mH_n} \qquad \text{(A.12d)}$$

<u>Equal matrices</u>

Two m-indexed matrices

$$^mA = [a_{j_1\dots j_m}]_{(j_1,\dots,j_m) \in D^mA} \quad ; \quad ^mB = [b_{j_1\dots j_m}]_{(j_1,\dots,j_m) \in D^mB} \qquad \text{(A.13)}$$

will be called equal matrices if $D^mA = D^mB$, and if for each sequence of indices $(j_1,\dots,j_m) \in D^mA$

$$a_{j_1\dots j_m} = b_{j_1\dots j_m} \tag{A.14}$$

Sum of multi-indexed matrices

Given the matrices (A.13), we shall say that the m-indexed matrix

$$^mG = [g_{j_1\dots j_m}]_{(j_1,\dots,j_m) \in D^mG} \tag{A.15a}$$

is the sum

$$^mG = {}^mA + {}^mB \tag{A.15b}$$

if $D^mG = D^mA = D^mB$, and if for each $(j_1,\dots,j_m) \in D^mA$

$$g_{j_1\dots j_m} \triangleq a_{j_1\dots j_m} + b_{j_1\dots j_m} \tag{A.15c}$$

Sum of block, multi-indexed matrices

Given two M-block (row), m-indexed matrices

$$^{\{M\}}A = [\,^mA\,]_{m=1,\dots,M} \quad ; \quad ^{\{M\}}B = [\,^mB\,]_{m=1,\dots,M} \tag{A.16a}$$

we shall say that the M-block (row), m-indexed matrix

$$^{\{M\}}G = [\,^mG\,]_{m=1,\dots,M} \tag{A.16b}$$

where ^mG is given by (A.15a), is the sum

$$^{\{M\}}G = {}^{\{M\}}A + {}^{\{M\}}B \tag{A.16d}$$

if for $m=1,\ldots,M$ we have $D^mA = D^mB = D^mG$, and if for each sequence of indices $(j_1,\ldots,j_m) \in D^mA$ the entries $g_{j_1\ldots j_m}$ are expresed by (A.15c).

Product of a scalar and a m-indexed matrix

Given a scalar $c \in \mathbb{R}$ and a m-indexed matrix ^mA (A.13), we shall say that the m-indexed matrix ^mG (A.15a) is the product of the scalar and the m-indexed matrix

$$^mG = c\cdot{}^mA \tag{A.17a}$$

if for each $(j_1,\ldots,j_m) \in D^mG$ (where $D^mG = D^mA$) we have

$$g_{j_1\ldots j_m} \overset{\Delta}{=} c\cdot a_{j_1\ldots j_m} \tag{A.17b}$$

Product of multi-indexed matrices

Let ^mA be given by (A.13), and let ^sB be given by (A.13) with m replaced by s, where $m \leq s$. Let υ, μ, ν be some given integers, satisfying

$$\upsilon + \mu = m \quad ; \quad \mu + \nu = s \tag{A.18a}$$

and moreover let $r = (\upsilon + \nu)$. Partitioning the m indices of the ^mA, and the s indices of the ^sB in accordance with (A.18a), and assuming that $D^{m}A = D^{\upsilon}A \times D^{\mu}A$ and $D^{s}B = D^{\mu}B \times D^{\nu}B$, we can write

$$ {}^{m}A = [a_{k_1 \dots k_\upsilon j_1 \dots j_\mu}] \quad (k_1,\dots,k_\upsilon) \in D^{\upsilon}A \; ; \; (j_1,\dots,j_\mu) \in D^{\mu}A \tag{A.18b} $$

$$ {}^{s}B = [b_{j_1 \dots j_\mu i_1 \dots i_\nu}] \quad (j_1,\dots,j_\mu) \in D^{\mu}B \; ; \; (i_1,\dots,i_\nu) \in D^{\nu}B \tag{A.18c} $$

We shall say that the $r = (\upsilon + \nu)$ -indexed matrix

$$ {}^{r=(\upsilon+\nu)}G = [g_{k_1 \dots k_\upsilon i_1 \dots i_\nu}] \quad (k_1,\dots,k_\upsilon) \in D^{\upsilon}A \; ; \; (i_1,\dots,i_\nu) \in D^{\nu}B \tag{A.18d} $$

whose domain is $D^{r}G = D^{\upsilon}A \times D^{\nu}B$, is the μ-product of the matrices (A.18b,c)

$$ {}^{r}G = {}^{m}A \cdot {}^{s}B = {}^{\upsilon \oplus \mu}A \cdot {}^{\mu \oplus \nu}B \tag{A.18e} $$

if $D^{\mu}A = D^{\mu}B = {}^{\mu}D$, and if for each $(k_1,\dots,k_\upsilon) \in D^{\upsilon}A$, and for each $(i_1,\dots,i_\nu) \in D^{\nu}B$

$$ g_{k_1 \dots k_\upsilon i_1 \dots i_\nu} \triangleq \sum_{{}^{\mu}D} a_{k_1 \dots k_\upsilon j_1 \dots j_\mu} b_{j_1 \dots j_\mu i_1 \dots i_\nu} \tag{A.18f} $$

where the sum in (A.18f) denotes the μ-fold summation with respect to $(j_1,\dots,j_\mu)$ over the μ-variate index-set ${}^{\mu}D$.

<u>Product of block, multi-indexed matrices</u>

Given the M-block (row) matrix ${}^{\{M\}}A$ (A.9), and the $(M \times M)$-block square-matrix ${}^{\{M \times M\}}H$ (A.11), we shall say that the M-block (row), u-indexed matrix

$$ {}^{\{M\}}G = [{}^{u}G]_{u=1,\dots,M} \quad ; \quad {}^{u}G = [g_{k_1,\dots,k_u}]_{(k_1,\dots,k_u)\in D^{u}G} \qquad \text{(A.19a)} $$

(where for $u=1,\dots,M$ we have $D^{u}G = D^{u}H$, with $D^{m\oplus u}H = D^{m}H \times D^{u}H$) is the block m-product of the matrices ${}^{\{M\}}A$ and ${}^{\{M\times M\}}H$

$$ {}^{\{M\}}G = {}^{\{M\}}A \cdot {}^{\{M\times M\}}H \qquad \text{(A.19b)} $$

if for $u=1,\dots,M$

$$ {}^{u}G \triangleq \sum_{m=1}^{M} {}^{m}A \cdot {}^{m\oplus u}H \qquad \text{(A.19c)} $$

with $\cdot$ denoting the product (A.18e). Using (A.18) with $\upsilon=0$, $\mu=m$ and $\nu=u$, we can rewrite (A.19) as

$$ g_{k_1\dots k_u} = \sum_{m=1}^{M} \sum_{{}^{m}D} a_{j_1\dots j_m} h_{j_1\dots j_m k_1\dots k_u} \qquad \text{(A.19d)} $$

where ${}^{m}D = D^{m}A = D^{m}H$ and $(k_1,\dots,k_u)\in D^{u}G$. Equivalently, using (A.12d), we can write

$$ g_{k_1\dots k_u} = {}^{\{M\}}A \cdot {}^{\{M\}}H_{k_1\dots k_u} \qquad \text{(A.19e)} $$

'Outer' or Kronecker product of multi-indexed matrices

Let ${}^{m}A$ be given by (A.13), and let ${}^{s}B$ be expressed by (A.13) with m replaced by s . We shall say that the $(m+s)$-indexed matrix

$$ {}^{m\oplus s}G = [g_{j_1\dots j_m k_1\dots k_s}]_{(j_1,\dots,j_m)\in D^{m}A \,;\, (k_1,\dots,k_s)\in D^{s}B} \qquad \text{(A.20a)} $$

(whose domain is $D^{m\oplus s}G = D^{m}A \times D^{s}B$) is the 'outer' (or Kronecker) product of the matrices ${}^{m}A$ and ${}^{s}B$

$$ {}^{m\oplus s}G = {}^{m}A \otimes {}^{s}B \qquad (A.20b) $$

if for each $(j_1,\dots,j_m) \in D^{m}A$ and $(k_1,\dots,k_s) \in D^{s}B$ we have

$$ g_{j_1\dots j_m k_1\dots k_s} \triangleq a_{j_1\dots j_m}\, b_{k_1\dots k_s} \qquad (A.20c) $$

From (A.20) it follows that if

$$ {}^{1}Y = [y_j]_{\; j\in D^{1}Y} \qquad (A.21a) $$

then

$$ {}^{m}Y = \overset{m}{\otimes}\, {}^{1}Y = [\, y_{j_1}\dots y_{j_m}\,]_{\; (j_1,\dots,j_m)\in D^{m}Y} \qquad (A.21b) $$

where

$$ D^{m}Y = D^{1}Y \underbrace{\times\dots\times}_{m} D^{1}Y \qquad (A.21c) $$

<u>'Outer' product of block, multi-indexed matrices</u>

Let

$$ {}^{\{M\}}Y = [\,{}^{m}Y\,]_{\; m=1,\dots,M} \qquad (A.22a) $$

where ${}^{m}Y$ is given by (A.21), and moreover let

$$ {}^{\{M\times M\}}G = [\,{}^{m\oplus u}G\,]_{\; m,u=1,\dots,M} \qquad (A.22b) $$

with

$$^{m\oplus u}G = [g_{j_1 \dots j_m k_1 \dots k_u}] \quad (j_1,\dots,j_m) \in D^m Y \; ; \; (k_1,\dots,k_u) \in D^u Y \tag{A.22c}$$

We shall say that the matrix $^{\{M\times M\}}G$ is the block, outer-product

$$^{\{M\times M\}}G = {}^{\{M\}}Y \otimes {}^{\{M\}}Y \tag{A.22d}$$

if for $m,u=1,\dots,M$

$$^{m\oplus u}G \triangleq {}^{m}Y \otimes {}^{u}Y \tag{A.22e}$$

or, equivalently, if for each $(j_1,\dots,j_m) \in D^m Y$ and $(k_1,\dots,k_u) \in D^u Y$

$$g_{j_1 \dots j_m k_1 \dots k_u} = y_{j_1} \cdots y_{j_m} y_{k_1} \cdots y_{k_u} \tag{A.22f}$$

APPENDIX 2

MULTI-VARIATE INDEX-SET RECURSIONS

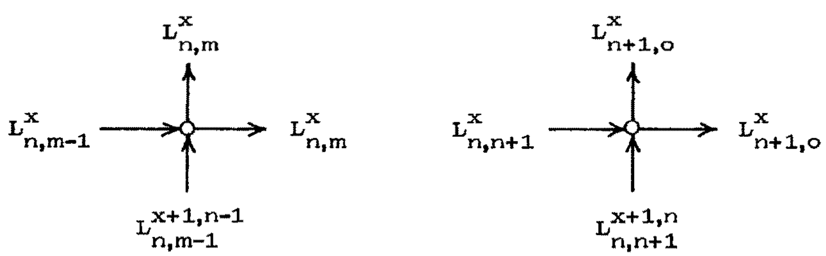
LB-recursions: 'local' order-updates for the 'bi-variate' part of the L-forward index-set, and for the 'uni-variate' part of the B-backward index-sets; i.e., for $m=2,\ldots,n+1$

$$L^{x}_{n,m} = L^{x}_{n,m-1} \cup L^{x+1,n-1}_{n,m-1} = \underbrace{\{x\} \cup L^{x+1,n-1}_{n,m-2}}_{L^{x}_{n,m-1}} \cup \{x+n+3-m,x+n+1\} = \{x\} \cup \overbrace{L^{x+1,n-1}_{n,m-2} \cup \{x+n+3-m,x+n+1\}}^{L^{x+1,n-1}_{n,m-1}} \quad \text{(A.23a)}$$

and for $m=n+2$

$$L^{x}_{n+1,o} = L^{x}_{n,n+1} \cup L^{x+1,n}_{n,n+1} = \underbrace{\{x\} \cup L^{x+1,n-1}_{n,n}}_{L^{x}_{n,n+1}} \cup \{x+1,x+n+1\} = \{x\} \cup \overbrace{L^{x+1,n-1}_{n,n} \cup \{x+1,x+n+1\}}^{L^{x+1,n}_{n,n+1}} \quad \text{(A.23b)}$$

These recursions can be schematically described as the LB index-set sections



(A.23c)

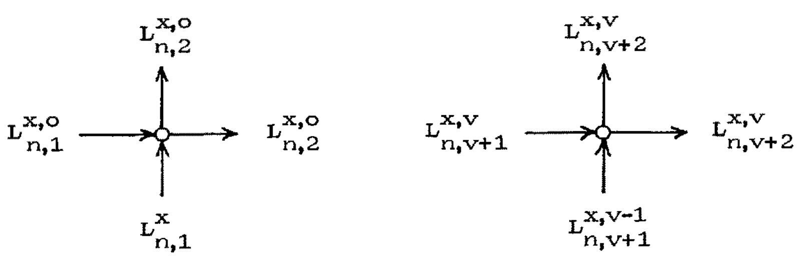
<u>BL-recursions:</u> 'local' order-updates for the 'uni-variate' part of the B-forward index-sets, and for the 'bi-variate' part of the L-backward index-set; i.e., for $v=0$

$$L^{x,o}_{n,2} = L^{x,o}_{n,1} \cup L^{x}_{n,1} = \rlap{\underbrace{\phantom{\{x,x\} \cup L^{x}_{n,o}}}_{L^{x,o}_{n,1}}}\{x,x\} \cup \overbrace{L^{x}_{n,o} \cup \{x+n+1\}}^{L^{x}_{n,1}} \qquad \text{(A.24a)}$$

and for $v=1,\dots,n$

$$L^{x,v}_{n,v+2} = L^{x,v}_{n,v+1} \cup L^{x,v-1}_{n,v+1} = \rlap{\underbrace{\phantom{\{x,x+v\} \cup L^{x,v-1}_{n,v}}}_{L^{x,v}_{n,v+1}}}\{x,x+v\} \cup \overbrace{L^{x,v-1}_{n,v} \cup \{x+n+1\}}^{L^{x,v-1}_{n,v+1}} \qquad \text{(A.24b)}$$

These recursions will be interpreted as the BL index-set sections



(A.24c)

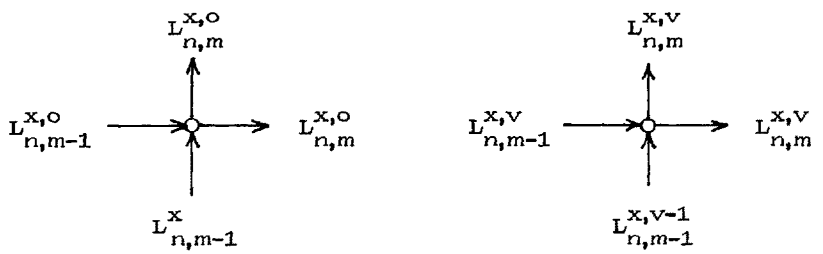
<u>BB-recursions:</u> 'local' order-updates for the 'bi-variate' parts of the B-forward and B-backward index-sets; i.e., for $v=0$ and $m=3,\dots,n+3$

$$L^{x,o}_{n,m} = L^{x,o}_{n,m-1} \cup L^{x}_{n,m-1} = \underbrace{\{x,x\} \cup \mathrlap{\overbrace{\phantom{L^{x}_{n,m-2} \cup \{x+n+4-m,x+n+1\}}}^{L^{x}_{n,m-1}}} L^{x}_{n,m-2}}_{L^{x,o}_{n,m-1}} \cup \{x+n+4-m,x+n+1\} \qquad \text{(A.25a)}$$

and for $v=1,\dots,n$ and $m=v+3,\dots,v+n+3$

$$L^{x,v}_{n,m} = L^{x,v}_{n,m-1} \cup L^{x,v-1}_{n,m-1} = \underbrace{\{x,x+v\} \cup \mathrlap{\overbrace{\phantom{L^{x,v-1}_{n,m-2} \cup \{x+n+4+v-m,x+n+1\}}}^{L^{x,v-1}_{n,m-1}}} L^{x,v-1}_{n,m-2}}_{L^{x,v}_{n,m-1}} \cup \{x+n+4+v-m,x+n+1\} \qquad \text{(A.25b)}$$

These recursions will result in the BB index-set sections



(A.25c)

'LOCAL' ORDER-UPDATE RECURSIONS

LB-recursions: for $m=2,\ldots,n+1$

$$\begin{bmatrix} A^{x}_{n,m}(Z) \\ \\ B^{x}_{n,m}(Z) \end{bmatrix} = \theta^{x}_{n,m} \begin{bmatrix} A^{x}_{n,m-1}(Z) \\ \\ Z\cdot B^{x+1,n-1}_{n,m-1}(Z) \end{bmatrix} \qquad \text{(A.26a)}$$

$$\rho^{x}_{n,m} = (A^{x}_{n,m}(Z)\ ,\ Z\cdot B^{x+1,n-1}_{n,m-1}(Z))_{\mathbb{Z}} \qquad \text{(A.26b)}$$

and for $m=n+2$

$$\begin{bmatrix} A^{x}_{n+1,o}(Z) \\ \\ B^{x}_{n+1,o}(Z) \end{bmatrix} = \theta^{x}_{n+1,o} \begin{bmatrix} A^{x}_{n,n+1}(Z) \\ \\ Z\cdot B^{x+1,n}_{n,n+1}(Z) \end{bmatrix} \qquad \text{(A.26c)}$$

$$\rho^{x}_{n+1,o} = (A^{x}_{n,n+1}(Z)\ ,\ Z\cdot B^{x+1,n}_{n,n+1}(Z))_{\mathbb{Z}} \qquad \text{(A.26d)}$$

BL-recursions: for $v=0$

$$\begin{bmatrix} A^{x,o}_{n,2}(Z) \\ \\ B^{x,o}_{n,2}(Z) \end{bmatrix} = \theta^{x,o}_{n,2} \begin{bmatrix} A^{x,o}_{n,1}(Z) \\ \\ B^{x}_{n,1}(Z) \end{bmatrix} \qquad \text{(A.27a)}$$

$$\rho^{x,o}_{n,2} = (A^{x,o}_{n,1}(Z)\ ,\ B^{x}_{n,1}(Z))_{\mathbb{Z}} \qquad \text{(A.27b)}$$

for $v=1,\ldots,n$

$$\begin{bmatrix} A^{x,v}_{n,v+2}(Z) \\ \\ B^{x,v}_{n,v+2}(Z) \end{bmatrix} = \Theta^{x,v}_{n,v+2} \begin{bmatrix} A^{x,v}_{n,v+1}(Z) \\ \\ B^{x,v-1}_{n,v+1}(Z) \end{bmatrix} \qquad (A.27c)$$

$$\rho^{x,o}_{n,m} = (A^{x,o}_{n,v+1}(Z) \; , \; B^{x,v-1}_{n,v+1}(Z))_{\mathbf{Z}} \qquad (A.27d)$$

<u>BB-recursions:</u> for $v=0$ and $m=3,\ldots,n+3$

$$\begin{bmatrix} A^{x,o}_{n,m}(Z) \\ \\ B^{x,o}_{n,m}(Z) \end{bmatrix} = \Theta^{x,o}_{n,m} \begin{bmatrix} A^{x,o}_{n,m-1}(Z) \\ \\ B^{x}_{n,m-1}(Z) \end{bmatrix} \qquad (A.28a)$$

$$\rho^{x,o}_{n,m} = (A^{x,o}_{n,m-1}(Z) \; , \; B^{x}_{n,m-1}(Z))_{\mathbf{Z}} \qquad (A.28b)$$

for $v=1,\ldots,n$ and $m=v+3,\ldots,v+n+3$

$$\begin{bmatrix} A^{x,v}_{n,m}(Z) \\ \\ B^{x,v}_{n,m}(Z) \end{bmatrix} = \Theta^{x,v}_{n,m} \begin{bmatrix} A^{x,v}_{n,m-1}(Z) \\ \\ B^{x,v-1}_{n,m-1}(Z) \end{bmatrix} \qquad (A.28c)$$

$$\rho^{x,v}_{n,m} = (A^{x,v}_{n,m-1}(Z) \; , \; B^{x,v-1}_{n,m-1}(Z))_{\mathbf{Z}} \qquad (A.28d)$$

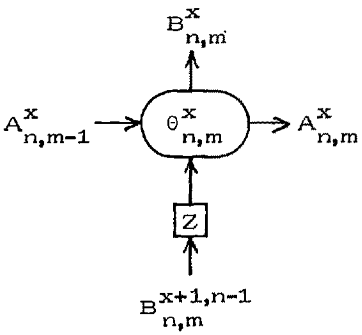
where

$$\Theta^{x}_{n,m} = (1-[\rho^{x}_{n,m}]^2)^{-\frac{1}{2}} \begin{bmatrix} 1 & -\rho^{x}_{n,m} \\ \\ -\rho^{x}_{n,m} & 1 \end{bmatrix} \qquad (A.29a)$$

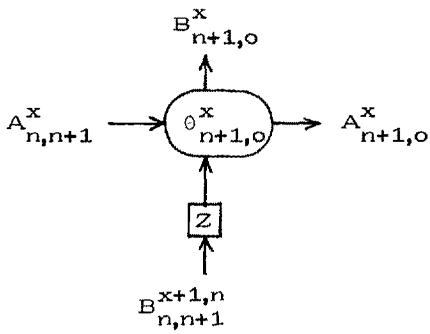
$$\Theta^{x,v}_{n,m} = (1-[\rho^{x,v}_{n,m}]^2)^{-\frac{1}{2}} \begin{bmatrix} 1 & -\rho^{x,v}_{n,m} \\ \\ -\rho^{x,v}_{n,m} & 1 \end{bmatrix} \qquad (A.29b)$$

'LOCAL' FILTER SECTIONS

LB-sections:

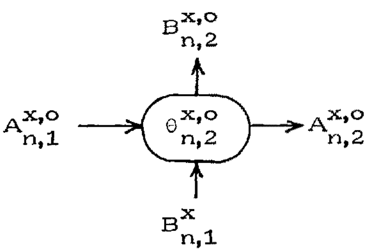


for m=2,...,n+1

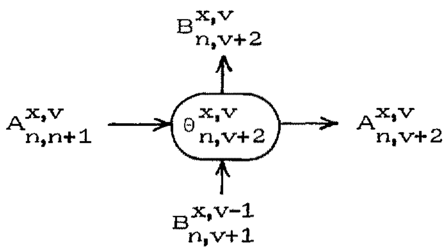


for m=n+2

BL-sections:

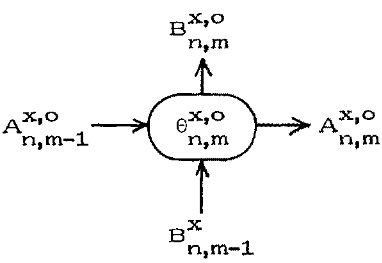


for v=0

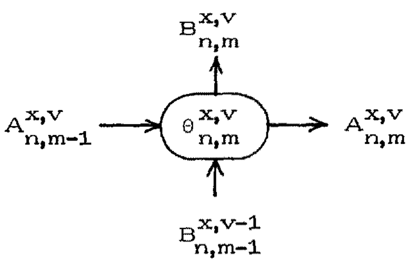


for v=1,...,n

BB-sections:



for v=0 and m=3,...,n+3



for v=1,...,n and m=v+3,...,v+n+3