# SUGGESTIONS FOR FURTHER READING

By its nature, the subject of arithmetic groups reaches out in many directions and has ill-defined boundaries. So it would be almost impossible to assemble a comprehensive bibliography. Instead, I have listed below some works which either develop or complement the topics treated in these notes. No claims of completeness (or even balance) are intended. Most of the papers cited presuppose some sophistication in the use of Lie groups or linear algebraic groups; but in his monograph, Borel [5] does attempt to ease the transition from special cases to general theory. Here are a few topics which the reader may wish to explore further:

Reduction theory for arithmetic subgroups of semisimple Lie groups is treated in Borel [5]; cf. Harder [1] for the function field case. Earlier work along these lines is also well worth consulting, e.g., Borel, Harish-Chandra [1], Godement [1], Mostow, Tamagawa [1]. The measure of a fundamental set is studied in various articles in Borel, Mostow [1], notably Langlands [1], as well as in Weil [1], Mars [1] - [3].

Cohomology of arithmetic groups can be looked at from many points of view. The work of Borel leads in a natural way to the results of Borel, Serre [1], as was mentioned briefly in 13.5. There are many other papers on cohomology, of which the following list is just a sample: Borel [6], [7], Borel, Wallach [1], Garland [2], Harder [2], Kazhdan [1], Raghunathan [1], Schwermer [1], Soulé [1].

As indicated in 13.4, reduction theory yields information about finite generation or finite presentation of arithmetic groups. Relevant papers include: Borel [1], Kneser [1], Behr [1] - [3], Stuhler [1]. Steinberg's work on central extensions (cf. §18 above) leads in a different way to explicit presentations of Chevalley groups over fields as well as over rings like $\mathbb{Z}$: see Steinberg [1], [2], Deodhar [1], Milnor [1], Behr [5].

The modular group $SL(2,\mathbb{Z})$ or $PSL(2,\mathbb{Z})$ has a life of its own. From the extensive literature we cite just a few sources, selected at random: Reiner [1], Newman [1], Jones [1]. Of special interest are congruence (and non-congruence) subgroups in $SL_2$ over the integers of various global fields, cf. Serre [3], [5], Mel'nikov [1].

The congruence subgroup problem has been attacked in greater and greater generality, but is still not entirely resolved for groups of rank at least 2 over global fields. For $SL(n,\mathbb{Z})$, $n \geq 3$, independent solutions were found by Bass, Lazard, Serre [1] and Mennicke [1].

Then Bass, Milnor, Serre [1] (cf. the exposition in Serre [2]) treated special linear and symplectic groups over arbitrary number fields, the case of $SL_2$ being handled separately by Serre [3]. Using results of Moore [1], Matsumoto [1] finished off the split (Chevalley type) groups. Non-split groups have been studied by a number of authors: Vaserstein [1] - [3], Kneser [5], Deodhar [1], Raghunathan [4]. The connections with algebraic K-theory have also been thoroughly explored, cf. Milnor [1], Keune [1].

In an entirely different direction, it is natural to ask whether every lattice in a semisimple Lie group G (discrete subgroup H for which G/H has finite invariant measure) is defined arithmetically, relative to some rational structure on G. In rank 1 there are known to be exceptions (cf. Vinberg [1], Mostow [3]), but in rank $\geq 2$ (suitably formulated) it turns out that all lattices are indeed arithmetic. Partial results in this direction were obtained by Prasad, Raghunathan [1], Raghunathan [5] (see Raghunathan [3] for further background on these matters). But the most general results are due to Margulis [1] - [3]; Tits [3] provides a very helpful exposition. For related questions about rigidity of lattices, see Mostow [1], [2], Prasad [1], [2].

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