APPENDIX

Since the writing of these notes, the author has improved (see [Da4]) the results of §2 (Chapter I) in some particular cases.

Before describing the results of the Appendix, let us recall the hypotheses of Chapter I

$$(H)\quad \begin{cases} u^\nu \overset{*}{\longrightarrow} u \text{ in } L^\infty_m(\Omega) \\ Au^\nu = \left(\sum_{j=1}^{m} \sum_{k=1}^{n} a_{ijk} \frac{\partial u_j^\nu}{\partial x_k} \right)_{1\leqslant i\leqslant q} \text{ bounded in } L^\infty_q(\Omega) \\ f(u^\nu) \overset{*}{\longrightarrow} \ell \text{ in } L^\infty(\Omega) \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuous function.

In §2 a necessary condition for weak lower semicontinuity (i.e., $\ell \geqslant f(u)$) was isolated and called A-quasiconvexity. This condition turned out to be sufficient in some particular cases. Recall that

<u>Definition</u>. A continous function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is said to be <u>A-quasiconvex</u> if

$$\int_D f(\mu + \zeta(x))\, dx \geqslant \int_D f(\mu)\, dx \tag{A.1}$$

for every $\mu \in \mathbb{R}^m$, for every hypercube $D \subset \mathbb{R}^n$ and for every $\zeta \in L(D)$ where

$$L(D) = \{\zeta \in L^\infty_m(D);\ \int_D \zeta(x)dx = 0 \text{ and } \zeta \in \text{Ker } A\}.$$

The aim of this Appendix is to show that for some special operators A (e.g., A = curl or A = div and hence for the variational case) one may further restrict the set L(D) by including a condition on the support of $\zeta \in L(D)$ (thus answering Remark (ii) p. 13); which therefore makes more precise the notion of quasiconvexity.

Before doing that we need to isolate a special class of functions which are in Ker A.

Notations. If A is defined as in Hypothesis (H), i.e.,

$$Au = \left(\sum_{j=1}^{m} \sum_{k=1}^{n} a_{ijk} \frac{\partial u_j}{\partial x_k} \right)_{1 \le i \le q} \tag{A.2}$$

we will denote by B

$$B : v(x_1, \ldots, x_n) = (v_1, \ldots, v_p) \longrightarrow Bv$$

the operator

$$Bv = \left(\sum_{\mu=1}^{p} \sum_{\nu=1}^{n} b_{\lambda\mu\nu} \frac{\partial v_\mu}{\partial x_\nu} \right)_{1 \le \lambda \le m} \tag{A.3}$$

where $b_{\lambda\mu\nu} \in \mathbb{R}$, $p \geq 1$ an integer and where B satisfies

$$ABv \equiv 0 \quad \text{for all } v \in C^2(\Omega; \mathbb{R}^p). \tag{A.4}$$

Remark. (A.4) implies that $Bv \in \operatorname{Ker} A$.

Examples of operators A and B.

(α) Let $m = nr$ ($r \geq 1$ an integer) and

$$u = (u_1, \ldots, u_r), \text{ with } u_j : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ 1 \le j \le r. \tag{A.5}$$

Let A be the operator

$$Au = (\operatorname{curl} u_1, \ldots, \operatorname{curl} u_r) \tag{A.6}$$

where, for $v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$,

$$\operatorname{curl} v = \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)_{\substack{1 \le i \le n \\ j > i}}$$

$$= \left(\left(\frac{\partial v_1}{\partial x_j} - \frac{\partial v_j}{\partial x_1} \right)_{j \ge 2} ; \left(\frac{\partial v_2}{\partial x_j} - \frac{\partial v_j}{\partial x_2} \right)_{j \ge 3} ; \ldots ; \frac{\partial v_{n-1}}{\partial x_n} - \frac{\partial v_n}{\partial x_{n-1}} \right). \tag{A.7}$$

Therefore for A as in (A.6) we have $q = n(n-1)r/_2$. We may then choose $p = r$ and B as follows

$$Bw = \nabla w = (\operatorname{grad} w_1, \ldots, \operatorname{grad} w_r) \tag{A.8}$$

where $w_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $1 \le j \le r$. Then A and B satisfy (A.2) - (A.4).

(β) Let m = n and

$$u(x_1,\dots,x_n) = (u_1,\dots,u_n),\ u_j : \mathbb{R}^n \longrightarrow \mathbb{R} \quad 1 \leqslant j \leqslant n, \tag{A.9}$$

with

$$Au = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \dots, \frac{\partial u_n}{\partial x_n}\right) \in \mathbb{R}^n \tag{A.10}$$

then the only B satisfying (A.3) and (A.4) is

$$Bv \equiv 0 \quad \text{for all } v \in C^2(\Omega;\mathbb{R}^p). \tag{A.11}$$

With the help of the notations above, we may introduce the following definition

Definition. A continuous function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is said to be A-B-quasiconvex, where A and B satisfy (A.2) - (A.4), if

$$\int_G f(\mu + B\zeta(x))\, dx \geqslant \int_G f(\mu)\, dx \tag{A.12}$$

for every $\mu \in \mathbb{R}^m$, for every bounded domain $G \subset \mathbb{R}^n$ and for every $\zeta \in W_o^{1,\infty}(G;\mathbb{R}^p)$.

We then have immediately

Proposition A.1. If f is A-quasiconvex, then f is A-B-quasiconvex.

Proof: Let G be a bounded domain of $\mathbb{R}^n$ and let K be a hypercube of $\mathbb{R}^n$ containing G. Let $\zeta \in W_o^{1,\infty}(G;\mathbb{R}^p)$. Extend ζ from G to the whole of K in the following way

$$\zeta \equiv 0 \quad \text{in } K - G. \tag{A.13}$$

We then deduce that $B\zeta \in L(K)$, i.e.,

$$\begin{cases} B\zeta \in L_m^\infty(K) \\ \int_K B\zeta(x)dx = 0 \\ B\zeta \in \operatorname{Ker} A. \end{cases} \tag{A.14}$$

Using the A-quasiconvexity of f we obtain

$$\int_K f(\mu + B\zeta(x))\,dx \geq \int_K f(\mu)\,dx, \tag{A.15}$$

and therefore

$$\int_G f(\mu + B\zeta(x))\,dx = \int_K f(\mu + B\zeta(x))\,dx - \int_{K-G} f(\mu)\,dx$$
$$\geq \int_G f(\mu)\,dx.$$

□

<u>Remark</u>. In the variational case, i.e., $m = nr$

$$\begin{cases} u = \nabla v = (\text{grad } v_1, \ldots, \text{grad } v_r),\ v_j : \mathbb{R}^n \longrightarrow \mathbb{R} \quad 1 \leq j \leq r \\ Au = \text{rot } \nabla v \equiv 0 \\ Bv = \nabla v, \end{cases}$$

the definition of the A-B-quasiconvexity corresponds exactly to that of Morrey ([Mo1], [Mo2]) given in §5 pp. 39 - 40.

We may now state the main theorem of this Appendix; recall first that

$$(H)\ \begin{cases} u^\nu \xrightarrow{*} u & \text{in } L^\infty_m(\Omega) \\ Au^\nu \xrightarrow{*} Au & \text{in } L^\infty_q(\Omega) \\ f(u^\nu) \xrightarrow{*} \ell & \text{in } L^\infty(\Omega) \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^n$ and $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is continuous

<u>Theorem A.2</u>. 1) <u>Necessity</u> : If, for every sequence $\{u^\nu\}$ satisfying (H), $\ell \geq f(u)$, then f is A-B-quasiconvex.

2) <u>Sufficiency</u> : If $\{u^\nu\}$, u satisfy Hypothesis (H) and if furthermore either

(α) f is A-quasiconvex and u^ν and u are such that

$$(H_o)\ u^\nu - u \in \text{Ker } A;$$

or

(β) f is A-B-quasiconvex and satisfies

$$|f(u) - f(v)| \leq a(1 + |u|^{\beta-1} + |v|^{\beta-1})\,|u-v|, \tag{A.16}$$

$a > 0$, $\beta \geq 1$, $u, v \in \mathbb{R}^m$; A and B satisfy (A.2) - (A.4) and

$$(H_{AB}) \begin{cases} \text{For every } u^\nu \overset{*}{\rightharpoonup} 0,\ Au^\nu \overset{*}{\rightharpoonup} 0, \text{ there exist } v^\nu \in W_o^{1,\beta}(\Omega;\mathbb{R}^p) \\ \text{and } w^\nu \in L_m^\beta(\Omega) \text{ such that} \\ u^\nu = Bv^\nu + w^\nu \\ v^\nu \rightharpoonup 0 \text{ in } W^{1,\beta}(\Omega;\mathbb{R}^p) \\ w^\nu \longrightarrow 0 \text{ in } L_m^\beta(\Omega); \end{cases}$$

then $\ell \geq f(u)$.

Proof: 1) Necessity : This is just Theorem 2.1 and Proposition A.1 above.

2) Sufficiency : The first part (α) is only Theorem 2.3. Part (β), once Hypothesis (H_{AB}) assumed, follows exactly the pattern of the proofs of Theorem 2.3 and Theorem 5.1 ((H_{AB}) replacing Step 2 of Theorem 5.1; for more details see [Da4]). □

Before proceeding further we need to make some remarks on Theorem A.2 Part (β)

Remarks: (i) Observe first that (A.16) is purely technical and comes from the fact that in (H_{AB}) we did not assume $\beta = \infty$. The important condition in the above theorem is obviously (H_{AB}). We will see below that the following operators satisfy (H_{AB})

1) A = curl, B = grad

2) A = div, B = curl

3) A = (curl, div), B = (grad, curl);

while those defined in (A.10), (A.11) do not satisfy (H_{AB}).

(ii) The definition of the operator B above and the Hypothesis (H_{AB}) imply that one may decompose u^ν into $Bv^\nu \in \text{Ker } A$ and w^ν, where w^ν is a sum of a boundary term (since v^ν is assumed to be 0 on $\partial\Omega$, while u^ν is not) and of a term in $(\text{Ker } A)^\perp$. For example in Theorem 5.1 (i.e., for the variational case) we have automatically that $u^\nu = Bv^\nu$, but in general $v^\nu \neq 0$ on $\partial\Omega$, by the use of Mac Shane's Lemma one is able to correct that and therefore to get (H_{AB}).

(iii) It is also interesting to compare the Hypothesis (H_{AB}) with that of constant rank used by Murat in [Mu3] (Murat's result is mentioned in Theorem 4.5 p. 37 of these notes). Using a theorem of Schulenberger and Wilcox ([SW1], [Ka1]) the hypothesis of constant rank of the operator A implies a condition very similar to (H_{AB}) (for more details see Lemma 3.6 in [Mu3]). However the method, we will use in Theorem A.4 below, is somehow different.

We now want to show that operators of the type div or curl (and hence for the variational case) satisfy Hypothesis (H_{AB}). This will result from well known theorems on the existence and regularity of elliptic operators.

We first introduce some notations.

<u>Notations</u>. (i) Let A be the operator defined in (A.2), i.e.,

$$Au = \left(\sum_{j=1}^{m} \sum_{k=1}^{n} a_{ijk} \frac{\partial u_j}{\partial x_k}\right)_{1 \leq i \leq q} .$$

We denote by A^* the operator defined as

$$\int_\Omega < A^* u(x); v(x) > dx = \int_\Omega < u(x); Av(x) > dx$$

for all $u \in C_o^\infty(\Omega;\mathbb{R}^q)$, $v \in C_o^\infty(\Omega;\mathbb{R}^m)$.

(ii) In particular we will denote by curl^* the operator A^* associated to $A = \text{curl}$, i.e.,

$$\text{curl } u = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)_{\substack{1 \leq i \leq n \\ j > i}}$$

$$= \left(\left(\frac{\partial u_1}{\partial x_j} - \frac{\partial u_j}{\partial x_1}\right)_{j \geq 2}; \left(\frac{\partial u_2}{\partial x_j} - \frac{\partial u_j}{\partial x_2}\right)_{j \geq 3}; \ldots; \frac{\partial u_{n-1}}{\partial x_n} - \frac{\partial u_n}{\partial x_{n-1}}\right),$$

therefore here $q = n(n-1)/_2$. If we denote by

$$\gamma(p) = (p-1)\, n - \frac{p(p-1)}{2}, \quad 1 \leq p \leq n$$

and if

$$v(x_1,\dots,x_n) = (v_1,v_2,\dots,v_{n(n-1)/2})$$

we then have

$$\operatorname{curl}^* v = \left(\sum_{\nu=1}^{p-1} \frac{\partial v_{p-\nu+\gamma(p)}}{\partial x_\nu} - \sum_{\nu=p+1}^{n} \frac{\partial v_{\nu-p+\gamma(\nu)}}{\partial x_\nu} \right)_{1\leqslant p\leqslant n} .$$

From the above definitions it is easy to deduce that :

Lemma A.3. (i) If $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then

$$\operatorname{curl}^* \operatorname{curl} u = -\Delta_n u + \operatorname{grad} \operatorname{div} u, \text{ for every } u \in C^2, \tag{A.17}$$

where Δ_n is defined as

$$\Delta_n u = (\Delta u_1,\dots,\Delta u_n) = \left(\sum_{j=1}^{n} \frac{\partial^2 u_1}{\partial x_j^2},\dots, \sum_{j=1}^{n} \frac{\partial^2 u_n}{\partial x_j^2} \right). \tag{A.18}$$

(ii) If $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then

$$\operatorname{curl} \Delta_n u = \Delta_{n(n-1)/2} \operatorname{curl} u, \text{ for every } u \in C^3. \tag{A.19}$$

(iii) If $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^{n(n-1)/2}$, then

$$\operatorname{div} \operatorname{curl}^* u \equiv 0, \text{ for every } u \in C^2 \tag{A.20}$$

$$\operatorname{curl}^* \Delta_{n(n-1)/2} u = \Delta_n \operatorname{curl}^* u, \text{ for every } u \in C^3. \tag{A.21}$$

We may now prove the following theorem

Theorem A.4. (i) If $m = n$, $A = \operatorname{curl}$, $B = \operatorname{grad}$, then A and B satisfy Hypothesis (H_{AB}) of Theorem A.2.

(ii) If $m = n$, $A = \operatorname{div}$, $B = \operatorname{curl}^*$, then A and B satisfy (H_{AB}).

Remark. If $m = nr$, $s \leqslant r$ and

$$u(x_1,\dots,x_n) = (u_1,\dots,u_s,u_{s+1},\dots,u_r)$$

$$Au = (\operatorname{curl} u_1,\dots,\operatorname{curl} u_s,\operatorname{div} u_{s+1},\dots,\operatorname{div} u_r)$$

$$Bu = (\operatorname{grad} u_1,\dots,\operatorname{grad} u_s,\operatorname{curl}^* u_{s+1},\dots,\operatorname{curl}^* u_r)$$

then the above theorem implies that A and B satisfy (H_{AB}).

Proof: (i) Let Ω be a bounded domain of $\mathbb{R}^n$, with a sufficiently regular boundary and let

$$\begin{cases} u^\nu \overset{*}{\longrightarrow} 0 \text{ in } L^\infty_n(\Omega) & \text{(A.22)} \\ Au^\nu = \operatorname{curl} u^\nu \overset{*}{\longrightarrow} 0 \text{ in } L^\infty_{n(n-1)/_2}(\Omega). & \text{(A.23)} \end{cases}$$

We want to show that, given $\beta \geq 1$, one may find $v^\nu \in W^{1,\beta}_o(\Omega)$ and $w^\nu \in L^\beta_n(\Omega)$ such that

$$\begin{cases} u^\nu = \operatorname{grad} v^\nu + w^\nu & \text{(A.24)} \\ v^\nu \rightharpoonup 0 \text{ in } W^{1,\beta}(\Omega) & \text{(A.25)} \\ w^\nu \longrightarrow 0 \text{ in } L^\beta_n(\Omega). & \text{(A.26)} \end{cases}$$

For this, let us consider the weak form of Laplace's equation

$$\int_\Omega < \operatorname{grad} v^\nu(x); \operatorname{grad} \varphi(x) > dx = \int_\Omega <u^\nu(x); \operatorname{grad} \varphi(x) > dx, \text{ for all } \varphi \in W^{1,\alpha'}_o(\Omega), \qquad \text{(A.27)}$$

where $\alpha' \geq 1$ is given.

By the classical results on uniformly elliptic equations (c.f., for example Theorem 7.2 in [Sr1]) we deduce the existence of a solution $v^\nu \in W^{1,\alpha}_o(\Omega)$ of (A.27), with α given by $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, such that

$$\| v^\nu\|_{W^{1,\alpha}} \leq K\| \operatorname{div} u^\nu\|_{W^{-1,\alpha}} \qquad \text{(A.28)}$$

where K is a constant independent of ν and $W^{-1,\alpha}$ denotes the dual of $W^{1,\alpha'}_o$.

From (A.22) (i.e., $\operatorname{div} u^\nu \in W^{-1,\infty}$), from (A.27) and (A.28) we deduce immediately that

$$v^\nu \rightharpoonup 0 \text{ in } W^{1,\alpha};$$

α being arbitrary we have indeed obtained (A.25).

We then define $w^\nu \in L^\alpha_m(\Omega)$ by

$$w^\nu = u^\nu - \operatorname{grad} v^\nu. \qquad \text{(A.29)}$$

Combining (A.22) and (A.25) we obtain

$$w^{\nu} \rightharpoonup 0 \text{ in } L^{\alpha}_{m}(\Omega), \text{ for all } \alpha \geq 1. \tag{A.30}$$

In order to conclude the proof of Part (i) of the theorem, we only need to show that in (A.30) the convergence is strong. Therefore let $\varphi \in C^{\infty}_{o}(\Omega;\mathbb{R}^{n})$ and observe that by Lemma A.3 one has

$$\int_{\Omega} < w^{\nu}; \Delta_{n} \varphi > dx = - \int_{\Omega} < w^{\nu}; \text{curl}^{*} \text{ curl } \varphi > dx + \int_{\Omega} < w^{\nu}; \text{grad div } \varphi > dx. \tag{A.31}$$

We now use (A.29) in (A.31) and integrate by parts to get

$$\begin{aligned}\int_{\Omega} < w^{\nu}; \Delta_{n} \varphi > dx = & \int_{\Omega} < \text{curl } u^{\nu}; \text{curl } \varphi > dx \\ & + \int_{\Omega} < \text{grad } v^{\nu}; \text{curl}^{*} \text{ curl } \varphi > dx \\ & + \int_{\Omega} < u^{\nu}; \text{grad div } \varphi > dx \\ & - \int_{\Omega} < \text{grad } v^{\nu}; \text{grad div } \varphi > dx.\end{aligned} \tag{A.32}$$

From (A.27) we immediately deduce that

$$\int_{\Omega} < w^{\nu}; \Delta_{n} \varphi > dx = \int_{\Omega} < \text{curl } u^{\nu}; \text{curl } \varphi > dx; \tag{A.33}$$

and therefore, for every $\varphi \in C^{\infty}_{o}(\Omega;\mathbb{R}^{n})$,

$$\begin{aligned}\left| \int_{\Omega} < w^{\nu}; \Delta_{n} \varphi > dx \right| & \leq \| \text{curl } u^{\nu} \|_{L^{\alpha}} \| \text{curl } \varphi \|_{L^{\alpha'}} \\ & \leq K \| \text{curl } u^{\nu} \|_{L^{\alpha}} \| \varphi \|_{W^{1,\alpha'}},\end{aligned} \tag{A.34}$$

with $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Recall also that by (A.23) curl $u^{\nu} \in L^{\infty}$.

Using again the local regularity of elliptic operators (c.f. for example, [Ag1] Theorem 6.2 if $\alpha = 2$ and [Sr1] Theorem 9.5 if $\alpha \neq 2$), we obtain, using (A.30), that

$$\| w^\nu \|_{W^{1,\alpha}(\Omega')} \leqslant K(\| \text{curl } u^\nu \|_{L^\alpha(\Omega)} + \| w^\nu \|_{L^\alpha(\Omega)}), \tag{A.35}$$

where Ω' is such that $\Omega' \subset\subset \Omega$ and K is a constant. Using Rellich's Theorem (see [Ad1]) we deduce that

$$w^\nu \longrightarrow 0 \text{ in } L^\alpha_m(\Omega'), \text{ for every } \Omega' \subset\subset \Omega. \tag{A.36}$$

We finally want to show that (A.30) and (A.36) imply (A.26), thus establishing Part (i) of the theorem.

For this, let α be large enough so that $1 \leqslant \beta < \alpha$, we therefore want to show

$$w^\nu \longrightarrow 0 \text{ in } L^\beta_m(\Omega). \tag{A.26}$$

Consider

$$\int_\Omega |w^\nu(x)|^\beta \, dx = \int_{\Omega-\Omega_\varepsilon} |w^\nu(x)|^\beta \, dx + \int_{\Omega_\varepsilon} |w^\nu(x)|^\beta \, dx, \tag{A.37}$$

where $\varepsilon > 0$ is arbitrary and $\Omega_\varepsilon \subset\subset \Omega$ is such that

$$\| w^\nu \|^\beta_{L^\alpha(\Omega)} \, (\text{meas } (\Omega-\Omega_\varepsilon))^{(\alpha-\beta)/\alpha} \leqslant \frac{\varepsilon}{2} . \tag{A.38}$$

Using (A.36) we deduce that for ν sufficiently large

$$\int_{\Omega_\varepsilon} |w^\nu(x)|^\beta \, dx \leqslant \frac{\varepsilon}{2} , \tag{A.39}$$

and by Hölder's inequality we have

$$\int_{\Omega-\Omega_\varepsilon} |w^\nu(x)|^\beta \, dx \leqslant \left(\int_{\Omega-\Omega_\varepsilon} dx \right)^{(\alpha-\beta)/\alpha} \left(\int_{\Omega-\Omega_\varepsilon} |w^\nu(x)|^\alpha \right)^{\beta/\alpha}$$

$$\leqslant \frac{\varepsilon}{2} . \tag{A.40}$$

Combining (A.39) and (A.40) we have indeed obtained (A.26) and this achieves the proof of Part (i) of the theorem.

(ii) This part is very similar to Part (i) and therefore we will leave out the details.

Let

$$\begin{cases} u^\nu \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty_m(\Omega) & \text{(A.41)} \\ \operatorname{div} u^\nu \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(\Omega), & \text{(A.42)} \end{cases}$$

we want to show that, given $\beta \geq 1$, we may find $v^\nu \in W^{1,\beta}_o(\Omega;\mathbb{R}^{n(n-1)/2})$ and $w^\nu \in L^\beta_n(\Omega)$ so that

$$\begin{cases} u^\nu = \operatorname{curl}^* v^\nu + w^\nu & \text{(A.43)} \\ v^\nu \rightharpoonup 0 \text{ in } W^{1,\beta}(\Omega;\mathbb{R}^{n(n-1)/2}) & \text{(A.44)} \\ w^\nu \longrightarrow 0 \text{ in } L^\beta_n(\Omega). & \text{(A.45)} \end{cases}$$

So let $v^\nu \in W^{1,\alpha}_o(\Omega;\mathbb{R}^{n(n-1)/2})$, $\alpha \geq 1$ given, be the weak solution of

$$\begin{cases} -\Delta_{n(n-1)/2}\, v^\nu = \operatorname{curl} u^\nu & \text{in } \Omega \\ v^\nu = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{(A.46)}$$

which satisfies

$$\|v^\nu\|_{W^{1,\alpha}} \leq K\|\operatorname{curl} u^\nu\|_{W^{-1,\alpha}} \qquad \text{(A.47)}$$

where K is a constant independent of ν. We then deduce from (A.41), (A.46) and (A.47) that

$$v^\nu \rightharpoonup 0 \text{ in } W^{1,\alpha}(\Omega;\mathbb{R}^{n(n-1)/2}); \qquad \text{(A.48)}$$

α being arbitrary we obtain (A.44).

Now let

$$w^\nu = u^\nu - \operatorname{curl}^* v^\nu. \qquad \text{(A.49)}$$

From (A.41) and (A.48) we deduce that

$$w^\nu \rightharpoonup 0 \text{ in } L^\alpha_n(\Omega). \qquad \text{(A.50)}$$

In order to conclude the proof, it only remains to show that in (A.50) the convergence is strong. From (A.49) we have that

$$\Delta_n w^\nu = \Delta_n u^\nu - \Delta_n \operatorname{curl}^* v^\nu, \text{ in the sense of distributions.} \qquad \text{(A.51)}$$

Using Lemma A.3 we deduce

$$\begin{aligned}\Delta_n w^\nu &= \Delta_n u^\nu - \operatorname{curl}^* \Delta_{n(n-1)/2} v^\nu \\ &= \Delta_n u^\nu + \operatorname{curl}^* \operatorname{curl} u^\nu \\ &= \operatorname{grad} \operatorname{div} u^\nu, \text{ in the sense of distributions,}\end{aligned} \qquad (A.52)$$

where we have used (A.46) in the second equality of (A.52). As in Part (i) from the regularity of elliptic operators, from (A.42) and (A.50) we obtain that

$$w^\nu \longrightarrow 0 \text{ in } W^{1,\alpha}_{loc}(\Omega;\mathbb{R}^n). \qquad (A.53)$$

Since in (A.52) α is arbitrary, by choosing α so that $1 \leqslant \beta < \alpha$, we deduce from Rellich's Theorem, from (A.50) and (A.53) that

$$w^\nu \longrightarrow 0 \text{ in } L^\beta_m(\Omega).$$ □

As in Chapter I §2 from the results on weak lower semicontinuity (i.e., $\ell \geqslant f(u)$) we deduce easily some results on weak continuity.

Definition. A continuous function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is said to be A-B-quasi-affine (resp. A-quasiaffine) if f and $-f$ are A-B-quasiconvex (resp. A-quasi-convex).

We then get immediately as a consequence of Theorem A.2 that if

$$(H) \begin{cases} u^\nu \overset{*}{\longrightarrow} u & \text{in } L^\infty_m(\Omega) \\ Au^\nu \overset{*}{\longrightarrow} Au & \text{in } L^\infty_q(\Omega) \\ f(u^\nu) \overset{*}{\longrightarrow} \ell & \text{in } L^\infty(\Omega) \end{cases}$$

Theorem A.5. Under the hypotheses of Theorem A.2

1) Necessity. If for every sequence $\{u^\nu\}$ satisfying (H), $\ell = f(u)$ then f is A-quasiaffine and thus f is A-B-quasiaffine.

2) Sufficiency. If $\{u^\nu\}$ and u satisfy (H) and either

(α) f is A-quasiaffine and u^ν and u are such that

$$(H_o)\ Au^{\nu} - Au \equiv 0;$$

or

(β) f is A-B-quasiaffine, A and B satisfy (H_{AB}) of Theorem A.2;

then $\ell = f(u)$.

Proof: The proof is a direct consequence of Theorem A.2 applied to f and - f. □

Corollary A.6: Let $g : \mathbb{R}^s \longrightarrow \mathbb{R}$ be convex and let

$$f(u) = g(\Phi_1(u),\ldots,\Phi_s(u))$$

where $\Phi_1,\ldots,\Phi_s$ are A-B-quasiaffine, then f is A-B-quasiconvex.

Proof: The proof is identical to that of Corollary 2.5 of Chapter I. □

Remark. As seen in Chapter I §5, if A = curl and B = grad, then the A-B-quasiaffine functions are just the subdeterminants of the matrix ∇u.

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