APPENDIX (An Abstract of a Result at Oberwalfach 1979)

This appendix contains a copy of the last page of the collection of abstracts presented at the international complexity conference in Oberwolfach (West Germany) in October 1979. In spite of his signature under the abstract, S. Winograd insisted (in writing to the present author) that he should not be credited for the exponent 2.522 because his participation was not original. Indeed, it was limited only to reconstruction of a theorem (see Theorem 7.2 in Section 7) after its sketchy presentation by A. Schonhage at that conference. However, the reconstruction did require substantial work since Schonhage's presentation of that theorem was indeed quite sketchy.

Complexity of Systems of
Algebraic Equations
Daniel Lazard

Let $f_1, \ldots, f_r$ be r polynomials in n indeterminates which have a finite number of common zeros in the algebraic closure of the ground field, counting the zeros at infinity.

An algorithm is given which computes all these zeros. If d is the ~~degree~~ highest degree of the polynomials, the computations needed by this algorithm consist in the resolution of one univariate polynomial whose degree is the number of solutions and a number of operations of the ground field which is polynomial in $(nd)^n$.

As of 21.24 hr. of October 26, 1979
the best known ~~example~~ exponent for
matrix multiplication is 2.521812716

VICTOR PAN & SHMUEL WINOGRAD

Index of Some Concepts

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