APPENDIX: PROGRESS REPORT ON THE VICTORIA DELFINO PROBLEMS

Since the Delfino problems were proposed in [1] two and a half years ago, one has been solved and there have been some interesting results related to three of the remaining four problems.

In connection with the first problem, Martin has established the conjectured lower bound for $\underset{\sim}{\delta}^1_5$ by proving (from AD + DC) that

$$\underset{\sim}{\delta}^1_5 \geq \aleph_{\omega^3+1} \; ;$$

moreover Martin showed (from AD) that the ultrapowers of $\underset{\sim}{\delta}^1_3 = \aleph_{\omega+1}$ under the three normal measures on $\underset{\sim}{\delta}^1_3$ are exactly $\underset{\sim}{\delta}^1_4 = \aleph_{\omega+2}$ (this was known to Kunen), $\aleph_{\omega\cdot 2+1}$ and $\aleph_{\omega^2+1}$ and that these three cardinals are measurable (and hence regular), so that (in particular), $\underset{\sim}{\delta}^1_5$ is not the first regular cardinal after $\underset{\sim}{\delta}^1_4$. We still have no upper bounds for $\underset{\sim}{\delta}^1_5$ from AD.

The second problem was solved by Moschovakis who showed (from AD + DC) that every coinductive pointset admits a scale. If we put

$$\Sigma^*_0 = \text{all Boolean combinations of inductive and coinductive sets}$$

and then define Σ^*_n by counting quantifiers over $\mathcal{R} = {}^\omega\omega$ in front of a Σ^*_0 matrix in the usual way, then the proof shows that every coinductive set admits a scale $\{\varphi_n\}_{n\in\omega}$, where each φ_n is a Σ^*_{n+1}-norm, uniformly in n.

Martin and Steel extended the method used by Moschovakis in this proof and showed that

$$ZF + DC + AD + V = L(R) \Rightarrow \text{Every } \Sigma^2_1 \text{ set admits a } \Sigma^2_1\text{-scale} \; ;$$

this combines with an earlier result of Kechris and Solovay to show that

$$ZF + DC + AD + V = L(R)$$
$$\Rightarrow \text{A pointset admits a scale if and only if it is } \underset{\sim}{\Sigma}^2_1 \, .$$

Martin then combined these ideas with the technique of the Third Periodicity Theorem ([2], 6E.1) and showed that under reasonable hypotheses of determinacy for games on $\mathcal{R}$, (AD_R), the scale property is preserved by the game quantifier $\Game^2$ on $\mathcal{R}$, where

$$(\Game^2\alpha)P(x,\alpha) \Leftrightarrow (\exists\alpha_0)(\forall\alpha_1)(\exists\alpha_2)(\forall\alpha_3) \ldots P(x,\langle\alpha_0,\alpha_1,\ldots\rangle) \, .$$

This result produces scales for sets that are not $\underset{\sim}{\Sigma}_1^2$ in $L(R)$ and leaves open the general question of the extent of scales in the presence of axioms stronger than AD.

In connection with the third problem, Kechris showed in [3] that if $T^3 = T^3(\overline{\varphi})$ is the tree associated with some Π_3^1-scale $\overline{\varphi}$ on a Π_3^1-complete set P and if

$$\tilde{L}[T^3] = \bigcup_{\alpha \in h} L[T^3, \alpha] \ ,$$

then

$$ZF + AD + DC + \underset{\sim}{\delta}_3^1 \to (\underset{\sim}{\delta}_3^1)^{\underset{\sim}{\delta}_3^1}$$
$$\Rightarrow \tilde{L}[T^3] \text{ is independent of the choice of } P, \overline{\varphi} .$$

This partial result emphasizes the importance of the question of <u>the strong partition property</u> for $\underset{\sim}{\delta}_3^1$ which is still open.

Finally, in connection with the fifth problem, it follows from unpublished results of Kechris and Solovay that

$$ZF + AD + DC + V = L(R)$$
$$\Rightarrow \text{Every function } f : D \to D \text{ on the degrees is representable} .$$

Although this has no direct bearing on a possible solution of the fifth problem, it underscores the generality of the question.

References

[1] APPENDIX: The Victoria Delfino problems, in Cabal Seminar 76-77, Lecture Notes in Mathematics #689, Springer 1978.

[2] Y. N. Moschovakis, Descriptive Set Theory, Studies in Logic, North Holland 1980.

[3] A. S. Kechris, Homogeneous trees and projective scales, this volume.