APPENDIX. SOME FUNCTIONAL ANALYTIC ASPECTS OF $C_c(X)$

We will be concerned with two "dualities" of $C_c(X)$, the linear duality and the Pontryagin's duality. Both indicate the very special character of the continuous convergence structure Λ_c in this part of functional analysis.

1. The c-Reflexivity of $C_c(X)$

For any convergence space X and any subset S let us denote by VS the linear space spanned by $i_X(S) \subset \mathcal{L}_c C_c(X)$. The man theorem, proved in [Bu 2], via an integral representation of positive linear functionals, is the following:

Theorem 88

For any convergence space X the linear space VX is dense in $\mathcal{L}_c C_c(X)$.

Proof:

First let us verify the assertion in case X is a compact topological space. The space $\mathcal{L}_c C_c(X)$ is locally compact, by the assumption just made. In fact the unit-ball U of $\mathcal{L} C_c(X)$ is compact and carries the topology of pointwise convergence (proposition 28). Hence U is the closed convex hull of $i_X(X) \cup (-i_X(X))$ (see [Du,Sch], V 8.6). Since U is absorbant, $VX \subset \mathcal{L}_c C_c(X)$ has to be dense.

Next let X be c-embedded. Choose $u \in \mathcal{L}_c C_c(X)$ By corollary 38 the seminorm $|u|$ can be majorized by a real multiple of a sup-seminorm taken over some compact set $K \subset X$. Since the restriction map

$r : C_c(X) \longrightarrow C_c(K)$ induces a continuous injection $r^* : \mathcal{L}_c C_c(K) \longrightarrow \mathcal{L}_c C_c(X)$ and since u factors over r, it is in the closure of VK. Thus $VX \subset \mathcal{L}_c C_c(X)$ is dense.

Finally let X be an arbitrary convergence space. The surjective map $i_X : X \longrightarrow Hom_c C_c(X)$ induces a bicontinuous isomorphism between $C_c(Hom_c C_c(X))$ and $C_c(X)$ and hence between $\mathcal{L}_c C_c(X)$ and $\mathcal{L}_c C_c(Hom_c C_c(X))$. Therefore $VX \subset \mathcal{L}_c C_c(X)$ is dense.

A convergence $\mathbb{R}$-vector space E is called <u>c-reflexive</u> if the canonical mapping

$$j_E : E \longrightarrow \mathcal{L}_c \mathcal{L}_c E,$$

defined by $j_E(t)(u) = u(t)$ for all $t \in E$ and for all $u \in \mathcal{L}_c E$, is a bicontinuous isomorphism.

Let us consider $j_{C_c}(X)$ for an arbitrary convergence space X. The map $i_X : X \longrightarrow \mathcal{L}_c C_c(X)$ induces a map

$$\tilde{i}_X : \mathcal{L}_c \mathcal{L}_c C_c(X) \longrightarrow C_c(X),$$

defined by $\tilde{i}_X(k) = k \circ i_X$ for each $k \in \mathcal{L}_c \mathcal{L}_c C_c(X)$, which obviously is continuous. Moreover

$$\tilde{i}_X \circ j_{C_c(X)} = id_{C_c(X)} .$$

Thus by theorem 88 we obtain the c-reflexivity of $C_c(X)$ (see [Bu 2]):

<u>Theorem 89</u>

For any convergence space X the convergence $\mathbb{R}$-algebra $C_c(X)$ is c-reflexive.

This result allows us to treat questions on the c-reflexivity of topological $\mathbb{R}$-vector spaces.

Let E be a topological $\mathbb{R}$-vector space. Since $\mathcal{L}_c E$ is locally compact, $\mathcal{L}_c \mathcal{L}_c E$ is a locally convex topological $\mathbb{R}$-vector space. Thus E can be c-reflexive only if E is locally convex. Let us assume that this is satisfied.

Any locally convex topological $\mathbb{R}$-vector space can be embedded in a $C_c(X)$ for some locally compact space X, (see [Kö], § 20, p.107). Call the embedding e.

Use the universal property of the continuous convergence and the Hahn-Banach theorem to verify that

$$e^*: \ \mathcal{L}_c C_c(X) \longrightarrow \mathcal{L}_c E$$

(sending each u into $u \circ e$) is a surjection, and that

$$e^{**}: \ \mathcal{L}_c \mathcal{L}_c E \longrightarrow \mathcal{L}_c \mathcal{L}_c C_c(X)$$

is a bicontinuous isomorphism onto a subspace.

The commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\ e\ } & C_c(X) \\ \Big\downarrow{\scriptstyle j_E} & & \Big\downarrow{\scriptstyle j_{C_c(X)}} \\ \mathcal{L}_c \mathcal{L}_c E & \xrightarrow{\ e^{**}\ } & \mathcal{L}_c \mathcal{L}_c C_c(X) \end{array}$$

yields, together with theorem 89, that j_E is a homeomorphism onto a subspace.

Let us show that j_E maps onto a dense subspace. To this end consider $\mathcal{L}_c E$. Clearly $\mathcal{L}_c \mathcal{L}_c E$ is a subspace of the locally convex topological space $C_c(\mathcal{L}_c E)$ and the restriction map

$$r : \mathcal{L}_c C_c(\mathcal{L}_c E) \longrightarrow \mathcal{L}_c \mathcal{L}_c \mathcal{L}_c E$$

is surjective (Hahn-Banach). Now $r \circ i_{\mathcal{L}_c E} = j_{\mathcal{L}_c E}$. Thus $r(V(\mathcal{L}_c E))$ is mapped onto a dense subspace of $\mathcal{L}_c \mathcal{L}_c \mathcal{L}_c E$ (theorem 88), which moreover has to be $j_{\mathcal{L}_c E}(\mathcal{L}_c E)$.

Since $j_E^* \circ r \circ j_{\mathcal{L}_c E} = id_{\mathcal{L}_c E}$, where j_E^* maps each $u \in \mathcal{L}_c \mathcal{L}_c \mathcal{L}_c E$ into $u \circ j_E$, we deduce that $j_{\mathcal{L}_c E}(\mathcal{L}_c E)$ is closed. Hence it is the whole space $\mathcal{L}_c \mathcal{L}_c \mathcal{L}_c E$.

This demonstrates that $\mathcal{L}_c E$ is c-reflexive. Thus E and $\mathcal{L}_c \mathcal{L}_c E$ have bicontinuously isomorphic dual spaces, which is only possible if $j_E(E) \subset \mathcal{L}_c \mathcal{L}_c E$ is dense. In conclusion let us state:

<u>Theorem 90</u>

For any locally convex topological $\mathbb{R}$-vector space E the map

$$j_E : E \longrightarrow \mathcal{L}_c \mathcal{L}_c E$$

is a homeomorphism onto a dense image. Hence a topological $\mathbb{R}$-vector space E is c-reflexive iff E is locally convex and complete.

The c-duals of topological $\mathbb{R}$-vector spaces are characterized as those locally compact convergence $\mathbb{R}$-vector spaces in which any compact subspace is topological and which possess point-separating continuous functionals.

For papers related to the c-duality we refer to [Bu 2] and [Bi,Bu,Ku]. The relation of theorem 90 to the classical results concerning the completion of locally convex $\mathbb{R}$-vector spaces can be made via an Ascoli-Arzelà theorem in [C,F].

2. On the Pontryagin reflexivity of certain convergence $\mathbb{R}$-vector spaces

Let E be a Hausdorff $\mathbb{R}$-vector space and T be the group (with the ususal topology) of all complex numbers of modulus one.

The collection of all group homomorphisms of E into T forms a group GE and if endowed with the continuous convergence structure a convergence group G_cE, then it is called the character group of E. Introducing the character group G_cG_cE of G_cE, we see that the canonical map

$$\hat{j}_E : E \longrightarrow G_cG_cE,$$

defined by $\hat{j}_E(e)(h) = h(e)$ for all $e \in E$ and all $h \in G_cE$, is continuous.

We now proceed to determine for which type of convergence $\mathbb{R}$-vector space E the map $\hat{j}_E$ is a bicontinuous isomorphism; that is to say which space E is <u>Pontryagin reflexive</u>. The methods we use here are based on the fact that $\mathbb{R}$ is a (the universal) covering of T. The covering projection $\kappa : \mathbb{R} \longrightarrow T$ sends each $r \in \mathbb{R}$ into $e^{2\pi i r}$.

For every Hausdorff convergence $\mathbb{R}$-vector space E, the convergence structure induces on any finite dimensional subspace the natural topology [Ku 3]. Now let $h \in G_cE$. The restriction of h to any finite dimensional subspace H lifts to a unique, continuous linear functional u_H of H. Thus h lifts uniquely to a linear functional u of E. To prove the continuity of u let us suppose that E is <u>balanced</u>, i.e. that with any filter Φ convergent to $o \in E$, the filter $[-1,1]\cdot\Phi$ converges too. Since $[-1,1]\cdot\Phi$ has a basis of path connected sets (defined in the obvious way), the convergence of $u([-1,1]\cdot\Phi)$ to zero is obvious. Hence u is continuous. For any $u \in \mathcal{L}E$ the map $\kappa \circ u$ is a character of E.

Hence

$$\kappa_E : \mathcal{L}E \longrightarrow GE,$$

sending each u into $\kappa \circ u$, is a group isomorphism. Since κ is a local homeomorphism and $\mathcal{L}_c E$ is path connected, κ_E is bicontinuous if both spaces are equipped with the continuous convergence structure.

By making explicit use of the theory of coverings, we have just obtained a short proof (and seen a variety of possible generalizations) of the following theorem (see [Kö] p.313 and [Bu 1]).

<u>Theorem 91</u>

For every balanced Hausdorff convergence $\mathbb{R}$-vector space E the covering map $\kappa : \mathbb{R} \longrightarrow T$ induces a bicontinuous group isomorphism

$$\kappa_E : \mathcal{L}_c E \longrightarrow G_c E.$$

Considering the following commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\hat{j}_E} & G_c G_c E \\ \Big\downarrow{\scriptstyle j_E} & & \Big\downarrow{\scriptstyle \kappa_E^*} \\ \mathcal{L}_c \mathcal{L}_c E & \xrightarrow{\kappa_{\mathcal{L}_c E}} & G_c \mathcal{L}_c E \end{array}$$

where $\kappa_E^*(k) = k \circ \kappa_E$ for each $k \in G_c G_c E$, we immediately deduce [Bu 1]:

Corollary 92

For every balanced Hausdorff convergence $\mathbb{R}$-vector space E the map

$$b : \mathcal{L}_c\mathcal{L}_c E \longrightarrow G_c G_c E,$$

where $b=\kappa_E^{*-1} \circ \kappa_{\mathcal{L}_E}$, is a bicontinuous isomorphism. Thus E is Pontryagin reflexive iff it is c-reflexive.

With the results of the last section applied to $C_c(X)$, the following corollary [Bu 1] is immediate:

Corollary 93

For any convergence space X the convergence $\mathbb{R}$-algebra $C_c(X)$ is Pontryagin reflexive. The character group $G_c C_c(X)$ is the closure of $\kappa_{C_c(X)}(VX)$.

Let us add to these results the general description of those topological $\mathbb{R}$-vector spaces which are Pontryagin reflexive (see [Bi 6]).

Theorem 94

A topological $\mathbb{R}$-vector space E is Pontryagin reflexive iff E is a complete locally convex space.

The proof is easily made by combining theorem 90 and corollary 92.

The general correspondence between complete subspaces of complete locally convex spaces and the whole character groups modulo annihilators, characteristic for Pontryagin's duality theory, are valid and are easy to verify.

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I N D E X

LIST OF SOME SYMBOLS

Sets with the indices c, co and s carry the continuous convergence structure, the topology of compact convergence and the topology of pointwise convergence, respectively.