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Index

An inequality for a class of harmonic functions in n-space

by

Christer Borell

1. Introduction

Let D be a bounded region in $\mathbb{R}^n$ for which the Dirichlet problem is solvable and let u be a solution of this problem with boundary data f. In this paper we shall give an upper estimate of u. More explicitely we perform symmetrizations of D and f and solve the corresponding Dirichlet problem. Let us call the solution u^o. We then try to find an upper estimate of u in terms of u^o. It shall be said at once that we make some restrictions on f.

We start with some definitions. Let L be an (n-k)-dimensional linear subspace of $\mathbb{R}^n$, $1 \leq k \leq n$. The k-dimensional Steiner symmetrization with respect to L then maps a compact subset M of $\mathbb{R}^n$ into a compact subset M^o of $\mathbb{R}^n$ characterized by the following: If $a \in L$ and $(a + L^{\perp}) \cap M$ has positive k-dimensional Lebesgue measure, then $(a + L^{\perp}) \cap M$ and $(a + L^{\perp}) \cap M^o$ have the same k-dimensional Lebesgue measure and $(a + L^{\perp}) \cap M^o$ is a closed k-dimensional ball of centre a. However, if $(a + L^{\perp}) \cap M$ is a set of k-dimensional Lebesgue measure zero, then $(a + L^{\perp}) \cap M^o$ is empty or the singleton set $\{a\}$ according as $(a + L^{\perp}) \cap M$ is empty or non-empty.

Let $e_1,\ldots,e_n$ be the standard basis in $\mathbb{R}^n$ and let the vector $x \in \mathbb{R}^n$ have the coordinates $x_1,\ldots,x_n$ with respect to this basis.

In the following we assume that $L = [e_1,\ldots,e_{n-k}]$, the linear subspace of $\mathbb{R}^n$ spanned by the first n - k basis vectors, and $1 \leq k \leq n-1$. Further, let $D \subseteq \mathbb{R}^n$ be a finite union of non-degenerated n-dimensional compact intervals, with sides parallel to the coordinate axes. We define $a_1(D),\ldots,\gamma(D)$, by

$$a_i = \min\{x_i | x \in D\} \quad , \quad b_i = \max\{x_i | x \in D\}$$
$$, \quad i = 1,\dots,n-k$$
$$\alpha_i = D \cap \{x_i = a_i\} \quad , \quad \beta_i = D \cap \{x_i = b_i\}$$

$$\alpha = \bigcup_1^{n-k} \alpha_i \quad , \quad \beta = \bigcup_1^{n-k} \beta_i \quad , \quad \gamma = \alpha \cup \beta$$

Let D^o be the k-dimensional Steiner symmetrization of D with respect to L. For short, let us write $a_1(D^o) = a_1^o,\dots,\gamma(D^o) = \gamma^o$. Then, clearly, $a_i^o = a_i$ and $b_i^o = b_i$.

Let $f : \partial D \to [0, +\infty[$ be a function such that $f_{|\partial D \setminus \gamma} = 0$ and set for fixed i, $1 \leq i \leq n-k$,

$$M_{\alpha_i}(f) = \{x \in \mathbb{R}^n | a_i - f(x_1,\dots,x_{i-1},a_i,x_{i+1},\dots,x_n) \leq x_i \leq a_i ,$$
$$(x_1,\dots,x_{i-1},a_i,x_{i+1},\dots,x_n) \in \alpha_i\}$$
$$M_{\beta_i}(f) = \{x \in \mathbb{R}^n | b_i \leq x_i \leq b_i + f(x_1,\dots,x_{i-1},b_i,x_{i+1},\dots,x_n) ,$$
$$(x_1,\dots,x_{i-1},b_i,x_{i+1},\dots,x_n) \in \beta_i\}$$

In the following we shall assume that the sets $M_{\alpha_i}(f)$ and $M_{\beta_i}(f)$ are finite unions of n-dimensional compact intervals, with sides parallel to the coordinate axes.

It is easy to see that there exists a unique function $f^o : \partial D^o \to [0, +\infty[$ such that $f^o_{|\partial D^o \setminus \gamma^o} = 0$ and

$$(M_{\alpha_i}(f))^o = M_{\alpha_i^o}(f^o) \; , \; (M_{\beta_i}(f))^o = M_{\beta_i^o}(f^o) \; , \; i = 1,\dots,n-k .$$

In the following let us write $x = (x_1,\dots,x_{n-k}|x_{n-k+1},\dots,x_n) = (x',x'')$, $a' = (a_1,\dots,a_{n-k})$, and $b' = (b_1,\dots,b_{n-k})$. The notation $x' > a'$ then means that $x_i > a_i$, $i = 1,\dots,n-k$.

Now let $\omega(x,f,D)$ be the solution of the Dirichlet problem in D with boundary data f. It will be convenient to set $\omega(x,f,D) = 0$ if

$x \notin D$, $a' \leq x' \leq b'$.

The purpose of this paper is to prove the following.

Theorem 1.1: If φ is any non-decreasing convex function on $\mathbb{R}$, then

(1.1) $\int \varphi(\omega(x,f,D))dx'' \leq \int \varphi(\omega(x,f^o,D^o))dx''$, $a' < x' < b'$.

In particular,

(1.2) $\max_{x''} \omega(x,f,D) \leq \omega((x',0''),f^o,D^o)$, $a' < x' < b'$.

Before the proof we will make a few remarks.

Above we have made several restrictions on D and f. The most important restrictions here are that D is bounded and that $f_{|\partial D \setminus \gamma} = 0$. Other than these restrictions Theorem 1.1 is obviously still true for much more general D and f: $\partial D \to [0, +\infty[$.

The characteristic function of a set $M \subseteq \mathbb{R}^n$ is denoted by χ_M. Note that if $f = \chi_\alpha$ or $f = \chi_\alpha + \chi_\beta$ in the Theorem 1.1, then $f^o = \chi_{\alpha^o}$ or $f^o = \chi_{\alpha^o} + \chi_{\beta^o}$, respectively. Theorem 1.1 is known in some special cases. In fact, Baernstein [1] proves the case $f = \chi_\alpha$, $n = 2$, $k = 1$, and Haliste [6, Theorem 8.2] proves the inequality (1.2) in the case $f = \chi_\alpha$, n arbitrary, $k = 1$. (Note, however, that Baernstein works with circular symmetrization with respect to the positive real axis.)

The inequality (1.1) is of the following general type;

$$\mu(\varphi) \leq \nu(\varphi) \text{ , all } \varphi', \varphi'' \geq 0 \text{ ,}$$

where μ and ν are positive Borel measures on the real line both with compact support. Inequalities of this kind have been studied in great detail and the interested reader may consult [10], which also explains the underlying geometrical nature of the inequality.

Our proof of Theorem 1.1 is divided into two steps. In the first step we prove the special case $k = 1$ following closely to [1]. In fact, by defining a certain transform $u^*(x)$ of $u(x) = \omega(x,f,D)$, we can argue exactly as in [1]. The transform u^* has been introduced by Baernstein in the plane in a slightly different way, and with this definition u^* makes sense for any subharmonic function $u: \mathbb{R}^2 \to [-\infty,+\infty[$. The general case of Theorem 1.1 then follows from the special case $k = 1$ already proved by performing a certain smoothing process.

Finally, in this section, I wish to express my gratitude to Matts Essén and Peter Sjögren for many valuable discussions and suggestions.

2. The case $k = 1$.

If $f: \mathbb{R} \to \mathbb{R}_+$ is a measurable function, the function $\tilde{f}$ denotes the rearrangement of f in decreasing order on the interval $[0, +\infty[$.

We need the following

Lemma 2.1: For any $f, g, \in L^1_+(\mathbb{R})$ it holds that

$$\int_0^\infty |\tilde{f}(t) - \tilde{g}(t)|dt \leq \int_{-\infty}^\infty |f(t) - g(t)|dt .$$

This inequality is just a special case of the more general inequalities appearing in [3] and [8].

Lemma 2.2: Let E be a finite union of compact intervals. Then there is a positive number $\delta(E)$, which only depends on E, so that

$$(2.1) \qquad \int_E f(t-a)dt + \int_E f(t+a)dt \leq \int_0^{|E|-2|a|} \tilde{f}(t)dt + \int_0^{|E|+2|a|} \tilde{f}(t)dt$$

for each $|a| < \delta(E)$, and each $f \in L^1_+(\mathbb{R})$.

Here $|E|$ denotes the one-dimensional Lebesgue measure of E.

Proof: Set $|(E-a) \cap (E+a)| = |E| - 2b$, $b \geq 0$. Then, clearly, the left-hand side of (2.1) is less or equal to

$$\int_0^{|E|-2b} \tilde{f}(t)dt + \int_0^{|E|+2b} \tilde{f}(t)dt ,$$

which is a non-increasing function of b. But since E is a finite union of compact intervals we have $b \geq |a|$ if $|a|$ is small enough. This proves Lemma 2.2.

Now suppose that $k = 1$ in Theorem 1.1 and set $u(x) = \omega(x,f,D)$ and $u_{x'}(x_n) = u(x',x_n)$. We define

$$u^*(x) = \int_0^{2x_n} \tilde{u}_{x'}(t)dt \ , \ a' \leq x' \leq b' \ , \ x_n \geq 0 \ .$$

<u>Lemma 2.3</u>: u^* <u>is continuous in</u> $a' \leq x' \leq b'$, $x_n \geq 0$ <u>and subharmonic in</u> $a' < x' < b'$, $x_n > 0$.

<u>Proof</u>: We first prove that u^* is continuous. Let $\bar{x} = (\bar{x}',\bar{x}_n)$ and $x = (x',x_n)$. Then

$$u^*(\bar{x}) - u^*(x) = u^*(\bar{x}) - u^*(\bar{x}',x_n) + \int_0^{2x_n} (\tilde{u}_{\bar{x}'}(t) - \tilde{u}_{x'}(t))dt \ .$$

Lemma 2.1 now gives

$$|u^*(\bar{x}) - u^*(x)| \leq |\int_{2x_n}^{2\bar{x}_n} \tilde{u}_{\bar{x}'}(t)dt| + \int |u_{\bar{x}'}(t) - u_{x'}(t)|dt$$

and the conclusion is obvious.

To prove that u^* is subharmonic let x be an arbitrary vector such that $a' < x' < b'$, $x_n > 0$. Since u is a real analytic function in the inerior of D we deduce that there exists a finite union of compact intervals E such that

$$u^*(x) = \int_E u_{x'}(t)dt \ , \ |E| = 2x_n \ .$$

But u is subharmonic in $a' < x' < b'$ and so we have the inequality

$$(2.2) \quad u_{x'}(t) \leq \frac{1}{2} \left(\int (u_{x'+\rho y'}(t-\rho y_n) + u_{x'+\rho y'}(t+\rho y_n))d\sigma(y)\right) ,$$

valid for all t , and all $\rho < \min_{1 \le i \le n-1} (|a_i - x_i| , |b_i - x_i|)$. Here σ is the normalized surface-area measure on $\{|y| = 1\}$.

Now let us in addition assume that $\rho < \delta(E)$. By integrating (2,2) over E and making use of Lemma 2.2 we obtain

$$u^*(x) < \frac{1}{2} \left(\int (u^*(x'+\rho y', x_n - \rho y_n) + u^*(x'+\rho y', x_n+\rho y_n))d\sigma(y)\right) ,$$

which proves that u^* is subharmonic.

<u>Proof of Theorem 1.1</u>, $k = 1$: Set $v(x) = \omega(x, f^o, D^o)$. To prove (1.1) it will be enough to show that $u^* \le v^*$. (Compare [7, Theorems 249 and 250].) To this end let us first define $D_+^o = D^o \cap \{x_n \ge 0\}$ and

$$v_1(x) = 2 \int_0^{x_n} v(x',t)dt , \quad x \in D_+^o .$$

The function v is clearly continuous and since $v(x', x_n) = v(x', -x_n)$ it follows that

$$\frac{\partial v}{\partial x_n} (x',0) = 0, a' < x' < b' .$$

Therefore v_1 is harmonic in the interior of D_+^o . Furthermore, the functions v^* and v_1 are equal on ∂D_+^o . Since v^* is subharmonic we deduce that $v^* \le v_1$ in D_+^o . But the opposite inequality is obvious from the definitions so $v^* = v_1$ in D_+^o . In particular,

$$\frac{\partial v}{\partial x_n} \le 0 \text{ in the interior of } D_+^o .$$

We will now show that $u^* \le v^*$. Therefore let $\delta > 0$ be arbitrary, and set $w = u^* - v^* - \delta x_n$ in D_+^o . We know that w is continuous in D_+^o and subharmonic in the interior of D_+^o . It is also readily seen that $w \le 0$ on $(\partial D_+^o \cap \{x_n = 0\}) \cup (\gamma^o \cap \{x_n \ge 0\})$. Let $(p', p_n) \in \partial D_+^o$, $a' < p' < b'$, $p_n > 0$, and suppose that $w(p',p_n) \ge w(p',p_n - h)$ for all $h > 0$ small anough. We will show that this is impossible. Since u and v vanish on $\partial D \setminus \gamma$ it can be assumed that $(p', p_n + \delta) \notin \partial D_+^o$ for each $\delta > 0$.

Set $q_n = \max \{t | (p',t) \in D\}$. Then since $v^* = v_1$ in D_+^o , we have

$$\int_{-\infty}^{q_n} u(p',t)dt - 2 \int_0^{p_n} v(p',t)dt - \delta p_n \geq$$

$$\geq \int_{-\infty}^{q_n-2h} u(p',t)dt - 2 \int_0^{p_n-h} v(p',t)dt - \delta(p_n-h) ,$$

for all $h > 0$ small enough. This yields $u(p',q_n) \geq \delta/2$ which is impossible. We conclude that w cannot attain a maximum on the upper boundary of D_+^o and therefore $w \leq 0$. Since $\delta > 0$ is arbitrary, $u^* \leq v^*$ in D_+^o and so $u^* \leq v^*$ everywhere in $a' \leq x' \leq b'$. This proves (1.1) and (1.2) follows from (1.1) and the fact that $\frac{\partial v}{\partial x_n} \leq 0$ in the interior of D_+^o . This concludes the proof of Theorem 1.1 in the special case $k = 1$.

3. <u>The general case</u>.

In the following we assume that $n - 1 \geq k \geq 2$ and we set $g_1 = e_{n-k+1},\ldots,g_k = e_n$. The linear subspace of $\mathbb{R}^n$ which is spanned by the vectors $g_1,\ldots,g_k$ will be denoted by $\mathbb{R}^k$. If A and B are non-empty compact subsets of $\mathbb{R}^k$ the Hausdorff distance d(A,B) of A to B equals

$$\inf \{\delta > 0 \mid A + \delta\bigcirc \supseteq B\} + \inf \{\delta > 0 \mid B + \delta\bigcirc \supseteq A\} ,$$

where $\bigcirc$ is the closed unit ball in $\mathbb{R}^k$.

Let S denote the k-dimensional Steiner symmetrization in $\mathbb{R}^k$. A theorem due to Sarvas [9, Theorem 4.32] then says that there exist two (k-1)-dimensional Steiner symmetrizations in $\mathbb{R}^k$, S_1 and S_2 , respectively, such that

$$S(A) = \lim_{j\to\infty} (S_1 \circ S_2)^j(A)$$

for any compact non-empty subset A of $\mathbb{R}^k$. Using this we can quickly prove Theorem 1.1. However, the proof of Theorem 4.32 in [9] is rather lengthy and to do this paper independent of [9] we will prove a result (Lemma 3.2) which is similar to Sarvas' result. Our proof of Lemma 3.2 is based on other ideas than [9] and we hope the proof can be independent interest.

We need

Lemma 3.1: *There exist* k *hyperplanes* $H_1,\dots,H_k$ *in* $\mathbb{R}^k$ *such that any convex body in* $\mathbb{R}^k$ *which is symmetric about each of these hyperplanes is a closed ball of centre* 0 .

Here and in the following a convex body means a compact convex set with non-empty interior.

Lemma 3.1 is, of course, well-known, see e.g. [2, pp. 86] and [4, pp. 98].

Let us assume that the vector $y \in \mathbb{R}^k$ has the coordinates $y_1,\dots,y_k$ with respect to the basis $g_1,\dots,g_k$. Further, let $Q_1,\dots,Q_k$ be isometric operators such that

$$Q_i(H_i) = \{y_k = 0\} \ , \ i = 1,\dots,k \ ,$$

where $H_1,\dots,H_k$ are as in Lemma 3.1. We let S_i denote the one-dimensional Steiner symmetrization with respect to the hyperplane $y_i = 0$ and set $S = S_1 \circ \dots \circ S_k$.

Lemma 3.2: *Let* A *be a compact subset of* $\mathbb{R}^k$ *with non-empty interior and define*

$$A_1 = S(A) \ , \ A_{j+1} = Q_j^{-1}\ (S(Q_j(A_j))) \ , \ j \geq 1 \ ,$$

where $Q_{k+1} = Q_1$, $Q_{k+2} = Q_2, \dots$.

Then $A_j \to \rho O$ *as* $j \to +\infty$, *where* $\rho > 0$ *fulfils* $m_k(\rho O) = m_k(A)$.

Here m_k is the k-dimensional Lebesgue measure.

If A is a convex body, Lemma 3.2 is a well-known result due to Blaschke. (see e.g. [2, pp. 86] and [4, pp. 98].) Our proof of Lemma 3.2 uses convexity methods and is a modification of the classical proof in the case when A is a convex body. The proof leans heavily on the fact that $d(\bar{\bar{A}},\bar{\bar{B}}) \leq d(A,B)$, where $\bar{\bar{A}}$ denotes the convex hull of A . This inequality follows directly from the identity $\overline{\overline{A+B}} = \bar{\bar{A}} + \bar{\bar{B}}$, which is valid for any subsets A and B of $\mathbb{R}^k$. Before the proof of Lemma 3.2 let us formulate

Definition 3.1. A subset A of $\mathbb{R}^k$ is said to be a k-convex symmetric body if

1) A is compact with non-empty interior.
2) A is symmetric about each hyperplane $y_i = 0$, $i = 1,\dots,k$.
3) The intersection of A and any straight line parallel to a coordinate axis is an interval.

We need a simple

Lemma 3.3:

a) If A is any compact subset of $\mathbb{R}^k$ with non-empty interior, then S(A) is a k-convex symmetric body.
b) If a sequence of k-convex symmetric bodies converges, then the limit set fulfils the conditions 2) and 3) of Definition 3.1.
c) If a sequence $\{A_j\}_1^\infty$ of k-convex symmetric bodies converges to a closed ball B of centre 0 , then $m_k(A_j) \to m_k(B)$ as $j \to +\infty$.

Proof: Everything here is trivial but let us prove c).

Note first that $\bar{\bar{A}}_j \to \bar{\bar{B}} = B$ as $j \to +\infty$. Hence $\overline{\lim_{j\to\infty}} m_k(A_j) \leq m_k(B)$.

It can be assumed that $m_k(B) > 0$ and let y belong to the interior of B. We claim that $y \in A_j$ if j is large enough. It can clearly be assumed that $y_1,\dots,y_k \geq 0$. Now choose $\delta > 0$ such that

$$[(\mathbb{R}^k \setminus \prod_1^k [y_k,+\infty[) + \delta O] \not\supseteq B .$$

If $y \notin A_j$, then Definition 3.1 implies that

$$(A_j + \delta O) \not\supseteq B .$$

Therefore, we conclude that $y \in A_j$ if j is large enough. Fatou's lemma now gives

$$m_k(B) \leq \underline{\lim}_{j\to\infty} m_k(A_j) ,$$

which proves c).

<u>Proof of Lemma 3.2</u>: Let $(j_p)_1^\infty$ be a strictly increasing sequence of natural numbers such that $A_{j_p} \to B$ as $p \to +\infty$. By the Blaschke selection theorem [5, p. 154] it suffices to prove that $B = \rho\bigcirc$.

First note that there exist a natural number r , $1 \leq r \leq k$, and a strictly increasing sequence $(p_\nu)_1^\infty$ of natural numbers such that $Q_{j_{p_\nu}} = Q_r$, and $j_{p_{\nu+1}} - j_{p_\nu} \geq k$ for all ν .

The surface-area of a convex body M is denoted by $\sigma(M)$. The Cauchy surface-area formula then implies that $\sigma(M_1) \leq \sigma(M_2)$ for all convex bodies M_1 and M_2 such that $M_1 \subseteq M_2$. Furthermore, if M is any compact subset of $\mathbb{R}^k$, we have $\overline{\overline{S(M)}} \subseteq S(\overline{\overline{M}})$. Using these two properties, we get

$$(3.1) \qquad \sigma(\overline{\overline{A}}_{j_{p_{\nu+1}}}) \leq \sigma(Q^{-1}_{j_{p_{\nu+1}}-1}(S(Q_{j_{p_{\nu+1}}-1}(\overline{\overline{A}}_{j_{p_{\nu+1}}-1})))) \leq$$

$$\leq \sigma(\overline{\overline{A}}_{j_{p_{\nu+1}}-1}) \leq \cdots \leq \sigma(Q_r^{-1}(S(Q_r(\overline{\overline{A}}_{j_{p_\nu}})))) .$$

By letting $\nu \to +\infty$ it follows that

$$(3.2) \qquad \sigma(\overline{\overline{B}}) \leq \sigma(Q_r^{-1}(S(Q_r(\overline{\overline{B}})))) = \sigma(S(Q_r(\overline{\overline{B}}))) .$$

Hence $\sigma(Q_r(\overline{\overline{B}})) \leq \sigma(S_k(Q_r(\overline{\overline{B}})))$, and we deduce that $Q_r(\overline{\overline{B}}) = S_k(Q_r(\overline{\overline{B}}))$. Using (3.2), we conclude that $\sigma(Q_r(\overline{\overline{B}})) \leq \sigma(S_1 \circ \ldots \circ S_{k-1}(Q_r(\overline{\overline{B}}))) \leq$ $\leq \sigma(S_{k-1}(Q_r(\overline{\overline{B}})))$. By repetition, we get $Q_r(\overline{\overline{B}}) = S(Q_r(\overline{\overline{B}}))$.

Now observe that $\bar{\bar{A}}_{j_{p_{\nu+1}}} \subseteq Q_r^{-1}(S(Q_r(\bar{\bar{A}}_{j_{p_\nu}})))$. The inequality (3.1) thus gives

$$\sigma(\bar{\bar{A}}_{j_{p_{\nu+1}}}) \leq \sigma(Q_{r+1}^{-1}(S(Q_{r+1}(\bar{\bar{A}}_{j_{p_\nu}+1})))) \leq$$

$$\leq \sigma(Q_{r+1}^{-1}(S(Q_{r+1}(Q_r^{-1}(S(Q_r(\bar{\bar{A}}_{j_{p_\nu}})))))))\ .$$

By letting $\nu \to +\infty$ we get

$$\sigma(\bar{\bar{B}}) \leq \sigma(Q_{r+1}^{-1}(S(Q_{r+1}(\bar{\bar{B}}))))\ ,$$

since $\bar{\bar{B}} = Q_r^{-1}(S(Q_r(\bar{\bar{B}})))$. By the same argument as above it follows that $Q_{r+1}(\bar{\bar{B}}) = S_k(Q_{r+1}(\bar{\bar{B}}))$ and $Q_{r+1}(\bar{\bar{B}}) = S(Q_{r+1}(\bar{\bar{B}}))$. By repetition, we have that the sets $Q_i(\bar{\bar{B}})$, $i = 1,\dots,k$, are symmetric about the hyperplane $y_k = 0$, and lemma 3.1 proves that $\bar{\bar{B}}$ is a closed ball of centre 0 .

In the next step it will be proved that $\bar{\bar{B}} = B$. Suppose to the contrary that $\bar{\bar{B}} \setminus B \neq \emptyset$. Since B is closed there exists a $\delta > 0$ such that $\bar{\bar{B}} \setminus (B + 2\delta O) \neq \emptyset$. Choose j_p so that $A_{j_p} + \delta O \supseteq B$ and $B + \delta O \supseteq A_{j_p}$. Then $B + 2\delta O \supseteq A_{j_p} + \delta O$ and $\bar{\bar{B}} \setminus B \supset \bar{\bar{B}} \setminus (A_{j_p} + \delta O) \neq \emptyset$. Since $Q_{j_p-1}(A_{j_p} + \delta O) = Q_{j_p-1}(A_{j_p}) + \delta O$ is a k-convex symmetric body we have a contradiction. Hence $\bar{\bar{B}} = B$. This also shows that $d(Q_{j_p-1}(A_{j_p}), B) = d(A_{j_p}, B) \to 0$ as $p \to \infty$. Lemma 3.3c) therefore gives $B = \rho O$, which was to be proved.

<u>Proof of Theorem 1.1</u>: We are now working in $\mathbb{R}^n$. Let $\bar{S}_i$ denote the one-dimensional Steiner symmetrization with respect to the hyperplane $x_{i+n-k} = 0$ for $i = 1,\dots,k$, and set $\bar{S} = \bar{S}_1 \circ \dots \circ \bar{S}_k$. Then there exists a unique function

$$\bar{S}_k(f) : \partial(\bar{S}_k(D)) \to [0, +\infty[\ ,$$

vanishing on $\partial\overline{S}_k(D)\setminus\gamma(\overline{S}_k(D))$ and such that

$$\overline{S}_k(M_{\alpha_i}(f)) = M_{\overline{S}_k(\alpha_i)}(\overline{S}_k(f)) \ , \ \overline{S}_k(M_{\beta_i}(f)) = M_{\overline{S}_k(\beta_i)}(\overline{S}_k(f)) \ ,$$

$i = 1,\ldots,n-k$.

The functions $\overline{S}_{k-1} \circ \overline{S}_k(f) = \overline{S}_{k-1}(\overline{S}_k(f)),\ldots,\overline{S}(f)$ are defined in the same way.

In the following let φ be an arbitrary non-decreasing convex function on $\mathbb{R}$. The special case $k = 1$ of Theorem 1.1 then yields

$$\int \varphi(\omega(x,f,D))\, dx_n \leq \int \varphi(\omega(x,\overline{S}_k(f),\overline{S}_k(D)))\, dx_n \ ,$$

$$\int\varphi(\omega(x,\overline{S}_k(f),\overline{S}_k(D)))dx_{n-1} \leq \int\varphi(\omega(x,\overline{S}_{k-1} \circ \overline{S}_k(f),\overline{S}_{k-1} \circ \overline{S}_k(D)))dx_{n-1} \ ,$$

$$\vdots$$

$$\int\varphi(\omega(x,\overline{S}_2\circ\ldots\circ\overline{S}_k(f),\overline{S}_2\circ\ldots\circ \overline{S}_k(D)))dx_{n-k+1} \leq \int\varphi(\omega(x,\overline{S}(f),\overline{S}(D)))dx_{n-k+1} \ .$$

Hence

$$(3.3) \qquad \int \varphi(\omega(x,f,D))dx'' \leq \int \varphi(\omega(x,\overline{S}(f),\overline{S}(D))dx'' \ .$$

Now let the isometric operators $Q_1,\ldots,Q_k$ be as above. These operators have unique isometric extensions to $\mathbb{R}^n$, denoted by $\overline{Q}_1,\ldots,\overline{Q}_k$, respectively, so that $\overline{Q}_i(e_j) = e_j$, $i=1,\ldots,k$, $j=1,\ldots,n-k$.

If $h\colon M \to [0,+\infty[$ $(M \subseteq \mathbb{R}^n)$ and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a 1-1 linear operator let Th be the mapping from $T(M)$ into $[0,+\infty[$ defined by $Th(x) = h(T^{-1}x)$, $x \in T(M)$.

We define

$$D_1 = \overline{S}(D) \ , \ D_{j+1} = \overline{Q}_j^{-1}(\overline{S}(\overline{Q}_j(D_j))) \ , \ j \geq 1 \ ,$$

$$f_1 = \overline{S}(f) \ , \ f_{j+1} = \overline{Q}_j^{-1}(\overline{S}(\overline{Q}_j(f_j))) \ , \ j \geq 1 \ .$$

Note that

$$\omega(\overline{Q}_1^{-1}x, f_1, D_1) = \omega(x, \overline{Q}_1(f_1), \overline{Q}_1(D_1)) .$$

Therefore (3.3) implies that

$$\int \varphi(\omega(\overline{Q}_1^{-1}x, f_1, D_1))dx'' \leq \int \varphi(\omega(x, \overline{S}(\overline{Q}_1(f_1)), \overline{S}(\overline{Q}_1(D_1)))dx'' .$$

But

$$\int \varphi(\omega(\overline{Q}_1^{-1}x, f_1, D_1))dx'' = \int \varphi(\omega(x, f_1, D_1))dx''$$

and a comparison with (3.3) gives

$$\int \varphi(\omega(x, f, D))dx'' \leq \int \varphi(\omega(x, f_2, D_2))dx'' .$$

By repetition we obtain

$$(3.4) \qquad \int \varphi(\omega(x, f, D))dx'' \leq \int \varphi(\omega(x, f_j, D_j))dx'' , \quad j \geq 1 .$$

Lemma 3.2 shows that $L^{\perp} \cap (D_j + a) \to L^{\perp} \cap (D^o + a)$ as $j \to +\infty$ for each $a \in L$. For fixed j the set-valued functions $a \to L^{\perp} \cap (D_j + a)$ and $a \to L^{\perp} \cap (D^o + a)$ only attain a finite number of sets and they have the same points of discontinuity. Hence $D_j \to D^o$ as $j \to \infty$.

Now let

$$M(f) = \bigcup_{1}^{n-k} (M_{\alpha_i}(f) \cup M_{\beta_i}(f)) ,$$

where $M_{\alpha_i}(f)$ and $M_{\beta_i}(f)$ are as in the introduction. It is then readily readily seen that

$$M(f_1) = \overline{S}(M(f)), M(f_{j+1}) = \overline{Q}_j^{-1}(\overline{S}(\overline{Q}_j(M(f_j)))) , \quad j \geq 1 .$$

Using Lemma 3.2 again we get that $M(f_j) \to M(f^o)$ as $j \to +\infty$. By letting $j \to +\infty$ in (3.4) we obtain (1.1). The inequality (1.2) follows in a similar way. This proves Theorem 1.1.

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