APPENDIX

Central extensions of affine group schemes

In this Appendix, the ground field k is arbitrary, and all group schemes are affine.

A.0. Our purpose here is to outline an elementary theory of central extensions of affine group schemes over an arbitrary ground field. Results summarized here are mostly known, but some are proven only over algebraically closed ground fields while others are known to be true in a more general context. (Cf. [5; Appendix], DG-III, §6 and [14; Chap. VII].) One of the two exact sequences below, A.8, is probably new, though it is modeled after an erroneously stated result by Miyanishi [5; Prop. 2, page 649].

A.1. A sequence $1 \to H \to E \to G \to 1$ of k-group schemes H, E, G is said to be k-*exact* or, plainly, *exact* if (i) all arrows represent k-homomorphisms, (ii) for any k-algebra R the sequence $1 \to H(R) \to E(R) \to G(R)$ of abstract groups is exact, and (iii) $E \to G$ is a faithfully flat epimorphism. The sequence, or often E itself, is referred to as an *extension of* G *by* H. If $1 \to H \to E \to G \to 1$ is k-exact and if the image of H is contained in the center of G, then we say the sequence is *central* k-*exact* or is a *central* *extension of* G *by* H. We also say that E is a central extension of G by H, by a slight abuse of language.

A.2. Let $1 \to H \to E \to G \to 1$ be k-exact. The sequence is said to be k-*split* if there is a k-homomorphism $G \to E$ such that $(G \to E \to G) = id_G$. The sequence is *geometrically* k-*split*, by definition, if there exists a k-morphism $\overline{G} \to \overline{E}$ such that $(\overline{G} \to \overline{E} \to \overline{G}) = id_{\overline{G}}$. Then, obviously, $\overline{E} \simeq \overline{H} \times G$.

SPLITTING LEMMA. *An exact sequence* $1 \to H \to E \to G \to 1$ *of* k-*group schemes is geometrically* k-*split whenever* H *is* k-*isomorphic to* G_a.

This is implied by DG-III, §4, 6.6. Within the classical framework, this was first proved by Rosenlicht [9; Th. 1, p.99]. Let us, however, prove it in a few lines by another method:

Proof. Let $\overline{H}$ = Spec k[T], $\overline{E}$ = Spec A and $\overline{G}$ = Spec B. Since $H \simeq G_a$ acts freely on $\overline{E}$ with the quotient $\overline{G}$, we have $A \simeq k[T] \otimes B \simeq B[T]$ by virtue of [4; Lemma 2, p. 403], and the ring homomorphism $A \to B$ given by $T \longmapsto 0$ evidently splits the sequence geometrically over K.

A.3. *Remarks*. The consequences of this Lemma are important. Firstly, we immediately derive the fact that the underlying scheme of a unipotent k-group of dimension n is k-isomorphic to $\mathbb{A}^n$ if k is *perfect*, or more generally if the group is k-*solvable* (see DG-IV, §4, 4.1 and [9; Cor. 2, p.101]). Secondly, this lemma is the key to make Serre's induction argument [14; VII, No. 10, pp. 175-177] valid over non-algebraically closed fields, thereby establishing Chevalley's theorem over a *perfect* field k: To wit, every commutative unipotent algebraic k-group is k-isogenous to a unique (up to order) product of Witt vector groups. (Analyzing Serre's argument [14; *loc. cit*], one can see directly that what fails the argument in case k is merely assumed perfect is the inability to affirm, in that set-up, that the extensions $0 \to G_a \to E \to W_n \to 0$ over k correspond to $H^2_{reg}(W_n, G_a)_s$. This one can now ascertain, possessing a regular cross section $W_n \to E$ defined over k, thanks to the Lemma. This gives a proof of Chevalley's theorem without resort to the theory of Dieudonné modules, as done in DG-V, §3, 6.11.)

A.4. If a central extension $0 \to H \to E \to G \to 1$ of k-group schemes is geometrically k-split, the group law on E is determined by 2-*cocycles* $G \times G \to H$ in the well-known manner. To wit, the group multiplication on $\overline{E} = \overline{G} \times \overline{H}$ is given by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2,\ h_1 + h_2 + \gamma(g_1, g_2))$$

for $g_1, g_2 \in G(R)$, $h_1, h_2 \in H(R)$, where $\gamma\colon G \times G \to H$ is a k-morphism satisfying

$$\gamma(g_1,g_2) + \gamma(g_1g_2,g_3) = \gamma(g_1,g_2g_3) + \gamma(g_2,g_3)$$

for all $g_1,g_2,g_3 \in G(R)$. Conversely, given a 2-cocycle $\gamma\colon G \times G \to H$, one can construct a central extension $0 \to H \to E \to G \to 1$ by defining a group law on $\overline{G} \times \overline{H}$ in the above fashion by making use of γ. <u>The extension thus obtained will be denoted by</u> $G \times_\gamma H$.

A.5. Let $1 \to H \to E_1 \to G \to 1$, $1 \to H \to E_2 \to G \to 1$ be extensions of G by H. We say that these extensions are <u>equivalent</u> if there is a k-homomorphism $E_1 \to E_2$ making the diagram

$$\begin{array}{ccccccccc} 1 & \to & H & \to & E_1 & \to & G & \to & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \to & H & \to & E_2 & \to & G & \to & 1 \end{array}$$

commutative. The set of equivalence classes of extensions of G by H is denoted by $\mathrm{Ext}(G,H)$. In case H is commutative, we consider the set of equivalence classes of central extensions of G by H and denote it by $\mathrm{Ext}_{\mathrm{cent}}(G,H)$, which is a subset of $\mathrm{Ext}(G,H)$. If in addition G is commutative, the set of all equivalence classes of commutative extensions is represented as $\mathrm{Ext}_{\mathrm{com}}(G,H)$.

A.6. Let G be a k-group scheme, H a commutative k-group scheme. Let $0 \to H \to E \to G \to 1$ be a central extension, and suppose given a k-homomorphism $\phi: G' \to G$ of k-group schemes. Then, one can construct a central extension $0 \to H \to E' \to G' \to 1$ unique up to equivalence subject to the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \to & H & \to & E' & \to & G' & \to & 1 \\ & & \| & & \downarrow & & \downarrow\phi & & \\ 0 & \to & H & \to & E & \to & G & \to & 1. \end{array}$$

We write ϕ^*E in place of E'. In a like manner, for a given k-homomorphism ψ; $H \longrightarrow H'$, one can construct ψ_*E uniquely up to equivalence subject to the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \to & H & \to & E & \to & G & \to & 1 \\ & & \psi\downarrow & & \downarrow & & \| & & \\ 0 & \to & H' & \to & \psi_*E & \to & G & \to & 1. \end{array}$$

Possessing ϕ^*E and ψ_*E, one can proceed to introduce a structure of additive group on the set $\mathrm{Ext}_{\mathrm{cent}}(G,H)$ in the usual fashion, and $\mathrm{Ext}_{\mathrm{cent}}(G,H)$ becomes a right $\mathrm{End}_{k\text{-}gr}(G)$-and left $\mathrm{End}_{k\text{-}gr}(H)$-bimodule. The constructions and verifications pertaining to the foregoing are found in [14; Chap. VII, §1], [5; Appendix], DG-III, §6 and SGAD-III, VI_A, XVII-App.1., though with various degrees of generality.

A.7. LEMMA. *Let* $\phi: G \to H$ *be a* k-*homomorphism of* k-*group schemes* G, H. *If the subgroup* $\phi(G(R))$ *is normal* (resp. *central*) *in* H(R) *for every* $R \in \mathrm{Alg}_k$, *then the image* $\phi(G)$ *is a* k-*closed normal* (resp. *central*) *subgroup of* H.

In fact, the image $\phi(G)$ is the (fpqc)-sheafication of the k-group functor $R \longmapsto \phi(G(R))$, so that if $x \in \phi(G)(R)$ and $h \in H(R)$ then there is a faithfully flat R-algebra R' such that $x' \in \phi(G(R'))$, where $x \longmapsto x'$ under $\phi(G)(R) \to \phi(G)(R')$. By assumption, $(h')^{-1}x'h' \in \phi(G(R')) \subseteq \phi(G)(R')$, and then, by (fpqc)-descent, one gets $h^{-1}xh \in \phi(G)(R)$. Because k is a field, $\phi(G)$ is clearly k-closed. Similar argument for the centrality of $\phi(G)$ when each $\phi(G(R))$ is central.

A.8. THEOREM. *Let* $0 \to H \xrightarrow{\tau} E \xrightarrow{\pi} G \to 1$ *be a central extension of* k-*group schemes*, H *being commutative*. *For every commutative* k-*group scheme* A, *consider the sequence of additive groups*

$$0 \to \mathrm{Hom}_{k\text{-}gr}(G,A) \to \mathrm{Hom}_{k\text{-}gr}(E,A) \to \mathrm{Hom}_{k\text{-}gr}(H,A)$$

$$\xrightarrow{\gamma} \mathrm{Ext}_{cent}(G,A) \xrightarrow{\pi^*} \mathrm{Ext}_{cent}(E,A) \xrightarrow{\tau^*} \mathrm{Ext}_{cent}(H,A) \quad (1)$$

where γ *sends* $\phi \in \mathrm{Hom}_{k\text{-}gr}(H,A)$ *to the extension class of*

ϕ_*E. Then:

(i) The sequence (1) is exact except possibly at $\mathrm{Ext}_{cent}(E,A)$, where only $\tau^*\pi^* = 0$ holds in general.

(ii) Assume that G, H and A are k-smooth and that $\mathrm{Hom}_{k_s\text{-}gr}(H,A) = \{0\}$. Then, the sequence (1) is exact throughout.

Proof. (i) We shall make quick verification of exactness at each spot, leaving out all routine procedures.

a) The exactness at $\mathrm{Hom}(G,A)$ and at $\mathrm{Hom}(E,A)$ is trivial.

b) Next, at $\mathrm{Hom}_{k\text{-}gr}(H,A)$, suppose $\phi \in \mathrm{Hom}(H,A)$ factors through τ so that $\phi = \psi\tau$ with $\psi: E \to A$. Then define a k-homomorphism $E \to A \times E$ by $x \longmapsto (-\psi x, x)$ for all $x \in E(R)$ and compose it with the canonical homomorphism $A \times E \to \phi_*E$. The composed homomorphism $E \to \phi_*E$ vanishes on the closed subgroup scheme H and thereby gives a k-homomorphism $G \to \phi_*E$. It is immediate that $(G \to \phi_*E \to G) = \mathrm{id}_G$, so ϕ_*E represents the zero in $\mathrm{Ext}_{cent}(G,A)$. Suppose now that ϕ_*E is trivial. Then we have a canonical projection $\rho: \phi_*E \simeq A \times G \to A$ with the property $(A \to \phi_*E \to A) = \mathrm{id}_A$. Thus, in the diagram

$$\begin{array}{ccccccccc} 0 & \to & H & \xrightarrow{\tau} & E & \xrightarrow{\pi} & G & \to & 1 \\ & & \phi\downarrow & & \downarrow\lambda & & \| & & \\ 0 & \to & A & \underset{\rho}{\overset{\tau'}{\rightleftarrows}} & \phi_*E & \to & G & \to & 1 \end{array}$$

we have $(\rho\lambda)\tau = \rho(\tau'\phi) = \phi$, whence ϕ factors through τ <u>via</u> $\rho\lambda$.

c) At $\mathrm{Ext}_{\mathrm{cent}}(G,A)$, first it is clear that the composition $\mathrm{Hom}(H,A) \to \mathrm{Ext}_{\mathrm{cent}}(G,A) \to \mathrm{Ext}_{\mathrm{cent}}(E,A)$ is zero, as $\phi \longmapsto \phi_*E \longmapsto \pi^*(\phi_*E) = \phi_*(\pi^*E)$ and π^*E is trivial. Next let $0 \to A \to X \to G \to 1$ be a central extension such that π^*X is trivial, and let us show that $X = \phi_*E$ for an appropriate $\phi\colon H \to A$. So, consider the diagram

$$
\begin{array}{ccccccccc}
0 & \to & A & \to & X & \to & G & \to & 1 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle\pi} & & \\
0 & \to & A & \to & \pi^*X & \to & E & \to & 1 \\
 & & {\scriptstyle -\phi}\nwarrow & & \uparrow & & \uparrow{\scriptstyle\tau} & & \\
 & & & & H & = & H & &
\end{array}
$$

in which $\pi^*X \simeq A \times E$ admits a projection to A. Set $-\phi$ to be the said projection preceded by the k-monomorphism $H \to \pi^*X$, as shown above. Then, under the identification of π^*X with $A \times E$, the monomorphism $H \to \pi^*X$ is given by $h \in H(R) \longmapsto (-\phi h, \tau h)$ and X is the quotient of $\pi^*X = A \times E$ by the image of H. This shows that $X = \phi_*E$.

d) Finally, at $\mathrm{Ext}_{\mathrm{cent}}(E,A)$, it is obvious that $\tau^*\pi^* = (\pi\tau)^* = 0$.

(ii) Assume now $\mathrm{Hom}_{k_s\text{-}gr}(H,A) = \{0\}$ and G,H,A (and hence E) are k-smooth. Let $0 \to A \to X \to E \to 1$ be a central extension such that τ^*X splits. Then we have a k-homomorphism $\psi : H \to X$ obtained by composing the section on H with the canonical homomorphism $\tau^*X \to X$, and clearly $\rho\psi = \tau$ where $\rho: X \to E$ is the given faithfully flat homomorphism. Let R be a k_s-algebra, $x \in X(k_s)$ and $h \in H(R)$. Then, as H is central in E, we have $\rho[x(\psi h)x^{-1}(\psi h)^{-1}] = (\rho x)(\tau h)(\rho x)^{-1}(\tau h)^{-1} = e \in E(R)$, so that we may write $x(\psi h)x^{-1} = (\alpha h)(\psi h)$ with $\alpha h \in A(R)$. One verifies right away the relation $\alpha(h + h') = \alpha h + \alpha h'$ and thereby obtains a k_s-homomorphism $\alpha: H_{k_s} \to A_{k_s}$ which is constant by assumption. Consequently, $\psi(H(R))$ commutes with $X(k_s)$. But, A and E being k-smooth, X, too, is k-smooth so that $X(k_s)$ is dense in X, whence follows that $\psi(H(R))$ is central in $X(R)$. Now, since the image functor $R \longmapsto \psi(H(R))$ is central in X, the image itself is central by virtue of A.6. Let $Y = X/\mathrm{Im}\psi$. It is routine to check that Y is a central extension of G by A such that $\pi^*Y = X$, as desired.

Q.E.D.

A.9. THEOREM. <u>Let</u> $0 \to B \xrightarrow{\tau} C \xrightarrow{\pi} A \to 0$ <u>be a commutative extension of commutative</u> k-<u>group schemes and let</u> G <u>be a</u> k-<u>group scheme</u>. <u>Then, the following sequence of additive groups</u>

$$0 \to \mathrm{Hom}_{k\text{-}gr}(G,B) \to \mathrm{Hom}_{k\text{-}gr}(G,C) \to \mathrm{Hom}_{k\text{-}gr}(G,A)$$

$$\xrightarrow{\gamma} \mathrm{Ext}_{cent}(G,B) \xrightarrow{\tau_*} \mathrm{Ext}_{cent}(G,C) \xrightarrow{\pi_*} \mathrm{Ext}_{cent}(G,A) \qquad (2)$$

is exact, where γ *sends* $\phi \in \mathrm{Hom}_{k\text{-}gr}(G,A)$ *to the extension class of* ϕ^*C.

Proof of this theorem is omitted, as it is routine, similar to that of A.8. Besides, the theorem is essentially known — cf. SGAD, *loc. cit.* and DG, *loc. cit.*

A.10. *Example*. Consider $0 \to \alpha_p \to G_a \overset{F}{\to} G_a \to 0$. Coupling the three group schemes with G_a, one obtains two complexes as in A.8 and A.9. The second is exact. As for the first, the part

$$\mathrm{Ext}_{cent}(G_a,G_a) \overset{F^*}{\to} \mathrm{Ext}_{cent}(G_a,G_a) \to \mathrm{Ext}_{cent}(\alpha_p,G_a)$$

is not exact. Indeed, note that $\mathrm{Ext}_{cent}(\alpha_p,G_a) \cong k$, while $\mathrm{Ext}_{cent}(G_a,G_a)$ is a free left $k[F]$-module with a countable basis $u_o, u_1, \cdots, u_n, \cdots$ such that $F^*(u_i) = Fu_i$ for all $0 \leq i < \infty$. (See 3.6.1.) The non-exactness of the sequence above is therefore evident.

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TABLE OF NOTATION

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