Appendix

Let W be a subvariety of a domain U . Let V be a purely k-dimensional subvariety of $U-W$. The assertion of Theorem (F) is that <u>if</u> <u>$\mathrm{Vol}_{2k}(V) < \infty$</u> <u>then</u> $\bar{V}$ (<u>closure in</u> U) <u>is a subvariety of</u> U . This can be proved by first demonstrating that $H_{2k}(\bar{V} \cap W) = 0$ and then proving Theorem (E) .

The proof that $H_{2k}(\bar{V} \cap W) = 0$ closely parallels that of Proposition 3 of Chapter III . However it is convenient to make the following reductions before proceeding. Firstly, we may localize and take U to be a ball. In that case W is cut out by global equations on U so we may replace W by some suitable $\{f = 0\}$. Finally, by embedding U in a higher-dimensional space via $(z_1, \ldots, z_n) \longrightarrow (z_1, \ldots, z_n, f(z_1, \ldots, z_n))$ we can arrange that W is a hyperplane.

Therefore, from now on we shall assume that W is the hyperplane $\{z_n = 0\}$. In proving that $H_{2k}(\bar{V} \cap W) = 0$, the assumption $\mathrm{Vol}_{2k}(V) < \infty$ plays the role of the upper bound. But besides that one needs a local lower bound of the following type.

(*) There is a constant $c(k,n) > 0$ such that if $p \in \bar{V}$ and $R > 0$ is so small that $B(p;R) \subset U$ then $\mathrm{Vol}_{2k}(V \cap B(p;R)) \geq c(k,n) \cdot R^{2k}$.

The case $k = 1$ is Theorem 2 of [1]. Here is a sketch of an argument of Bishop for the general case, by induction on k.

Remark. By integrating, it suffices to find $d(k,n) > 0$ so that $\mathrm{Vol}_{2k-1}(V \cap S(p;R)) \geq d(k,n) \cdot R^{2k-1}$ where $S(p;R) = \{ p: \|p\| = R \}$. By translation and a change of scale we can restrict to the case $p = O$ and $R = 1$. Let $S = S(O;1)$.

LEMMA 1. For each $0 < r < 1$ let L_r be the hyperplane $\{ z_1 = \sqrt{1-r^2} \}$ in $\mathbb{C}^n$ and for each $w = (w_1, \ldots, w_n) \in S$ let P_w be the hyperplane $\{ \sum_{i=1}^{n} w_i z_i = 0 \}$. There is a constant $c(n) > 0$ such that

$$(1) \quad \mathrm{Vol}_{2n-3}(L_r \cap S) \geq c(n) \int \mathrm{Vol}_{2n-5}(L_r \cap S \cap P_w)\, dS(w).$$

Proof: We have

$$(2) \quad \mathrm{Vol}_{2n-3}(L_r \cap S) = c_1(n) \cdot r^{2n-3}$$

where $c_1(n)$ is the volume of the unit $2n-3$ sphere.

$L_r \cap S \cap P_w$ is a $2n-5$ sphere with equations

$$z_1 = \sqrt{1-r^2} \quad , \quad \sum_{j=2}^{n} |z_j|^2 = r^2 \quad , \quad \sum_{j=2}^{n} w_j z_j = -w_1 \cdot \sqrt{1-r^2} \ .$$

Its radius is therefore $\leq r$ if $|w_1| \leq r$ and 0 otherwise. Hence, for $c_2(n)$ = the volume of the unit 2n-5 sphere, we have

$$\int \mathrm{Vol}_{2m-5}(L_r \cap S \cap P_w)\, dS(w) \leq c_2(n)\, r^{2n-5} \int_{|w_1| \leq r} dS(w) \ .$$

But for some $c_3(n) > 0$,

$$\int_{|w_1| \leq r} dS(w) \leq c_3(n) \cdot r^2 \ .$$

Then (1) holds with $c(n) = c_1(n)/(c_2(n) \cdot c_3(n))$.

DEFINITION . A 2k-1 <u>volume element</u> U of S is an open subset of $L \cap S$ where L is a complex linear variety in $\mathbb{C}^n$ of (complex) dimension k .

LEMMA 2. <u>If</u> U <u>is a</u> 2n-3 <u>dimensional volume element of</u> S <u>then</u>

(3) $\underline{\mathrm{Vol}}_{2n-3}(U) \geq c(n) \int \underline{\mathrm{Vol}}_{2n-5}(U \cap P_w)\, dS(w)$.

<u>Proof:</u> For a proper choice of coordinates $z_1, \ldots, z_n$ and

$r > 0$ L is the L_r of Lemma 1. We may decompose $L \cap S$ into a large number N of disjoint pieces which are nearly congruent under the unitary group, in such a way that U is very nearly the union of some M of these pieces. Then the two sides of (3) are very nearly M/N times the corresponding sides of (1). Thus (3) follows from (1) in the limit.

LEMMA 3. If U is a 2k-1 dimensional volume element of S, $1 \leq k \leq n-1$, then

$$(4) \quad \underline{\mathrm{Vol}}_{2k-1}(U) \geq c(k+1) \int \underline{\mathrm{Vol}}_{2k-3}(U \cap P_w)\, dS(w) .$$

Proof: In an appropriate coordinate system $L = \{ z_1 = \sqrt{1-r^2}, z_{k+2} = 0, \dots, z_n = 0 \}$. Therefore (4) reduces to (3) with n replaced by k+1.

Now, by the remark at the beginning of this discussion what we require is a constant $d(k,n) > 0$ such that $\mathrm{Vol}_{2k-1}(V \cap S) \geq d(k,n)$. By Theorem 2 of [1] we may take $d(1,n) = 2\pi$. Suppose we already have $d(k-1, n-1)$. Then approximate $V \cap S$ by the union of finitely many 2k-1 dimensional volume elements $U_1, \dots, U_m$ of S. By (4)

$$\mathrm{Vol}_{2k-1}(V \cap S) \approx \sum_{j=1}^{m} \mathrm{Vol}_{2k-1}(U_j) \geq$$

$$\sum_{j=1}^{m} c(k+1) \int \mathrm{Vol}_{2k-3}(U_j \cap P_w)\, dS(w) \approx$$

$$c(k+1) \int \mathrm{Vol}_{2k-3}(V \cap S \cap P_w)\, dS(w) \geq$$

$$c(k+1) \int d(k-1, n-1)\, dS(w) = c(k+1)\, d(k-1, n-1)\, \mathrm{Vol}_{2n-1}(S) .$$

Passing to the limit gives

$$\mathrm{Vol}_{2k-1}(V \cap S) \geq d(k,n)$$

with $d(k,n) = c(k+1)\, d(k-1, n-1)\, \mathrm{Vol}_{2n-1}(S)$.

This completes our sketch of Bishop's argument for (*) . It is now a simple matter to modify the proof of Proposition 3 to get $H_{2k}(\bar{V} \cap W) = 0$.

As for the proof of Theorem (E) I will make only the following comment. The condition $H_{2k}(\bar{V} \cap W)$ yields (similar to Proposition 4) , locally, a projection

$$\bar{V} \cap (N_k \times N_{n-k}) \xrightarrow{\Pi} N_k \subset \mathbb{C}^k$$

that is <u>proper</u> , and such that the image of $\bar{V} \cap W \cap (N_k \times N_{n-k})$ is <u>closed</u> and of <u>measure zero</u> in N_k . Bishop than makes a close analysis of this projection Π , using Rado's Theorem and some elementary properties of representing measures for uniform algebras, to get analytic equations for $\bar{V}$ on $N_k \times N_{n-k}$. This is Lemma 9 of [1] . The reader is now invited to turn to that paper.

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Also, as a further reference on volumes of analytic varieties there is

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