Appendix

In order to develop the theory of flags and of decompositions, we introduce the notion of an "Ω cone-block bundle", where Ω is a partially ordered set. Our treatment of Ω cbb's now follows Rourke and Sanderson's [19, I and II]. Thus our Result X, Y. Z is a reformulation of [19, X, Result Y. Z]. It is usually left to the reader to reformulate Rourke and Sanderson's proof; we confine ourselves to details that did not arise in the case of block bundles.

Let Ω be a finite partially-ordered set with a unique minimal element 0 and a unique maximal element α. An Ω cone-block bundle $\{\xi\}$ over a cell complex K is a family $\{\xi_\omega : \omega \in \Omega\}$ of cone-block bundles over K such that:

Ω cbb 1 if $\omega < \omega'$ in Ω, then $|\xi_\omega| \subseteq |\xi_{\omega'}|$;

Ω cbb 2 $\xi_0 = K$;

Ω cbb 3 for each $\sigma \in K$, there are a compact polyhedron with a family of subpolyhedra indexed by Ω, say $(F_\alpha(\sigma), \{F_\omega(\sigma)\})$, and a block structure $h : \xi_\alpha(\sigma) \to \sigma \times cF_\alpha(\sigma)$ which, for every $\omega \in \Omega$, restricts to a block structure $h : \xi_\omega(\sigma) \to \sigma \times cF_\omega(\sigma)$.

We write: $\{\xi\}/K$ is an Ω cbb.

We leave the reader to define the restriction $\{\xi\} | L$ of an Ω cbb $\{\xi\}/K$ to a subcomplex $L < K$.

If $\{\xi\}/K$ and $\{\eta\}/L$ are Ω cbb's, then an Ω cbb isomorphism

$f : \{\xi\} \approx \{\eta\}$ is a p.ℓ. isomorphism $f : |\xi_\alpha| \to |\eta_\alpha|$ which restricts to a cone-block bundle isomorphism $f : \xi_\omega \approx \eta_\omega$ for every $\omega \in \Omega$.

Given Ω as before; then a family of compact polyhedra $(F_\alpha, \{F_\omega\})$ indexed by Ω is an Ω family if $F_o = \emptyset$; $F_\omega \subseteq F_{\omega'}$ whenever $\omega < \omega'$; and $F_\omega \cap F_{\omega'} =$ some $F_{\omega''}$ whenever $\omega, \omega' \in \Omega$. An Ω cbb $\{\xi\}/K$ is a c{F}-block bundle if for every $\sigma \in K$, $(F_\alpha(\sigma), \{F_\omega(\sigma)\}) = (F_\alpha, \{F_\omega\})$. The proof of Lemma 1.7 shows that given an Ω cbb $\{\xi\}/K$ and $\sigma \in K$, then $(F_\alpha(\sigma), \{F_\omega(\sigma)\})$ is unique. By Theorem I, 1.1 (below), any $\{\xi\}$ is a disjoint union of some $c\{F^i\}$-block bundles $\{\xi^i\}/K^i$, where the K^i are the components of K. If $\{F\}$ is an Ω family and K is a cell complex, then we have the trivial c{F}-bb $K \times c\{F\}$ (see Chapter 3 for the product of blocks). A trivialization of an Ω cbb $\{\xi\}/K$ is an isomorphism with some trivial c{F}-bb over K; $\{F\}$ is then uniquely determined.

Let $\{\xi\}/K$ be an Ω cbb, σ a cell of K. Then a chart for $\{\xi\}$ at σ is a trivialization $h(\sigma)$ of $\{\xi\} \vdash \langle\sigma\rangle$. An atlas for $\{\xi\}$ is a family $\{h(\sigma) : \sigma \in K,\ h(\sigma) \text{ is a chart at } \sigma\}$.

Let $(F_\alpha, \{F\})$ be an Ω-family. Then we have the disjunction $\{\mathfrak{F}_\omega : \omega \in \Omega\}$ of F_α defined by $\mathfrak{F}_\omega = F_\omega - \cup\{F_{\omega'} : F_{\omega'} \subsetneqq F_\omega\}$. Henceforth we shall always assume that $\{\mathfrak{F}_\omega\}$ is a variety of F_α. Thus if $\{\xi\}/K$ is an Ω cbb, and we write $\{\xi\}$ as a disjoint union of $c\{F^i\}$-block bundles, then the disjunction $\{\mathfrak{F}^i_\omega\}$ associated to each $\{F^i\}$ is a variety.

Theorem I, 1.1(n). Let $\{\xi\}/K$ be an Ω cbb with $|K| \underset{p.\ell.}{\approx} D^n$ (an n-disk). Then $\{\xi\}$ is trivial.

We use induction on n. For $n = 0$, K is a single vertex, and $\{\xi\}$ is trivial by Ω cbb 3. Now assume Theorem I, 1.1(n-1). We mark with an asterisk the results that depend on this assumption.

Lemma I, 1.2. Given an Ω family $\{F\}$, cells σ^1 and σ^2, and a p.ℓ. isomorphism

$$f : (\partial\sigma^1 \times c\{F\}) \cup (\sigma^1 \times \{F\}) \cup \sigma^1 \longrightarrow$$
$$\longrightarrow (\partial\sigma^2 \times c\{F\}) \cup (\sigma^2 \times \{F\}) \cup \sigma^2$$

with $f^{-1}(\sigma^2) = \sigma^1$. Then f extends to a p.$\ell$. isomorphism $f'' : \sigma^1 \times c\{F\} \to \sigma^2 \times c\{F\}$.

Pick points $x^j \in \operatorname{int} \sigma^j$ for $j = 1, 2$, and regard $\sigma^j \times c\{F\}$ as the cone from x^j on $\partial\sigma^j * \{F\}$. Then the restriction of f to $g : \partial\sigma^1 * \{F\} \to \partial\sigma^2 * \{F\}$ extends to an isomorphism $g'' : \sigma^1 \times c\{F\} \to \sigma^2 \times c\{F\}$ by extending conically to x^1 and x^2. Now g'' need not equal f on σ^1; however $g''(\sigma^1) = \sigma^2$ and g'' equals f on $\partial\sigma^1$. So there is an isotopy h_t of σ^2 rel $\partial\sigma^2$ such that $h_1 \circ g''$ equals f on σ^1. The proof of Akin's Covering Isotopy Theorem [1] shows that h_t extends to an isotopy h''_t of $\sigma^2 \times c\{F\}$ rel $\partial\sigma^2 * \{F\}$. Then $f'' = h''_1 \circ g''$ is the required isomorphism.

Proposition I, 1.3*. Let $\{\xi\}/K$ be an Ω cbb such that $|K| \underset{p.\ell.}{\approx} D^n$ and K has just one n-cell, σ^n. Let ρ^{n-1} be any (n-1)-cell in K, and let $L = K - \{\sigma, \rho\}$. Given a trivialization $t : \{\xi\} | L \approx L \times c\{F\}$. Then t extends to a trivialization $t'' : \{\xi\} \approx K \times c\{F\}$.

By Ω cbb 3 there is a block structure $h : \xi_\alpha(\sigma) \to \sigma \times cF_\alpha$

which restricts to a structure $h : \xi_\omega(\sigma) \to \sigma \times cF_\omega$ for all $\omega \in \Omega$. Let $V_\alpha = h^{-1}(\partial\sigma \times cF_\alpha \cup \sigma \times F_\alpha)$, $V_\omega = h^{-1}(\partial\sigma \times cF_\omega \cup \sigma \times F_\omega)$. Then $|\{\xi\} \restriction L|$ is a regular neighbourhood of $|L|$ rel bdy $|L|$ in $(V_\alpha, \{V_\omega\})$. Pick points $x \in \text{int}\,|L|$ and $y \in \text{int}\,\rho$. Then there is a p.ℓ. isomorphism $p : (V_\alpha, \{V_\omega\}) \to (x \cup y) * (\text{bdy}\,|L| * (F_\alpha, \{F_\omega\}))$ and we may assume p corresponds $|\{\xi\} \restriction L| \longleftrightarrow x * (\text{bdy}|L| * F_\alpha)$. Let $(W_\alpha, \{W_\omega\}) = p^{-1}(y * (\text{bdy}|L| * (F_\alpha, \{F_\omega\})))$; then $W_\alpha \cap |K| = \rho$. By Lemma I, 1.2 the p.ℓ. isomorphism $t : p^{-1}(\text{bdy}|L| * (F_\alpha, \{F_\omega\})) \longrightarrow$
$\longrightarrow \partial\rho * (F_\alpha, \{F_\omega\})$ together with the identity on ρ extend to an isomorphism $t'' : (W_\alpha, \{W_\omega\}) \to \rho \times c(F_\alpha, \{F_\omega\})$. This, together with t, gives an isomorphism (different from p)

$t_1 : (V_\alpha, \{V_\omega\}) \longrightarrow (\partial\sigma \times c(F_\alpha, \{F_\omega\})) \cup (\sigma \times (F_\alpha, \{F_\omega\}))$.

The proof now follows that given by Rourke and Sanderson, except that one uses Cohen's Regular Neighbourhood Theorem (our Theorem 1.1) instead of the regular neighbourhood theorem they quote.

Proposition I, 1.4*. Any Ω cbb $\{\xi\}/K$ with $\dim|K| \leq n$ has an atlas.

Let $\{\xi\}/K$, $\{\eta\}/L$ be Ω cbb's. Then $\{\eta\}$ is a subdivision of ξ if η_ω/L is a subdivision of ξ_ω/K (as cone block bundles) for all $\omega \in \Omega$.

Theorem I, 1.5*. 1. Let $\{\xi\}/K$ be an Ω cbb with $\dim|K| \leq n$, and let L be a cell complex subdivision of K. Then there is a subdivision $\{\eta\}/L$ of $\{\xi\}$.

2. Let $\{\xi\}/K$ be an Ω cbb with $\dim|K| \leq n$. Given $K' < K$ and a

subdivision $\{\eta'\}/L'$ of $\{\xi\}\upharpoonright K'$. Set $\{\eta\}/L = (\{\xi\}-\{\xi\}\upharpoonright K) \cup \{\eta'\}$ (as a set of blocks. Then $\{\eta\}$ is an Ω cbb and subdivides $\{\xi\}$.

Theorem I, 16.*. Let $\{\xi\}/K$ and $\{\xi'\}/K'$ be Ω cbb's with $\dim|K| \leq n$. Assume $|K| \searrow |L|$ (collapses geometrically), where L is a subcomplex of K. Given a cell complex isomorphism $h : K \approx K'$ and an extension of h to an isomorphism of Ω cbb's $h' : \{\xi\}\upharpoonright L \approx \{\xi'\}\upharpoonright hL'$. Then h and h' extend to an Ω cbb isomorphism $h'' : \{\xi\} \approx \{\xi'\}$.

Corollary I, 1.7*. Given an Ω cbb $\{\xi\}/K$ such that $\dim|K| \leq n$ and $|K| \searrow 0$. Then $\{\xi\}$ is trivial.

The proofs are the same as in [8, I, §1.] In particular, Theorem I, 1.1(n) holds, and the inductive step is proved. Theorem I, 1.1 and the starred results now hold without restriction on n.

Corollary I, 1.8. Given Ω cbb's $\{\xi\}/K \times I$ and $\{\eta\}/K \times I$, a subcomplex $L < K$, and an isomorphism $h : \{\xi\}\upharpoonright(K\times 0 \cup L\times I) \approx \{\eta\}\upharpoonright(K\times 0 \cup L\times I)$ which is the identity on $K \times 0 \cup L \times I$. Then h extends to an isomorphism $h' : \{\xi\} \approx \{\eta\}$ which is the identity on $K \times I$.

Theorem I, 1.9. Let $\{\eta\}/L$ and $\{\eta'\}/L$ be subdivisions of an Ω cbb $\{\xi\}/K$. Then $\{\eta\} \approx \{\eta'\}$.

Given an Ω cbb $\{\xi'\}/K'$ and a cell complex K of which K' is a subdivision. For each $\sigma \in K$, let $h : \{\xi'\}\upharpoonright K'\langle\sigma\rangle \to K'\langle\sigma\rangle \times c\{F\}$ be a trivialization, given by Theorem I, 1.1. For each $\omega \in \Omega$ define a block $\xi_\omega(\sigma)$ so that $h : \xi_\omega(\sigma) \to \sigma \times cF_\omega$ is a block structure. Then

$\{\xi\} = \{\xi_\omega(\sigma) : \sigma \in K,\ \omega \in \Omega\}$ is an Ω cbb over K, called the amalgamation of $\{\xi'\}$ over K. Note that $\{\xi'\}$ is a subdivision of $\{\xi\}$.

Given Ω cbb's $\{\xi\}/K$ and $\{\eta\}/L$ with $|K| = |L|$. Then $\{\xi\}$ and $\{\eta\}$ are equivalent if there are subdivisions $\{\xi'\}/K'$ of $\{\xi\}$, $\{\eta'\}/L'$ of $\{\eta\}$, and an isomorphism $h : \{\xi'\} \approx \{\eta'\}$ which is the identity on $|K|$. We write: $\{\xi\} \sim \{\eta\}$.

Now for each cell complex K and each fixed Ω family $\{F\}$, let $I(K, \{F\})$ be the set of isomorphism classes of $c\{F\}$-bb's $\{\xi\}/K$ (here all isomorphisms are understood to be the identity on K). For each polyhedron X and each fixed Ω family $\{F\}$, let $I(X, \{F\})$ be the set of equivalence classes of $c\{F\}$-bb's $\{\xi\}/K$ with $|K| = X$.

Theorem I, 1.10. The map $a : I(K, \{F\}) \to I(|K|, \{F\})$, which assigns to each isomorphism class of Ω cbb's its equivalence class, is a bijection.

Let u/X be an element of $I(X, \{F\})$, and let $Y \subseteq X$ be a subpolyhedron. Take a cell complex triangulation K of X in which Y is covered by a subcomplex $L < K$. Represent u by $\{\xi\}/K \in I(K, \{F\})$. Then the equivalence class v/Y of $\{\xi\}\restriction L$ depends only on u and Y, and is called the restriction of u to Y. We write: $v = u\restriction Y$, and sometimes, if u is represented by $\{\xi'\}/K'$, we refer to v loosely as: $\{\xi'\}\restriction Y$.

If Ω and Ω' are partially-ordered sets with unique minimal elements 0 and 0' , and unique maximal elements α and α', then $\Omega \times \Omega'$ has the partial ordering: $\omega \times \varphi' < \chi \times \psi'$ if $\omega < \chi$ and $\varphi' < \psi'$. $0 \times 0'$ is the unique minimal element, and $\alpha \times \alpha'$ the unique maximal element of $\Omega \times \Omega'$. Now let $\{\xi\}/K$ be an Ω cbb, $\{\eta'\}/L'$ an Ω'cbb. Then $\{\xi \times \eta'\}$, defined as $\{\beta \times \gamma' : \beta \in \{\xi\}, \gamma \in \{\eta'\}\}$ (see Chapter 3 for the definition of the block $\beta \times \gamma'$), is an $\Omega \times \Omega'$ cbb over $K \times L'$. $\{\xi \times \eta'\}$ is the cartesian product of $\{\xi\}$ and $\{\eta'\}$. This product gives rise to a well-defined product on equivalence classes, $I(X, \{F\}) \times I(Y, \{F'\}) \to I(X \times Y, \{F * F'\})$. Note that if $\{F\}$ is an Ω-family and $\{F'\}$ an Ω'-family such that the associated disjunctions $\{\mathfrak{F}\}$ and $\{\mathfrak{F}'\}$ are varieties, then the disjunction associated to the $\Omega \times \Omega'$ family $\{F * F'\}$ is again a variety.

Hence we can define the Whitney sum of equivalence classes u/X and v/X as $u \times v \pitchfork \Delta$, where Δ, the diagonal of $X \times X$, is identified with X by $(x, x) \longleftrightarrow x$. We write: $u \oplus v/X$.

Given $f : X \to Y$ a p. ℓ. map, and u/Y. Then the induced class f^*u/X is defined as $X \times u \pitchfork \Gamma f$, where X/X is regarded as the only element of $I(X, \{\emptyset\})$, and Γf, the graph of f, is identified with X by $(x, f(x)) \longleftrightarrow x$.

Theorem I, 1.11. Given $u/(X \times I)$; then $u = (u \pitchfork (X \times 0)) \times I$.

Corollary I, 1.12. Given homotopic maps $f, g : X \to Y$, and a class u/Y Then $f^*u = g^*u$.

Let Ω be a partially ordered set with unique minimal element 0 and unique maximal element α. Define $\Omega+$ to have as underlying set the disjoint union of Ω and an element λ, with partial ordering generated by the partial ordering on Ω together with: $\alpha < \lambda$. If $(F_\alpha, \{F_\omega\})$ is an Ω family and S^q is a sphere of sufficiently large dimension, then there is a p.ℓ. embedding $i : F_\alpha \hookrightarrow S^q$, and the $\Omega+$ family $(S^q, \{iF_\alpha, iF_\omega\})$ is unique up to p.ℓ. isomorphism. We abbreviate it to: $(S^q, \{F_\alpha, F_\omega\})$.

<u>Proposition I, 2.1.</u> Given an Ω cbb $\{\xi\}/K$ with fibre $(F_\alpha, \{F_\omega\})$. Then there is an $\Omega+$ cbb $\{\xi^*\}/K$ with fibre $(S^q, \{F_\alpha, F_\omega\})$, for some sufficiently large q, such that ξ^*_λ/K is the trivial cS^q-block bundle over K, and $\{\xi^*_\alpha, \xi^*_\omega\}/K = \{\xi\}/K$.

Take $q \geq 2(\dim|\xi_\alpha| + 1)$. Then we have $q \geq 2(\dim F_\alpha + 1)$, so there is a unique embedding of F_α in S^q. The proof of [8, I, Proposition 2.1] gives a p.ℓ. embedding $h : |\xi_\alpha| \hookrightarrow |K| \times D^{q+1}$ such that:
$h : |K| \approx |K| \times 0$ is the obvious map, where 0 is an interior point of D^{q+1}; $h^{-1}(\sigma \times D^{q+1}) = |\xi_\alpha(\sigma)|$ for every $\sigma \in K$. Let $(V, \{V_\omega(\sigma)\})$ be a regular neighbourhood of $|K| \times 0$ in $(|K| \times D^{q+1}, \{h \mid \xi_\omega(\sigma)|\})$, where ω ranges over Ω, and σ over K. Then we can define an $\Omega+$ cbb $\{''\xi^*\}/K$ by the rules:

$|''\xi^*_\lambda(\sigma)| = (\sigma \times D^{q+1}) \cap V;$

$\mathcal{G}''\xi^*_\lambda(\sigma) = \sigma;$

$\partial''\xi^*_\lambda(\sigma) = (\partial\sigma \times D^{q+1}) \cap V;$

$''\xi^{*\bullet}_\lambda(\sigma) = (\sigma \times D^{q+1}) \cap \mathrm{fr}\, V;$

$|''\xi_\omega^*(\sigma)| = V_\omega(\sigma)$;

for all $\sigma \in K$ and for all $\omega \in \Omega$. Let $\{''\xi\}/K$ be the Ω cbb $\cup\{''\xi_\omega^* : \omega \in \Omega\}$. Then there is an isomorphism of Ω cbb's $f : \{''\xi\} \approx \{\xi\}$ which is the identity on K. (This follows from the fact that $(|''\xi_\alpha|, \{|''\xi_\omega(\sigma)|\})$ is a regular neighbourhood of $|K|$ in $(|\xi_\alpha|, \{|\xi_\omega(\sigma)|\})$, which is itself a weak regular neighbourhood of $|K|$.) Under this identification f, $\{''\xi^*\}$ becomes the required $\Omega+$ cbb $\{\xi^*\}/K$.

Now let $f : L \to K$ be a cell complex map. Then we have $f \times \text{id.} : |L| \times D^{q+1} \to |K| \times D^{q+1} = |\xi_\lambda^*|$. Define the Ω cbb $\{f^\# \xi\}/L$ to have blocks:

$|f^\# \xi_\omega(\tau)| = (f\times\text{id.})^{-1}|\xi_\omega| \cap (\tau \times D^{q+1})$;

$\mathcal{C}(f^\#\xi_\omega(\tau)) = \tau$;

$\partial(f^\#\xi_\omega(\tau)) = (f\times\text{id.})^{-1}|\xi_\omega| \cap (\partial\tau \times D^{q+1})$;

$f^\#\xi_\omega^\bullet(\tau) = (f\times\text{id.})^{-1}(\xi_\omega^\bullet) \cap (\tau \times D^{q+1})$;

for all $\tau \in L$, and for all $\omega \in \Omega$. Then $\{f^\#\xi\}$ has fibre $\{F\}$ (we are assuming $\{\xi\}$ is a $c\{F\}$-bb.)

Proposition I, 1.13. If $\{\xi\}/K$ represents $u \in I(X, \{F\})$, then $\{f^\#\xi\}/L$ represents f^*u.

Corollary I, 1.14. $(f\circ g)^* = g^*\circ f^*$, for any maps f and g.

Theorem I, 4.1. Given a $c\{F\}$-bb $\{\xi\}/K$, and subdivisions $\{\xi'\}/K'$, $\{\eta'\}/K'$ of $\{\xi\}$. Let $L < K$ have induced subdivision $L' < K'$ and be such that $\{\xi'\}|L' = \{\eta'\}|L'$. Then there is an isotopy f_t of $|\xi_\alpha|$ rel $|K|$ such that:

$f_1 : \{\xi'\} \approx \{\eta'\}$;

$f_t(\{\xi'\})$ is a subdivision of $\{\xi\}$ for all t;

f_t is the identity on $|\xi_\alpha \restriction L|$.

Further, if $\Omega^* \subseteq \Omega$ is a subset, and $\xi'_{\omega^*} = \eta'_{\omega^*}$ for every $\omega^* \in \Omega^*$, then we may add: f_t is the identity on every $|\xi'_{\omega^*}|$. (Note that $\xi'_\omega = \eta'_\omega$ whenever $\omega <$ some $\omega^* \in \Omega^*$.)

The proof of this extra condition, and others like it, is an application of Akin's Covering Isotopy Theorem [1], quoted as our Theorem 1.6.

<u>Theorem I, 4.4.</u> 1. Let $(X_n, \ldots, X_o)$ be a stratified polyhedron, and let $(\xi)/K$, $(\eta)/K$ be n-flag neighbourhoods of X_o in $(X_n, \ldots, X_1)$. Then there is an isotopy f_t of $(X_n, \ldots, X_o)$ rel X_o such that

$f_1 : (\xi) \approx (\eta)$.

2. Further if $X^* <$ bdy X, and (ξ) and (η) respect X^*, with $(\xi)|K^* = (\eta)|K^*$, then we may add:

f_t is the identity on X^*.

3. Further if for some $m \leq n$, $(\xi_m, \ldots, \xi_1)/K = (\eta_m, \ldots, \eta_1)/K$ then we may add: f_t is the identity on X_m.

We shall also need a slightly different version of [8, I, Theorem 4.4]:

<u>Theorem I, 4.A.</u> 1. Let $\{\xi\}/K$ and $\{\eta\}/K$ be $c\{F\}$-bb's such that $|K|$ is a manifold, $|\xi_\omega| = |\eta_\omega|$ and $\xi^\bullet_\omega = \eta^\bullet_\omega$ for all $\omega \in \Omega$. Then

there is an isotopy f_t of $(|\xi_\alpha|, \{|\xi_\omega|, \xi_\alpha^\bullet\})$ rel $|K|$ such that $f_1 : \{\xi\} \approx \{\eta\}$.

2. Further, if $K^* < \text{bdy } K$ and $\{\xi\}|K^* = \{\eta\}|K^*$, then we may add: f_t is the identity on $|\xi_\alpha | K^*|$.

3. Further, if $\Omega^* \subseteq \Omega$ is a subset, and $\xi_{\omega^*} = \eta_{\omega^*}$ for every $\omega^* \in \Omega^*$, then we may add: f_t is the identity on $\cup\{|\xi_{\omega^*}|\}$. (Note that $\xi_\omega = \eta_\omega$ whenever $\omega <$ some $\omega^* \in \Omega^*$.)

Proposition II, 4.4. Any block decomposition over a disk is trivial. Hence a block decomposition is in face a decomposition.

Theorem II, 4.5. Given block decompositions $((\eta_1)) = [(\xi_1); (\zeta_1)]/K$ and $((\eta_2)) = [(\xi_2); (\zeta_2)]/K$, and isomorphisms $g : \xi_1 \approx \xi_2$, $h : \zeta_1 \approx \zeta_2$. Then g and h extend to an isomorphism $f : ((\eta_1)) \approx ((\eta_2))$.

Proposition II, 4.6. Given flags $(\xi)/K$ and $(\zeta)/K$. Then we have block decompositions $[(\xi) \times K; K \times (\zeta)]/K \times K$ and $[K \times (\zeta); (\xi) \times K]/K \times K$.

Let P be a cell complex triangulation of $|(\xi)|$ in which every block of (ξ) and its rim are covered by subcomplexes. Let Q be a similar triangulation of $|(\zeta)|$. Then $P \times (\zeta)/P \times K$ and $(\xi) \times Q/K \times Q$ are complementary blockings of $[(\xi) \times K; K \times (\zeta)]$ and of $[K \times (\zeta); (\xi) \times K]$.

Theorem II, 4.7. Given flags $(\xi)/K$, $(\zeta)/K$. Then there is a block decomposition $(\xi) \oplus (\zeta) = [(\xi_1); (\zeta_1)]$ such that $(\xi_1) \approx (\xi)$ and $(\zeta_1) \approx (\zeta)$; and $(\xi) \oplus (\zeta) = [(\zeta_1); (\xi_1)]$ is also a block decomposition.

For if $t : (\xi) \times (\zeta) \approx (\zeta) \times (\xi)$ is the isomorphism (of Ω cbb's, where $\Omega = \{0, \ldots, p\} \times \{0, \ldots, q\}$) which exchanges the factors, then t induces an equivalence $t^* : (\xi) \oplus (\zeta) \sim (\zeta) \oplus (\xi)$, by restriction to the diagonal Δ of $|K \times K|$. Then in the notation of the previous proposition, $(\xi) \oplus (\zeta)$ has the complementary blockings $P \times (\zeta) \upharpoonright \Delta$ and $(\xi) \times Q \upharpoonright \Delta$.

<u>Theorem II, 4.8.</u> If $((\eta)) = [(\xi); (\zeta)]/K$ is a block decomposition, then there is an isomorphism $h : ((\eta)) \approx (\xi) \oplus (\zeta)/K$; and $((\eta))$ has the block decomposition structure $[(\zeta); (\xi)]$ given by $h^{-1} \circ t^* \circ h$.

If $(\mu)/R$ is a blocking of $[(\xi); (\zeta)]$, then we can choose h to give an isomorphism $h : (\mu) \approx P \times (\zeta) \upharpoonright \Delta$. Then $(h^{-1} \circ t^* \circ h)^{-1}((\xi) \times Q \upharpoonright \Delta)$ is a blocking of $[(\zeta); (\xi)]$ complementary to (μ).

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