# APPENDIX

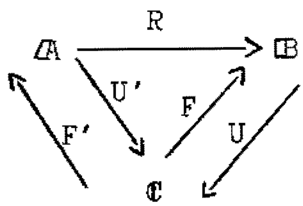
The notion of cotensor is independent of the notion of V-limit, but it is clear by now that cotensors behave in all respects as if they were V-limits. In introducing cotensors in this paper (actually, in introducing the dual concept, Chapter I Section 2) we made the (trivial) observation that in the ordinary set-based world this is actually the case. Explicitly, given any object A in a (locally small) category $\mathbb{A}$ and any set S, the formula (1) $\bar{\mathbb{A}}(S,A) = \prod_S A$ holds. This formula holds only in the set-based context, but for many closed categories $\mathbb{V}$, it is still the case that the concept of cotensor is not independent of that of the limit. That is, for those $\mathbb{V}$, cotensors are real limits, and can be constructed by means of a similar but more complicated formula generalizing formula (1) above.

In this appendix we give general conditions which (when satisfied by a closed category $\mathbb{V}$) imply that in the V-based world cotensors are real limits. More explicitly, if a closed category $\mathbb{V}$ satisfies these conditions, then any V-category $\mathbb{A}$ which has small limits is cotensored, and cotensors are constructed in terms of limits by means of a specific formula. Clearly, the same conditions imply the dual result, that is, any V-category $\mathbb{A}$ which has small colimits is tensored, and tensors are constructed in terms of colimits by means of a specific formula.

Basic in proving this result is a theorem of [8] that we now state in its generalized V-version.

Theorem A.1.

Given a V-adjoint triangle,




$(\epsilon, \eta) : F \dashv_V G$

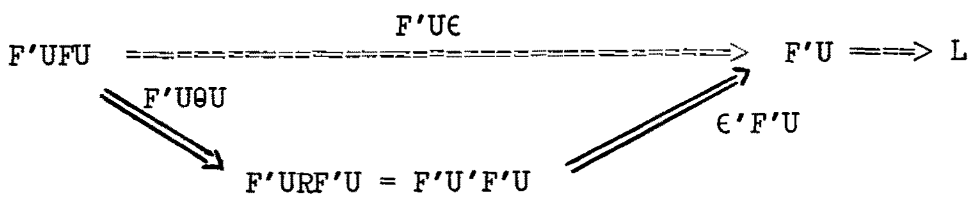
$(\epsilon', \eta') : F' \dashv_V G'$

(the vertices V-categories and the arrows V-functors, UR = U') such that the diagram:

$$FUFU \underset{\epsilon FU}{\overset{FU\epsilon}{\rightrightarrows}} FU \xrightarrow{\epsilon} id$$

is a V-coequalizer of V-functors, then, if it exists, the following coequalizer (1) of V-functor is a V-left adjoint $\mathbb{B} \xrightarrow{L} \mathbb{A}$ of $\mathbb{A} \xrightarrow{R} \mathbb{B}$.

(1)




$$\theta = (F \xrightarrow{F\eta'} FU'F' = FURF' \xrightarrow{\epsilon RF'} RF')$$

The proof given in [8] translates word by word into this general V-context, and so we do not give a proof here. ∎

Notice that the V-coequalizer (1) will exist (pointwise) if $\mathbb{A}$ has V-coequalizers. More generally, it would be enough to assume that $\mathbb{A}$ has V-coequalizers of reflexive pairs.

(the double arrow $F'U \xrightarrow{F'\eta U} F'UFU$ is a reflection for the pair of double arrows in (1)).

Now we record explicitly an observation needed to prove our next result.

Observation A.1

Let $\mathbb{V}$ be a closed category such that the base functor $\mathbb{V} \xrightarrow{\mathbb{V}_o(I,-)} \mathbb{S}$ reflects isomorphisms. Then, given any V-functor $\mathbb{B} \xrightarrow{G} \mathbb{A}$ ($\mathbb{B}$, $\mathbb{A}$ any V-categories), a functor $\mathbb{A} \xrightarrow{F} \mathbb{B}$, left adjoint to G, $(\theta_o, \epsilon, \eta) : F \dashv G$, has a structure of V-functor V-left adjoint to G.

Proof:

For any given $A \in \mathbb{A}$ (fixed) define:

$$\theta = (\mathbb{B}(FA, -) \xrightarrow{G} \mathbb{A}(GFA, G(-)) \xrightarrow{\mathbb{A}(\eta A, \square)} \mathbb{A}(A, G(-)))$$

It is clear that $\theta$ is V-natural. On the other hand, it is immediate that for any $B \in \mathbb{B}$, $\mathbb{V}_o(I, \theta B) = \theta_o AB$, so $\theta$ is an isomorphism. The result follows then from Proposition 0.2. ∎

If $\mathbb{V}$ is a closed category with small coproducts, then the base functor $\mathbb{V} \xrightarrow{\mathbb{V}_o(I, -)} \mathbb{S}$ has a left adjoint

$\mathcal{S} \xrightarrow{-\otimes_{\mathfrak{V}} I} \mathbb{V}$, $(S \otimes_{\mathfrak{V}} I = \coprod_S I)$, $id \overset{\eta}{\Longrightarrow} \mathbb{V}_o(I, -\otimes_{\mathfrak{V}} I)$,

$\mathbb{V}_o(I, -)\otimes_{\mathfrak{V}} I \overset{\epsilon}{\Longrightarrow} id$ .

Theorem A.2

Let $\mathbb{V}$ be a closed category with small coproducts such that for every $V \in \mathbb{V}$ the diagram:

$$\mathbb{V}_o(I, \mathbb{V}_o(I,V)\otimes_{\mathfrak{V}} I)\otimes_{\mathfrak{V}} I \underset{\epsilon\mathbb{V}_o(I,V)\otimes_{\mathfrak{V}} I}{\overset{\mathbb{V}_o(I, \epsilon V)\otimes_{\mathfrak{V}} I}{\rightrightarrows}} \mathbb{V}_o(I,V)\otimes_{\mathfrak{V}} I \xrightarrow{\epsilon V} V$$

is a coequalizer in $\mathbb{V}$, then

a) Any V-category $\mathbb{A}$ with small colimits is tensored, moreover, given any $A \in \mathbb{A}$ and $V \in \mathbb{V}$, there are explicitly determined maps such that the following diagram is a coequalizer in $\mathbb{A}$.

$$\text{(1.a)} \quad \mathbb{V}_o(I, \mathbb{V}_o(I,V) \otimes_{\mathfrak{V}} I)\otimes_{\mathfrak{A}} A \rightrightarrows \mathbb{V}_o(I,V)\otimes_{\mathfrak{A}} A \longrightarrow V \otimes_{\mathfrak{A}} A ,$$

Where $S \otimes_{\mathfrak{A}} A = \coprod_S A$ (for any set S).

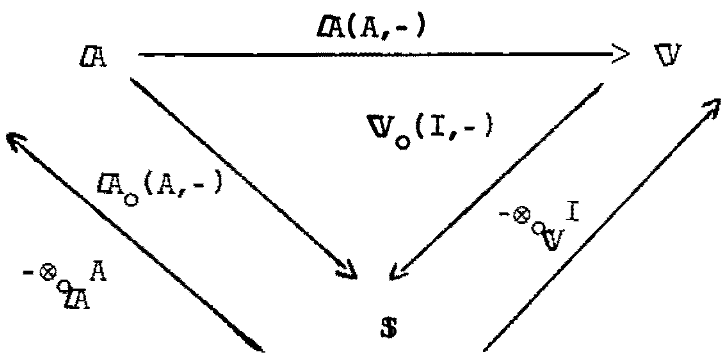
b) Dually, any V-category $\mathcal{A}$ with small limits is cotensored, moreover, given any $A \in \mathcal{A}$ and $V \in \mathcal{V}$, there are explicitly determined maps such that the following diagram is an equalizer in $\mathcal{A}$.

$$\overline{\mathcal{A}}(V,A) \longrightarrow \overline{\mathcal{A}}_o(\mathcal{V}_o(I,V),A) \rightrightarrows \overline{\mathcal{A}}_o(\mathcal{V}_o(I,\mathcal{V}_o(I,V)\otimes_{\mathcal{V}} I),\ A)$$

where $\overline{\mathcal{A}}_o(S,A) = \prod_S A$ (for any set S).

<u>Proof</u>:

a) Consider the adjoint triangle:




then, by the set-based version of Theorem A.1 $\mathcal{A}(A,-)$ has a left adjoint which is computed as the coequalizer (1.a) above. It only remains to see then that this left adjoint is actually a V-left adjoint. This follows from Observation A.1 once we notice that the conditions imposed in $\mathbb{V} \xrightarrow{\mathbb{V}_o(I,-)} \mathbb{S}$ imply that $\mathbb{V}_o(I, -)$ reflects isomorphisms.

Part b) is just part a) applied to the V-category $\mathbb{A}^{op}$ (which is cocomplete). ∎

Finally, let us remark that it is possible to see (using Theorem A.1 for example) that the condition imposed on $\mathbb{V}$ in Theorem A.2 implies that $\mathbb{V}$ is a full reflexive sub-category of the category of algebras over the monad in S determined by the pair of adjoint functors $-\otimes_{\mathbb{V}} I \dashv \mathbb{V}_o(I, -)$ .

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