N O T E S   A N D   R E M A R K S

PRELIMINARIES

For the result of P7 see Bourbaki [6], chap.I, §10, ex. 1.

The results of P8 can be found in Effros [8].

P11 can for instance be proved by applying the well known idea in the proof of 6.B of Halmos [9].

Many of the elementary properties of τ-smooth and tight measures can be found in Varadarajan [27].

P16. The fact that a τ-smooth measure on a locally compact space is tight, follows from the observation that the paving $\mathcal{G}$ of open, relatively compact sets filters to the right to X (also employ the regularity of μ). The corresponding fact for a space provided with a complete metric is proved by an "$\varepsilon \cdot 2^{-n}$-argument" where, for each n, we work with the paving $\mathcal{G}_n$ consisting of all sets that are finite unions of open spheres with radius 1/n; to get through along these lines, we just have to recall that a totally bounded subset of a complete metric space is relatively compact.

The first result of P19 is taken from Topsøe [24]. This result may be generalized, considering a class of functions instead of a class of sets, to the following result: Let $\mathcal{F}$ be a lattice of functions containing the two constant functions 0 and 1 and separating points T_2 in the sense that, for every pair (x,y) of distinct points in X, there exists a function f in $\mathcal{F}$ with $f_*(x) = 1$ and $f^*(y) = 0$; if μ_1 and μ_2 are tight, and if $\mu_1(f_*) \leq \mu_2(f^*)$ for all $f \in \mathcal{F}$, then $\mu_1 \leq \mu_2$ holds.

SECTION 2

Theorem 2.2 for X a topological space and $\mathcal{K} = \mathcal{K}(X)$ is due to Kisyński (cf. Theorem 1.2 of [14]). For a topological space, $\mathcal{G}(\mathcal{K}(X))$ is the class of open sets forthe associated k-space. Thus, if X is a k-space, for instance if X is first-countable or locally compact, we have $\mathcal{G}(\mathcal{K}(X)) = \mathcal{G}(X)$. We do not know if it is always true that $\mathcal{B}(\mathcal{K}(X)) = \mathcal{B}(X)$.

The proof of Lemma 2.4, (ii) is taken from Kisyński (cf. Theorem 1.1 of [14]).

SECTION 3

Lemma 3.5 is due to Ditlev Monrad. This lemma allowed us to drop the extra assumption that $k\setminus 1 \in \mathcal{K}$ for all $k \in \mathcal{K}^{\cdot}$, an assumption which was imposed at an ealier stage of the work when a lemma analogous to Lemma 4.7 played an important role.

Note that if $I_*(1) < \infty$ holds then the considerations following Lemma 3.5 can be simplified.

Lemma 3.7 was pointed out to us by N. Holger Petersen.

Comparing with Lemma 2.1, it is natural to conjecture that if $\mathcal{K}$ and λ satisfy the assumptions of Theorem 3.13 and if λ is τ-smooth at 0 then λ will be τ-smooth (and hence I will be τ-smooth w.r.t. $\mathcal{U}_+(\mathcal{K})$). We have proved this conjecture if we further assume that λ is locally finite (this notion beeing defined in analogy with the corresponding notion from section 2).

SECTION 4

Theorem 1,§7 is due to A. Markoff (cf. Theorem of [1]).

SECTION 5

As already mentioned, I am indebted to E.T. Kehlet for a helpful discussion. The actual arrangement of the steps (i)-(vi) in the proof of Theorem 5.1 is due to Tue Tjur (an ealier version worked with an extra assumption of local finiteness).

A similar result as Theorem 5.1 may be derived by taking as starting point a $(\emptyset,\cup f,\cap f)$-paving $\mathcal{K}_o$ and a tight and σ-smooth set-function λ_o on $\mathcal{K}_o$; then $\mathcal{K}$ should denote the $(\emptyset,\cup f,\cap c)$-closure of $\mathcal{K}_o$ and λ the set-function on $\mathcal{K}$ defined by

$$\lambda K = \inf_{K_o \supseteq K} \lambda_o K_o ; \; K \in \mathcal{K} .$$

Returning to Theorem 5.1 as it stands, note that, since $\mathcal{K}$ is a $(\emptyset,\cup f,\cap a)$-paving, $\mathcal{G}(\mathcal{K})$ defines a topology on X, and note that μ is a τ-smooth measure in this space (here, we better allow the topological space to be non-Hausdorff). Thus we have seen that "all abstract τ-smooth measure theory is topological". Had we assumed that $\mathcal{K}_o$ were a compact paving, we would find that "all abstract compact measure theory is topological"; for this to be true, we should allow non-Hausdorff topological spaces and then base the "compact" or "tight" measure theory on set-functions defined on the compact paving consisting of all closed and compact subsets.

SECTION 6

Those results of this section depending on set-functions defined on $\mathcal{G}(X)$ can be generalized by replacing $\mathcal{G}(X)$ with a $(\emptyset,\cup f,\cap f)$-paving of open sets separating points.

In the approach of Halmos, Theorem 6.2, (ii) plays an important role (cf. 53.E of [9]).

SECTION 7

The investigations of this section were stimulated by discussions with E. Alfsen.

Note that condition (iv) of Theorem 7.1 is fulfilled if X is completely regular and if the mappings $\mu \to \mu(f)$ with f bounded and continuous are lower semi-continuous.

Lemma 7.5 is a simple consequence of Theorem 2.5.2 in Michael [16], however, the proof in the text is very direct.

In establishing Lemma 7.7 we have been influenced by LeCam, who employed the idea behind the proof of (i) to establish a particular case of Theorem 9.3 (cf. Theorem 4 of [15]).

If X does not have property (*), $\mathcal{P}_+(X;t)$ need not be sequentially Hausdorff in the topology ϑ; this follows by inspection of Varadarajans example, p.225 of [27].

SECTION 8

The definition of the weak topology (or, more correctly, the topology of weak convergence) is, in a sense, new. However, due to previous research (Alexandroff [1] and Varadarajan [27]), it is a very natural definition. The definition is expressed in terms of semi-continuity, but may as well be expressed in terms of continuity, provided we change the topology of the real line.

We have chosen to work entirely in a topological set-up in part II, even though a more axiomatic setting is possible (cf. [26]). Recall, that the notes and remarks to section 5 implies that it is very likely that we will find ourselves in a topological setting anyhow.

The proof of Theorem 8.1 does not reveal anything new.

SECTION 9

The proof cf. (iv), Theorem 9.1 resembles closely that given in the notes and remarks to P16.

To my mind, the most interesting unsolved problem on compactness in the weak topology is the problem to characterize those spaces X for which the relatively compact subsets of $\mathcal{M}_+(X;t)$ and the tight subsets of $\mathcal{M}_+(X;t)$ (with μX bounded) are the same. In [27] Varadarajan claimes that every metrizable space has this property; unfortunately, his proof only works in the locally compact case (see the remark preceeding Theorem 16.3). Even if one considers the space of rationals, it does not seem to be known if the desired property holds; the best result we have been able to obtain in this special case is the following: If $\mathcal{P}$ is a compact subset of the set of measures on the rationals, and if ε is positive then there exists a closed, totally bounded and nowhere dense set F such that $\mu(\complement F) < \varepsilon$ for every $\mu \in \mathcal{P}$.

SECTION 10

Some of the results have been taken from [24].

The second condition in Theorem 10.1 may of course be replaced by the condition

$$\liminf \mu_\alpha \bar{A} \geq \mu \mathring{A}; \quad A \in \mathcal{A} .$$

Theorem 10.1 may be generalized to a class of functions, see the notes and remarks to P19.

Theorem 10.4 is, essentially, due to Varadarajan (see Theorem II.5 of [27]). In case X is regular, we may replace the main condition of Theorem 10.4 by the condition

$$\liminf \mu_\alpha \bar{A} \geq \mu \mathring{A}; \quad A \in \mathcal{A} ,$$

since, in that case, $\mathcal{A}$ will separate points and closed sets T_2.

Corollary 10.5 is a convenient generalization of Theorem 2.2 of Billingsley [3]. It is, for instance, easy to obtain the usual characterization of weak convergence on Euclidean spaces from this corollary.

SECTION 11

Theorem 11.1 is partially, if not entirely, known.

In Theorem 11.2, (ii) and (v) are known (see Varadarajan [27]) and, what we only observed recently, (iii) is known too (see Blau [5]). For similar properties see Kallianpur [12] and Varadarajan [27]. Note that the result that $\mathcal{M}_+(X;\tau)$ is separable and metrizable if and only if X is so follows from (iii) and (iv). One could add to Theorem 11.2 the result already proved that $\mathcal{M}_+^1(X;t)$ (or $\mathcal{M}_+^1(X;\tau)$) is compact if and only if X is so.

SECTIONS 12,13,14

The results given generalize those of Billingsley and Topsøe [4] and Topsøe [22] and, furthermore, contain a result announced in Topsøe [23].

The reader should observe that in our main results we assume that the measure μ is tight. This is indeed a very convenient assumption, but it is not necessary to make it (τ-smoothness will do, but it is not entirely trivial to see this).

The paper [25] contains some applications of the results of section 13 to Glivenko-Cantelli problems.

SECTION 17

Compare with the results of [23].

R E F E R E N C E S

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