Appendix

In this appendix we collect some properties of the Legendre Polynomials.

Statements, which are quoted here but not proved, may be found in standard textbooks [E 4] [S 10] :

The Legendre Polynomials are solutions of the differential equation

$$(A\ 1) \qquad (1-x^2)\, P_\ell''(x) - 2x\, P_\ell'(x) + \ell\cdot(\ell+1)\, P_\ell(x) = 0$$

with the boundary condition

$$(A\ 2) \qquad P_\ell(\pm 1) = (\pm 1)^\ell .$$

They satisfy the following recurrence formula

$$(A\ 3) \qquad (\ell+1)\, P_{\ell+1}(x) = (2\ell+1)\, x\, P_\ell(x) - \ell\, P_{\ell-1}(x) .$$

The following closed form holds:

$$(A\ 4) \qquad P_\ell(\cos\Theta) = \sum_{k=0}^{\ell} g_k\, g_{\ell-k} \cos(\ell-2k)\Theta$$

$$g_k = 2^{-2k} \binom{2k}{k}$$

which expresses P_ℓ in terms of Tchebichef polynomials.

Also

$$(A\ 5) \qquad P_\ell(-x) = (-1)^\ell\, P_\ell(x) .$$

All zero's of P_ℓ lie in the interval [-1,1] . In this interval $|P_\ell(x)| \leq 1$. Another estimate is

$$(A\ 6) \qquad |P_\ell(\cos\Theta)| \leq 2\sqrt{\frac{1}{\pi(2\ell+1)\sin\Theta}} \quad ; \; 0 < \Theta < \pi .$$

An integral representation is given by

(A 7) $$P_\ell(\cos\Theta) = \frac{1}{\pi}\int_0^\pi (\cos\Theta + i\sin\Theta\cos\phi)^\ell \, d\phi$$

and a generating function is given by

(A 8) $$(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{\ell=0}^{\infty} P_\ell(x)\, z^\ell .$$

The Legendre Polynomials satisfy the following orthogonality relation in $[-1,1]$

(A 9) $$\int_{-1}^{+1} P_\ell(x)\, P_{\ell'}(x)\, dx = \frac{\delta_{\ell,\ell'}}{\ell + \frac{1}{2}} .$$

Complete induction, using A (3) , gives

(A 10) $$\sum_{\ell=0}^{L} (2\ell+1)\, P_\ell(x) = (L+1)\,\frac{P_{L+1}(x) - P_L(x)}{x-1} .$$

Similarly

(A 11) $$P_{\ell+1}(x) > P_\ell(x) \quad ; \quad x > 1$$

(A 12) $$\frac{P_{\ell+1}(x)}{P_{\ell+1}(x')} < \frac{P_\ell(x)}{P_\ell(x')} \quad ; \quad 1 \leqslant x < x' .$$

Put $x = \cosh\Psi$ for $x > 1$. Then (A 4) gives

$$P_\ell(\cosh\Psi) \geqslant 2^{-2\ell}\binom{2\ell}{\ell}\cosh\ell\Psi$$
$$\geqslant 2^{-(2\ell+1)}\binom{2\ell}{\ell} e^{\ell\Psi} = 2^{-(2\ell+1)}\binom{2\ell}{\ell}\left(x+\sqrt{x^2-1}\right)^\ell .$$

Therefore

(A 13) $$P_\ell(x) \geqslant \frac{c_0}{\sqrt{2\ell+1}}\left(x+\sqrt{x^2-1}\right)^\ell \geqslant \frac{c_0}{\sqrt{2\ell+1}}\left(1+\sqrt{2(x-1)}\right)^\ell \qquad x \geqslant 1 ,$$

where Stirling's formula and the trivial estimate

$$\left(x + \sqrt{x^2-1}\right) \geqslant \left(1 + \sqrt{2(x-1)}\right)$$

has been used.

In chapter III b we used the following estimate

(A 14) $$\left| P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) \right| < C \, \frac{\sqrt{|\theta_1 - \theta_2|}}{(\sin\theta_1 \sin\theta_2)^{\frac{1}{4}}} \quad ; \quad 0 < \theta_1, \theta_2 < \pi$$

where C may be chosen independently of ℓ.

For the proof we first use (A 6) to obtain

(A 15) $$\left| P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) \right| < 2\sqrt{\frac{1}{\pi(2\ell+1)}} \left\{ \frac{1}{(\sin\theta_1)^{\frac{1}{2}}} + \frac{1}{(\sin\theta_2)^{\frac{1}{2}}} \right\}.$$

Now we write

$$P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) = \int_{\cos\theta_2}^{\cos\theta_1} P_\ell'(\cos\theta) \, d\cos\theta$$

where we have

(A 16) $$\left| P_\ell'(\cos\theta) \right| < C \, \frac{\sqrt{(2\ell+1)}}{(\sin\theta)^{\frac{3}{2}}} \, .$$

Therefore for θ_1, θ_2 in the interval $(0, \frac{2}{3}\pi)$, where $0.7\,\theta < \sin\theta < \theta$

(A 17) $$\left| P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) \right| < C\sqrt{2\ell+1} \, \left| \theta_1^{\frac{1}{2}} - \theta_2^{\frac{1}{2}} \right| .$$

Multiplying (A 15) and (A 17) we obtain

(A 18) $$\left| P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) \right| < C \, \frac{\sqrt{|\theta_1 - \theta_2|}}{\theta_1^{\frac{1}{4}} \, \theta_2^{\frac{1}{4}}} \, .$$

Now we may replace (A 18) by

(A 19) $$\left| P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) \right| < C \, \frac{\sqrt{|\theta_1 - \theta_2|}}{(\sin\theta_1 \sin\theta_2)^{\frac{1}{4}}}$$

which holds for $0 < \theta_{1,2} < \pi$; $0 < \theta_{2,1} < \frac{2}{3}\pi$ and

$\frac{\pi}{3} < \Theta_{1,2}$; $0 < \Theta_{2,1} < \pi$ where we used (A 5) .

For the remaining case $0 < \Theta_{1,2} < \frac{\pi}{3}$; $\frac{2}{3}\pi < \Theta_{2,1} < \pi$ we use

$$| P_\ell (\cos\Theta_1) - P_\ell (\cos\Theta_2) | < 2$$

which proves (A 14) .

We want to get some estimates on $P_\ell(z)$ if z becomes complex and varies in an ellipse with foci ± 1 . Let $z = \cos(\Theta_1 + i\Theta_2)$ (Θ_1 , Θ_2 real) .

Then

$$\mathrm{Re}\, z = \cos\Theta_1 \cosh\Theta_2$$

$$\mathrm{Im}\, z = -\sin\Theta_1 \sinh\Theta_2$$

Therefore, if we choose Θ_2 to be constant, we just obtain the equation for an ellipse with foci ± 1 and semimajor axis $\cosh\Theta_2$:

$$\left(\frac{\mathrm{Re}\, z}{a}\right)^2 + \left(\frac{\mathrm{Im}\, z}{b}\right)^2 = 1$$

$$a = \cosh\Theta_2 , \quad b = \sinh\Theta_2 , \quad a^2 - b^2 = 1 .$$

Now define

(A 20) $$d^\ell_{\mu\nu}(\Theta) = \langle \ell\mu | \exp - i\Theta J_y | \ell\nu \rangle$$

where the $|\ell\mu\rangle$ $(-\ell \leq \mu \leq \ell)$ are the basis vectors of the representation space of $SU(2)$ with angular momentum ℓ and where J_y is the representation of the generator of the rotations around the 2-axis.

Then in particular for integer ℓ

(A 21) $$d^\ell_{00}(\Theta) = P_\ell(\cos\Theta)$$

so estimates on $d^\ell_{\mu\nu}$ will give estimates on P_ℓ . Now we may write

$$d^\ell_{\mu\nu}(\Theta) = \sum_\lambda \langle \ell\mu | \ell, J_y = \lambda \rangle \langle \ell, J_y = \lambda | \ell\nu \rangle \, e^{-i\Theta\lambda}$$

which, when combined with Schwartz's inequality, leads to

$$| d^{\ell}_{\mu\lambda}(\theta)| \leq \exp \ell \theta_2 \sum_{\lambda} |\langle \ell\mu | \ell J_y = \lambda\rangle| |\langle \ell, J_y = \lambda | \ell\nu\rangle|$$
$$\leq \exp \ell \theta_2$$

So if $\cos \Theta$ varies in an ellipse with foci ± 1 and semimajor axis we have

(A 22) $$| d^{\ell}_{\mu\nu}(\Theta)| \leq \left(x + \sqrt{x^2 - 1}\right)^{\ell}$$

The combination of (A 13) and (A 22) proves the statement made on page 16.

Finally, using (A 4) , it is easy to see that $|P_\ell(z)|$ takes its maximum at the right extremity of the ellipse. Generally this is seen to be true for $|d^{\ell}_{\lambda\lambda}(\Theta)|$ if ℓ is integer and for $|\cos\frac{\Theta}{2} \cdot d^{\ell}_{\lambda\lambda}(\Theta)|$ if ℓ is half-integer. This proves the statement made on page 36 and page 114 . Assume e.g. that

$$g(z) = \sum_{\ell} a_\ell P_\ell(z) \quad , \quad a_\ell \geq 0$$

is a function analytic in an ellipse with foci ± 1 and semimajor axis $x > 1$.
Then

$$|g(z)| \leq g(x)$$

for all z in that ellipse.

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