Appendix A. The FKG inequality

Here we will prove the inequality stated in §2.2, which is due to Fortuin, Kasteleyn, and Ginibre (1971). Since it is no doubt useful for the study of percolation processes on lattices other than the square lattice, we will prove it in its original formulation as a general result about correlations on a finite distributive lattice.

A subset Γ' of a lattice Γ is called a sublattice of Γ if for any x and y in Γ', $x \vee y$ and $x \wedge y$ also lie in Γ'. A subset Γ' of a lattice is called a lower layer (semi-ideal in the terminology of [15]) if whenever $x \in \Gamma'$ and $y \in \Gamma$ with $y \leq x$, then $y \in \Gamma'$. The length of a totally ordered set of n elements is defined as the least upper bound of the lengths of the totally ordered subsets of Γ. If Γ is finite and non-empty, it has a least element 0 and a greatest element I; a minimal element $x \neq 0$ of a lattice is called an atom.

The lemma below is needed in the induction step of the proof of the main result:

> Lemma A.1 Let Γ be a finite distributive lattice with an atom a; let Γ'_a, Γ_a'', and Γ_a be the sets $\{x \in \Gamma\colon x \geq a\}$, $\{x \in \Gamma\colon x \not\geq a\}$, $\{x \in \Gamma\colon x = x' \vee a$ for some $x' \in \Gamma_a''\}$, respectively. Then

(i) Γ'_a, Γ''_a, and Γ_a are finite distributive lattices;

(ii) Γ''_a and Γ_a are (lattice) isomorphic; and

(iii) Γ_a is a lower layer of Γ'_a.

<u>Proof</u> (i). It is trivial that Γ'_a is a sublattice of Γ(and hence a lattice). If $x \in \Gamma''_a$ then $x \wedge a < a$ and therefore, since a is an atom, $x \wedge a = 0$; conversely, if $x \wedge a = 0$ then $x \in \Gamma''_a$. If x and y are in Γ''_a then $(x \wedge y) \wedge a = x \wedge (y \wedge a) = 0$; also, $(x \vee y) \wedge a = (x \wedge a) \vee (y \wedge a) = 0 \vee 0 = 0$, so Γ''_a is a sublattice of Γ. Finally, if $x', y' \in \Gamma''_a$, $x = x' \vee a$ and $y = y' \vee a$, then $x \wedge y = (x' \vee a) \wedge (y' \vee a) = (x' \wedge y') \vee a$ and $x \vee y = (x' \vee a) \vee (y' \vee a) = (x' \vee y') \vee a$. So both $x \vee y$ and $x \wedge y$ lie in Γ_a, i.e., Γ_a is a sublattice of Γ. Since Γ is finite and distributive, so are Γ'_a, Γ''_a, and Γ_a.

(ii). If $x \in \Gamma''_a$ then $x \vee a \in \Gamma_a$. Conversely, if $x \in \Gamma_a$ there exists an $x' \in \Gamma''_a$ such that $x = x' \vee a$. Suppose that also $x = x'' \vee a$ with $x'' \in \Gamma''_a$. Then $x' = x' \wedge (x' \vee a) = x' \wedge (x'' \vee a) = (x' \wedge x'') \vee (x' \wedge a) = (x' \wedge x'') \vee 0 = x' \wedge x''$, and by symmetry, $x'' = x' \wedge x''$. Hence $x' = x''$ and it follows that the mapping $x \to x \vee a$ from Γ''_a onto Γ_a is one-to-one; since $x \wedge y \to (x \wedge y) \vee a$ and $x \vee y \to (x \vee y) \vee a$, this mapping is a lattice isomorphism.

(iii). Let $x = x' \vee a$ be an element of Γ_a and suppose that $y \in \Gamma'_a$ with $y \leq x$. Then $(y \wedge x') \vee a = (y \vee a) \wedge (x' \vee a) = y \wedge x = y$;

so $y = y' \vee a$ with $y' = y \wedge x' \leq x'$, and therefore $y' \in \Gamma''_a$. Thus $y \in \Gamma_a$ by definition, and Γ_a is a lower layer of Γ'_a. If I'' is the greatest element of Γ''_a, the greatest element of Γ_a is $I'' \vee a$, and for any $x \in \Gamma_a$ the corresponding element in Γ''_a is $x' = x \wedge I''$.

In the proof of Theorem 2.4 below, no generality is lost in taking μ to be a probability measure on the power set of Γ. If Γ' is a subset of Γ, let

$$E'_\mu(f) \equiv \frac{\sum_{x \in \Gamma'} \mu(x) f(x)}{\sum_{x \in \Gamma'} \mu(x)}$$

Proof of Theorem 2.3. Note first that it is enough that f and g be increasing on the support of μ, i.e., $\{x \in \Gamma: \mu(x) > 0\} \equiv \Gamma_0$, since E_μ is determined only by this set; and (2.3) is trivial unless both x and y are in Γ_0. So we may assume that $\Gamma_0 = \Gamma$, i.e., that μ is strictly positive on Γ.

If Γ has a single element then $\ell(\Gamma) = 0$, and (2.4) holds with equality. Otherwise $\ell(\Gamma) \geq 1$ and Γ contains at least one atom. Assume now that the theorem holds for any lattice of length $\leq n-1$, and let Γ be a lattice of length $n \geq 1$ and μ a strictly positive probability measure on Γ. Let f and g be increasing functions on Γ. Let

(A.1) $\quad \theta = E_\mu(fg) - E_\mu(f)E_\mu(g) = \sum_{x,y \in \Gamma} \mu(x)\mu(y) \; [f(x)g(x) - f(x)g(y)]$.

Let a be an atom of Γ, and let Σ' and Σ'' denote the sum in (A.1) taken over all elements of Γ'_a and Γ''_a, respectively. We can rewrite θ as:

$$\text{(A.2)} \qquad \theta = \sum_x{}' \sum_y{}' \mu(x)\mu(y)\,[f(x)g(x) - f(x)g(y)]$$
$$+ \sum_x{}'' \sum_y{}'' \mu(x)\mu(y)\,[f(x)g(x) - f(x)g(y)]$$
$$+ \sum_x{}' \sum_y{}'' \mu(x)\mu(y)\,[f(x)g(x) - f(x)g(y) + f(y)g(y) - f(y)g(x)].$$

Clearly μ satisfies (2.3) on the sublattices Γ'_a, Γ''_a, and Γ_a, and f and g are increasing on these lattices. Now $\ell(\Gamma'_a) = n-1$ and $\ell(\Gamma''_a) \leq n-1$, so by the induction hypothesis, the first two sums in (A.2) are non-negative. In the third sum we can again use the induction hypothesis on the first and third terms; putting all of this together we have (omitting the summation variables):

$$\text{(A.3)} \qquad \theta \geq \frac{\Sigma'\mu f\ \Sigma'\mu g\ \Sigma''\mu}{\Sigma'\mu} + \frac{\Sigma'\mu\ \Sigma''\mu f\ \Sigma''\mu g}{\Sigma''\mu}$$
$$- \Sigma'\mu f\ \Sigma''\mu g - \Sigma'\mu g\ \Sigma''\mu f,$$

which becomes

$$\text{(A.4)} \qquad \theta \geq (\Sigma'\mu\ \Sigma''\mu)^{-1}\,(\Sigma'\mu f\ \Sigma''\mu - \Sigma'\mu\ \Sigma''\mu f)\,(\Sigma'\mu g\ \Sigma''\mu - \Sigma'\mu\ \Sigma''\mu g).$$

We now claim that all factors on the right-hand side of (A.4) are non-negative, i.e., that

$$\text{(A.5)} \qquad E''_\mu(f) \leq E'_\mu(f), \quad E''_\mu(g) \leq E'_\mu(g).$$

If this can be established it will follow that $\theta \geq 0$, and the theorem will be proved.

To do this, we will show that

(A.6) $\quad E''_\mu(f) \le E^a_\mu(f) \le E'_\mu(f)$,

where $\quad E^a_\mu(f) \equiv \dfrac{\sum_{\Gamma_a} \mu(x)f(x)}{\sum_{\Gamma_a} \mu(x)}$.

Consider the first inequality in (A.6); if $x \in \Gamma''_a$ and $y \in \Gamma''_a$ with $y \le x$, then (2.3) implies that

(A.7) $\quad \mu(x)\mu(y\vee a) \le \mu(x\wedge(y\vee a))\mu(x\vee(y\vee a)) = \mu(y)\mu(x\vee a)$.

Therefore, if $\mu_a(x) \equiv \mu(x\vee a)$ for all $x \in \Gamma''_a$, the function $\frac{\mu}{\mu_a}$ is decreasing on Γ''_a. On the other hand, the function $f_a(x) \equiv f(x\vee a)$ is increasing on Γ''_a, since f increases on Γ_a. We apply the induction hypothesis to Γ''_a with the probability measure $\frac{\mu_a}{\Sigma''\mu_a}$, which satisfies (2.3); then

(A.8) $\quad \Sigma''\mu_a \; \Sigma''\mu f_a \le \Sigma''\mu \; \Sigma''\mu_a f_a$.

Since f is increasing on Γ, $f \le f_a$ on Γ''_a; this and (A.8) give the first inequality in (A.6).

For the second, observe that

(A.9) $\quad E^a_\mu(f) = \dfrac{E'_\mu(f1_{\Gamma_a})}{E'_\mu(1_{\Gamma_a})}$.

Since Γ_a is a lower layer of Γ'_a, 1_{Γ_a} is decreasing on Γ'_a, so that

(A.10) $$E'_{\mu}(f1_{\Gamma_a}) \leq E'_{\mu}(f)\, E'_{\mu}(1_{\Gamma_a}),$$

which by (A.9) gives the second inequality of (A.6).

The authors observe in [15] that the condition in (2.3) is not a necessary condition for increasing functions on Γ to have positive correlations, when $\ell(\Gamma) > 2$.

Appendix B. Ergodic theorems for subadditive processes

We give here a proof of Theorem 2.7 and a sketch of the proof of Theorem 2.8.

> Lemma B.1 Let $\{x_{mn}\}$ be subadditive with time constant γ, and let $\xi = \limsup_n \frac{x_{on}}{n}$.
> Then ξ is a.s. finite and $E(\xi) = \gamma$; further, $\frac{x_{on}}{n} \to \xi$ in L^1.

Proof. Here we follow Kingman (1968). By subadditivity, for any fixed n,

$$(B.1) \qquad x_{ot} \leq \sum_{r=1}^{[t/n]} x_{(r-1)n,\, rn} + x_{[t/n]n,t}$$

$$\leq \sum_{r=1}^{[t/n]} x_{(r-1)n,rn} + w_{[t/n]n,t} \, ,$$

where $w_{[t/n]} = \sum_{u=0}^{n-1} |x_{[t/n]n+u\,,\,[t/n]n+u+1}|$.

By the Birkhoff ergodic theorem (using (2.6) and (2.7) of Chapter 2), the limit

$$z_n = \lim_N \frac{1}{N} \sum_{r=1}^{N} x_{(r-1)n,rn}$$

exists a.s., and $E(z_n) = E(x_{on}) = g_n$.

Also, for any $\epsilon > 0$,

$$\sum_{N=1}^{\infty} P(w_N \geq N\epsilon) = \sum_{N=1}^{\infty} P(w_o \geq N\epsilon) \leq \frac{1}{\epsilon} E(w_o) < \infty,$$

so that by the Borel-Cantelli lemma,

$$\frac{W_N}{N} \to 0 \text{ a.s.}$$

Inequality (B.1) then gives

$$\xi \leq \limsup_{t\to\infty} \frac{x_{ot}}{[t/n]n} \leq \limsup_{t\to\infty} \frac{1}{[t/n]n} \{ \sum_{r=1}^{[t/n]} x_{(r-1)n,rn} + w_{[t/n]} \}$$

$$= \frac{z_n}{n}.$$

In particular, $\xi < \infty$ and $E(\xi) \leq \frac{g_n}{n}$. Since this holds for all n,

(B.2) $\quad E(\xi) \leq \gamma$.

Now let

$$b_{st} = \sum_{r=s+1}^{t} x_{r-1,r} - x_{st} ;$$

b_{st} is then a non-negative superadditive process. If

$$B_n = \inf_{t\geq n} \frac{b_{ot}}{t} ,$$

then B_n is an increasing sequence, converging as $n \to \infty$ to

$$\liminf_{t\to\infty} \frac{b_{ot}}{t} = \liminf_{t\to\infty} \frac{1}{t} (a_{ot} - x_{ot}) = z_1 - \xi .$$

By monotone convergence,

$$\lim_{n\to\infty} E(B_n) = E(\lim_n B_n) = E(z_1 - \xi) = g_1 - E(\xi) .$$

But we also have

$$\lim_{n\to\infty} E(B_n) = \lim_{n\to\infty} E(\frac{b_{on}}{n}) = \lim_{n\to\infty} \frac{ng_1 - g_n}{n} = g_1 - \gamma .$$

Hence $g_1 - E(\xi) \leq g_1 - \gamma$, so that

(B.3) $E(\xi) \geq \gamma$.

From (B.2) and (B.3) we see that $E(\xi) = \gamma$.

Finally,

$$E\left|\frac{b_{ot}}{t} - B_t\right| = E\left(\frac{b_{ot}}{t} - B_t\right) = g_1 - \frac{g_t}{t} - E(B_t)$$

$$\rightarrow (g_1 - \gamma) - (g_1 - \gamma) = 0,$$

so that $\frac{b_{ot}}{t} - B_t \rightarrow 0$ in L^1 as $t \rightarrow \infty$. Since B_t is monotone and converges a.s. to $z_1 - \xi$, it converges in mean, and so $\frac{b_{ot}}{t} \rightarrow z_1 - \xi$ in mean. But by Birkhoff's theorem, $\frac{a_{ot}}{t} \rightarrow z_1$ in mean, and so

$$\frac{x_{ot}}{t} = \frac{a_{ot} - b_{ot}}{t} \rightarrow \xi \quad \text{in } L^1 \text{ as } t \rightarrow \infty .$$

For the other half of the proof we follow the argument given by Burkholder in the discussion of [35].

Lemma B.2 There is a stationary sequence $f_o, f_1, \ldots$ such that $E(f_o) = \gamma$ and

$$\sum_{k=s}^{t-1} f_k \leq x_{st} \qquad 0 \leq s < t.$$

Proof. Let

(B.4) $$f_{kn} = \frac{1}{n} \sum_{r=1}^{n} (x_{k,k+r} - x_{k+1,k+r}) .$$

By (2.6), the sequence $\underset{\sim}{f}_o = (f_{on})$, $\underset{\sim}{f}_1 = (f_{1n})$, ..., is stationary.

Take t with $t \geq k+1 \geq s+1$, and $n > t$. Since $x_{kr} - x_{k+1,r} \leq x_{k,k+1}$ for all $k+1 \leq r$,

$$nf_{kn} = \sum_{r=k+1}^{k+n} (x_{kr} - x_{k+1,r}) \leq \sum_{r=t+1}^{r} (x_{kr} - x_{k+1,r}) + t\, x_{k,k+1}.$$

Therefore,

$$n \sum_{k=s}^{t-1} f_{kn} \leq \sum_{r=t+1}^{n} (x_{sr} - x_{tr}) + t \sum_{k=s}^{t-1} x_{k,k+1} .$$

By (2.5), the first sum on the right-hand side is dominated by

$\sum_{r=t+1}^{n} x_{st} - (n-t)x_{st}$. Thus

$$\text{(B.5)} \qquad \sum_{k=s}^{t-1} f_{kn} \leq x_{st} + \frac{1}{n} w_{st} \quad \text{provided that } n > t,$$

where $w_{st} = t\,[\sum_{k=s}^{t-1} x_{k,k+1} - x_{st}]$.

Now $f_{on} \leq x_{o1}$ and by (B.4),

$$E(f_{on}) = \frac{1}{n} \sum_{r=1}^{n} (g_r - g_{r-1}) = \frac{1}{n} g_n \geq \gamma.$$

So $E|f_{on}| \leq E|x_{o1}| + E(x_{o1} - f_{on}) \leq E|x_{o1}| + g_1 - \gamma$

and the sequence (f_{on}) is L^1-bounded. By a theorem of Komlòs (1967) there is a sequence $n_1 < n_2 < \ldots$ of positive integers and an integrable function f_o such that

$$A_o^j = \frac{1}{j} \sum_{i=1}^{j} f_{on_i} \to f_o \text{ a.s. as } j \to \infty.$$

By stationarity,

$$A_k^j = \frac{1}{j} \sum_{i=1}^{j} f_{kn_i} \to f_k \text{ a.s. as } j \to \infty$$

and $(f_o, f_1, \ldots)$ is a stationary sequence. Given $t \in N$, let

$$i_t = \inf \{i: t < n_i\}$$

and define

$$A_o^j(t) = \frac{1}{j} \sum_{i=i_t}^{i_t+j-1} f_{on_i}$$

with a similar definition of $A_k^j(t)$. Clearly $A_k^j(t) \to f_k$ a.s. as $j \to \infty$ for any t and $k \in N$. By (B.5),

$$\sum_{k=s}^{t-1} A_k^j(t) \leq x_{st} + w_{st} \left[\frac{1}{j} \sum_{i_t}^{i_t+j-1} \frac{1}{n_i}\right] .$$

Letting $j \to \infty$, we get $\sum_{k=s}^{t-1} f_k \leq x_{st}$.

Let $\{x_{mn}\}$ be subadditive with time constant γ; putting Lemmas B.1 and B.2 together we can prove that $\frac{x_{on}}{n}$ converges a.s. Let $y_{mn} = \sum_{k=m}^{n-1} f_k$, as in Lemma B.2. Then $x_{mn} = y_{mn} + z_{mn}$, where $\{z_{mn}\}$ is non-negative and subadditive. Also,

$$E(\frac{z_{on}}{n}) = E(\frac{x_{on}}{n}) - E(\frac{y_{on}}{n}) = \frac{g_n}{n} - \gamma \to 0 \quad \text{as} \quad n \to \infty,$$ so the process $\{z_{mn}\}$ has time constant zero.

Applying Lemma B.1, if $\eta = \limsup_{n\to\infty} \frac{z_{on}}{n}$, we have $\eta \geq 0$, since $z_{on} \geq 0$, but $E(\eta) = 0$. Hence $\eta = 0$ a.s. and $\frac{z_{on}}{n} \to 0$ a.s. By the ergodic theorem, $\frac{\;\;_{on}}{n} \to \xi$ a.s. and in L^1, so that $\frac{x_{on}}{n} \to \xi$ a.s. and in L^1.

Finally suppose that the process $\{x_{mn}\}$ is independent. Let $\mathcal{T}$ be the σ-field of events generated by $\{x_{mn}\}$ which are invariant under the shift θ: $x_{mn} \to x_{m+1,n+1}$. It is easy to see that ξ is measurable in $\mathcal{T}$; since $\{x_{mn}\}$ is independent, $\mathcal{T}$ is trivial, and it follows that $\xi = \gamma$ a.s. This completes the proof of Theorem 2.7.

The proof of Theorem 2.8 rests on Theorem B.3 below. As observed by Dunford (1951), this theorem is essentially a consequence of a mean ergodic theorem of F. Riesz (1938); the proof is implicit in the proof of the main theorem in Dunford's paper.

<u>Theorem B.3</u> Let $\{x_{\underset{\sim}{j}}\}_{\underset{\sim}{j} \in N^2}$ be an array of random variables stationary under the shifts θ_1: $x_{i,j} \to x_{i+1,j}$ and θ_2: $x_{i,j} \to x_{i,j+1}$. Let $S_{\underset{\sim}{n}} = \sum_{\underset{\sim}{j} \leq \underset{\sim}{n}} x_{\underset{\sim}{j}}$ and, if $\underset{\sim}{n} = (n_1, n_2)$, let $|\underset{\sim}{n}| = n_1 n_2$. Then if $x_{(1,1)}$ is integrable, $\frac{S_{\underset{\sim}{n}}}{|\underset{\sim}{n}|}$ converges in L^1.

The proof of Theorem 2.8 proceeds as in Lemma B.1 , using Theorem B.3 in place of the Birkhoff theorem, and a standard diagonal argument; the details may be found in [47].

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