Concluding Remarks

1° For the proof of the unique continuation theorem (Proposition 1.4) we referred to the unique continuation theorem for partial differential equation. On the other hand Jäger [3] gives the unique continuation theorem for an abstract ordinary differential operator with operator-valued coefficients which can be referred to in our case.

2° Our proof of the limiting absorption principle was along the line of Saitō [3]. In Ikebe-Saitō [1] and Lavine [1] the Schrödinger operator was directly treated and the limiting absorption principle for the Schrödinger operator with a long-range potential was proved.

3° In §5 we introduced the kernel $Z(y,k)$ of the stationary modifier $\lambda(y,k)$ as a solution of the equation

$$(13.1) \qquad D^j \{2kZ(y) - Q_0(y) - Z(y)^2 - \varphi(y)\} = O(|y|^{-j-\tilde{\varepsilon}}) \quad (j=0,1,\ \tilde{\varepsilon}>1)$$

Noting that $Z(y)^2 + \varphi(y) = (\text{grad } \lambda)^2$, we can rewrite (13.1) as

$$(13.2) \qquad D^j \{2k \frac{\partial\lambda}{\partial|y|} - Q_0(y) - (\text{grad } \lambda)^2\} = O(|y|^{-j-\tilde{\varepsilon}}) \quad (j=0,1,\ \tilde{\varepsilon}>1)$$

4° Our proof of Theorems 5.3 and 5.4 is a unification of the ones in Saitō [3] and [6]. The method of the proof has its origin in Jäger [3] in which he treated an differential operator with operator-valued, short-range coefficients.

5° In §5 we defined the stationary modifier by a sort of successive approximation method. The condition $(\tilde{Q}_0)$ was assumed so that the successive approximation process may be effective. As a result our long-range potential $Q_0(y)$ is assumed to satisfy the estimates for the derivatives $D^j Q_0$, $j=0,1,2,\ldots,m$ and m is a rather large number. Here it should be mentioned that our proof of Theorems 5.3 and 5.4 is effective as far as we can construct a stationary modifier which satisfies (1) of Remark 5.9. Hörmander [1] constructed the time-dependent modifier $W(y,t)$ for a type of elliptic operators with more general long-range coefficients than ours. Kitada [2] showed that a stationary modifier satisfying (1) of Remark 5.9 can be constructed by starting with Hörmander's time-dependent modifier. If we use Kitada's stationary modifier we can replace $(\tilde{Q}_0)$ by

$(\hat{Q}_0)$. There exist constants C_0 and $0 < \varepsilon \leqq 1$ such that $Q_0(y)$ is a C^4 function and

$$|D^j Q_0(y)| \leqq C_0(1+|y|)^{-d(j)} \qquad (y \in \mathbf{R}^N, j=0,1,2,3,4) \quad ,$$

where D^j denote an arbitrary derivative of j-th order and $d(j) = j+\varepsilon_0$ $(j=1,2,3)$, $d(4) > 0$, $d(1) + d(4) > 5$.

But in this lecture I adopted the primitive method of successive approximation, because we need further preparations with respect to the theory of partial differential equation in order to introduce the more minute method which starts with Hörmander [1].

6° Theorem 8.7 was first stated and proved explicitly by Kitada [2].

7° Ikebe [4] gave the proof of the orthogonality of the generalized Fourier transforms by treating the Schrödinger operator directly and making use of the Lippmann-Schwinger equation. Our proof of the orthogonality of $F_{\pm}$ is different in using Proposition 8.5 instead of the Lippmann-Schwinger equation.

8° The modified wave operators. The time-dependent modified wave operators $W_{D,\pm}$ for the Schrödinger operator with a long-range potential we defined by Alsholm-Kato [1], Alsholm [1] and Buslaev-Matveev [1] as

$$(13.3) \qquad W_{D,\pm} = \text{s-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0-iX_t} ,$$

where X_t is a function of H_0 . On the other hand from the viewpoint of the stationary method the stationary wave operator $\tilde{W}_{D,\pm}$ should be defined by

$$(13.4) \qquad \tilde{W}_{D,\pm} = \tilde{F}^*_{\pm} \tilde{F}_{0,\pm} ,$$

$\tilde{F}_{0,\pm}$ being the generalized Fourier transforms associated with H_0 . From the orthogonality of the generalized Fourier transforms the completeness of $\tilde{W}_{D,\pm}$ follows immediately. Recently the relation $\tilde{W}_{D,\pm} = \tilde{W}_{D,\pm}$ is shown by Kitada [1], [2], [3] and Ikebe-Isozaki [1], whence follows the completeness of the time-independent modified wave operator $W_{D,\pm}$.

9° In §12 we treated the case that $Q_1(y)$ is a general short-range potential. Then we approximated $Q_1(y)$ by a sequence $\{Q_{1n}(y)\}$, where $Q_{1n}(y)$ has compact support in $\mathbf{R}^N$. But there is another method which starts with the relations

$$(13.5) \qquad (L-k^2)^{-1} = (L_1-k^2)^{-1} \{1-C_1(L-k^2)^{-1}\}$$

where $L_1 = -\frac{d^2}{dr^2} + B(r) + C_0(r)$, 1 is the identity operator and

$v = (L-k^2)^{-1}f$ means the radiative function for $\{L,k,\ell[f]\}$. Then $\Gamma(k)$ can be defined by

$$\Gamma(k) = F_1(k)\ \{1-C_1(L-k^2)^{-1}\} \tag{13.6}$$

This method was adopted in Ikebe [3].

10° In Theorem 11.9 we introduced the generalized Fourier transforms $\tilde{F}_{\pm}$ associated with H. Let $F_{0,\pm}$ be the generalized Fourier transforms associated with H_0, the self-adjoint realization of $-\Delta$. In this case the Green kernel $G_0(r,s,k)$ can be represented by the use of the Hankel function and the exact form of $F_{0,\pm}$ is known as

$$(F_{0,\pm}\phi)(\xi) = C_I(N)(2\pi)^{-\frac{N}{2}} \underset{R\to\infty}{\text{l.i.m.}} \int_{|y|<R} e^{\mp iy\xi}\phi(y)dy \tag{13.7}$$

in $L_2(\mathbf{R}^N,d\xi)$ where

$$C_I(N) = -e^{\mp\frac{\pi}{4}(N-1)i} \tag{13.8}$$

For proof see Saitō [2], §7. This means that $\tilde{F}_{0,\pm}$ are essentially the usual Fourier transforms and the $\tilde{F}_{\pm}$ are the generalization of the usual Fourier transforms in this sense.

11° As was stated in the Introduction, an oscillating long-range potential $Q_0(y)$ such as $Q_0(y) = \frac{1}{|y|}\sin|y|$ does not satisfy any assumption. As for the Schrödinger operator with an oscillating long-range potential we can refer to Mochizuki-Uchiyama [1], [2].

12° Finally let us give two remarks on the condition (5.2) in Assumption 5.1 on a long-range potential Q_0. In order to give a unified treatment for $0 < \varepsilon \leqq 1$ we assumed the condition (5.2). But in the case of $\frac{1}{2} < \varepsilon \leqq 1$ we can adopt a weaker condition $m_0 = 2$. In fact in this case we have

$$(13.9) \qquad Z(y) = \frac{1}{2k} \int_0^r \mathcal{Q}_0(t\omega)dt \qquad (y = r\omega),$$

and the first condition of (5.41) can be weakened as

$$(13.10) \qquad |(D^j Z)(y)| \leqq C(1+|y|)^{-j-\varepsilon} \qquad (j=0,1,2),$$

because the condition

$$(13.11) \qquad |D^3 Z(y)| \leqq C(1+|y|)^{-3-\varepsilon}$$

is used to estimate the term $((Z' + P)u, Bu)_X$ in the proof of Lemma 6.5 only and we can directly estimate this term without making use of (13.11) in the case of $\frac{1}{2} < \varepsilon \leqq 1$. Thus (5.2) can be replaced by

$$(13.12) \qquad m_0 > \frac{2}{\varepsilon} - 1 \quad (0 < \varepsilon \leqq \tfrac{1}{2}) \quad \text{and} \quad m_0 = 2 \quad (\tfrac{1}{2} < \varepsilon \leqq 1) .$$

Next let us consider the case the $\mathcal{Q}_0(y)$ is spherically symmetric, i.e. $\mathcal{Q}_0(y) = \mathcal{Q}_0(|y|)$. In this case the stationary modifier $Z(y)$ is also spherically symmetric and the operator M, the functions $\varphi(y;\lambda)$, $P(y;\lambda)$ are all identically zero. Therefore the proof of Theorem 5.4 becomes much simpler. For example, we do not need Lemma 6.6. Moreover (5.35), by which $Z(y)$ is defined, takes the following simpler form:

$$(13.13) \qquad \begin{cases} Z_1(y) = \dfrac{1}{2k} \mathcal{Q}_0(y) , \\ Z_{n+1}(y) = \dfrac{1}{2k}\{\mathcal{Q}_0(y) + (Z_n(y))^2\} , \end{cases}$$

and we set $Z(y) = Z_{n_0}$ where n_0 is the least integer such that

$(n_0 + 1)\varepsilon > 1$. Noting that the right-hand side of (13.13) does not contain any derivative of $Z_n(y)$, we can replace (5.2) by

(13.14) $$m_0 = 2 \quad (\text{if } \mathcal{Q}_0(y) \text{ is spherically symmetric}).$$

REFERENCES

S. Agmon

[1] Spectral properties of Schrödinger operators and scattering theory, Ann. Scuola Nor. Sup. Pisa (4) 2 (1975), 151-218.

P.K. Alsholm

[1] Wave operators for long-range scattering, Thesis, UC Berkeley (1972).

P.K. Alsholm and T. Kato

[1] Scattering with long-range potentials, Proc. Symp. Pure Math. Vol XIII (1973), 393-399.

N. Aronszajn

[1] A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, J. de Math. 36 (1957), 235-247.

E. Buslaev and V. B. Matveev

[1] Wave operators for the Schrödinger equations, with a slowly decreasing potential, Theoret. and Math. Phys. 2(1970), 266-274.

J.D. Dollard

[1] Asymptotic convergence and the Coulomb interaction, J. Mathematical Phys. 5 (1964), 729-738.

A. Erdélyi and others

[1] Higher transcendental function II, McGraw-Hill, New York, 1953.

L. Hörmander

[1] The existence of wave operators in scattering theory; Math. Z. 149(1976), 69-91.

T. Ikebe

[1] Eigenfunction expansions associated with the Schrödinger operators and their application to scattering theory, Arch. Rational Mech. Anal. 5(1960), 1-34.

[2] Spectral representations for the Schrödinger operators with long-range potentials, J. Functional Analysis 20 (1975), 158-177.

[3] Spectral representations for the Schrödinger operators with long-range potentials, II, Publ. Res. Inst. Math. Sci. Kyoto Univ. 11 (1976), 551-558.

[4] Remarks on the orthogonality of eigenfunctions for the Schrödinger operator in $\mathbf{R}^n$ J. Fac. Sci. Univ. Tokyo 17 (1970), 355-361.

T. Ikebe and H. Isozaki

[1] Completeness of modified wave operators for long-range potentials, Preprint.

T. Ikebe and T. Kato

[1] Uniqueness of the self-adjoint extension of singular elliptic differential operators, Arch. Ratinal Mech. Anal. 9 (1962), 77-92.

T. Ikebe and Y. Saitō

[1] Limiting absorption method and absolute continuity for the Schrödinger operator, J. Math. Kyoto Univ. 7(1972), 513-542.

W. Jäger

[1] Über das Dirichletsche Aussenraumproblem für die Schwingungsgleichung, Math. Z. 95 (1967), 299-323.

[2] Zur Theorie der Schwingungsgleichung mit variablen Koeffizienten in Aussengebieten, Math. Z. 102 (1967). 62-88.

[3] Das asymptotische Verhalten von Lösungen eines typus von Differential gleichungen, Math. Z. 112 (1969), 26-36.

[4] Ein gewöhnlicher Differentialoperator zweiter Ordnung für Funktionen mit Werten in einem Hilbertraum, Math. Z. 113 (1970) 68-98.

T. Kato

[1] Perturbation Theory for Linear Operators, Second Edition, Springer, Berlin, 1976.

H. Kitada

[1] On the completeness of modified wave operators, Proc. Japan. Acad. 52 (1976), 409-412.

[2] Scattering theory for Schrödinger operators with long-range potentials, I, J. Math. Soc. Japan 29 (1977), 665-691.

[3] Scattering theory for Schrödinger operators with long-range potentials, II, J. Math. Soc. Japan 30 (1978), 603-632.

S. T. Kuroda

[1] Construction of eigenfunction expansions by the perturbation method and its application to n-dimensional Schrödinger operators, MRC Technical Summary Report No. 744 (1967).

R. Lavine

[1] Absolute continuity of positive spectrum for Schrödinger operators with long-range potentials, J. Functional Analysis 12 (1973), 30-54.

K. Mochizuki and J. Uchiyama

[1] Radiation conditions and spectral theory for 2-body Schrödinger operators with "oscillatory" long-range potentials, I, the principle of limiting absorption, to appear in J. Math. Kyoto Univ.

[2] Radiation conditions and spectral theory for 2-body Schrödinger operators with "oscillatory" long-range potentials, II, spectral representation, to appear in J. Math. Kyoto Univ.

Y. Saitō

[1] The principle of limiting absorption for second-order differentia equations with operator-valued coefficients, Publ. Res. Inst. Math. Sci. Kyoto Univ. 7 (1971/72), 581-619.

[2] Spectral and scattering theory for second-order differential operators with operator-valued coefficients, Osaka J. Math. 9(1972), 463-498.

[3] Spectral theory for second-order differential operators with long-range operator-valued coefficients, I, Japan J. Math. New Ser. 1 (1975), 311-349.

[4] Spectral theory for second-order differential operators with long-range operator-valued coefficients, II, Japan J. Math. New Ser. 1 (1975), 351-382.

[5] Eigenfunction expansions for differential operators with operator-valued coefficients and their applications to the Schrödinger operators with long-range potentials, International Symposium on Mathematical Problems in Theoretical Physics. Proceedings 1975, Lecture Notes in Physics 39(1975), 476-482, Springer.

[6] On the asymptotic behavior of the solutions of the Schrödinger equation $(-\Delta + Q(y)-k^2)V = F$, Osaka J. Math. 14(1977), 11-35.

[7] Eigenfunction expansions for the Schrödinger operators with long-range potentials $Q(y) = O(|y|^{-\varepsilon})$, $\varepsilon > 0$, Osaka J. Math. (1977), 37-53.

D. W. Thoe

[1] Eigenfunction expansions associated with Schrödinger operators in $\mathbb{R}^n$, $n \geqq 4$, Arch. Rational Mech. Anal. 26(1967), 335-356.

K. Yosida

[1] Functional Analysis, Springer, 1964.

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