

A P P E N D I X

(1) Proposition. If E is an admissible vector space, then the scalar multiplication: $\mathbb{R} \times E \rightarrow E$ is equably continuous.

Proof. Let $\mathfrak{X} \downarrow E$. Then by (2.1.1), (2.4.2) and (7.1.1)

$$\mathfrak{Y} = (\mathfrak{X} \vee (-\mathfrak{X}))^\wedge \downarrow E.$$

If $Y \in \mathfrak{Y}$, then $Y \supset (X \cup (-X))^\wedge$, $X \in \mathfrak{X}$. Thus for any $x \in X$ the segment $[-x, x] \subset (X \cup (-X))^\wedge$ and consequently $\lambda x \in (X \cup (-X))^\wedge$ for each $|\lambda| \leq 1$ and $x \in X$, hence $I_1 X \subset Y$ where $I_1 = [-1, 1]$, and therefore $I_1 \mathfrak{X} \leq \mathfrak{Y} \downarrow E$. By (2.4.2) also $\beta \cdot I_1 \mathfrak{X} = I_\beta \mathfrak{X} \downarrow E$ for each $\beta \neq 0$. This proves the equable continuity of the scalar multiplication by (2.8.8) and (2.5.1).

(2) Proposition. Let \mathfrak{X} be a filter on a vector space. Then the filter $\mathfrak{X}^* = \bigvee_{\delta \neq 0} (\delta \cdot \mathfrak{X})$ is the finest one among all equable filters coarser than \mathfrak{X} .

Proof. Obviously \mathfrak{X}^* is equable. Let $M \in \mathfrak{X}^*$. Then $M \supset V \cdot A$, where $V \in V$, $A \in \bigvee_{\delta \neq 0} (\delta \mathfrak{X})$. Choose $\alpha \in V$, $\alpha \neq 0$. Since $A \in \delta \mathfrak{X}$ for all $\delta \neq 0$, we have $\alpha \cdot A \in \mathfrak{X}$, and therefore $M \in \mathfrak{X}$, because $M \supset \alpha \cdot A$. This proves that \mathfrak{X}^* is coarser than \mathfrak{X} .

Let now \mathfrak{Y} be any equable filter coarser than \mathfrak{X} : $\mathfrak{X} \leq \mathfrak{Y} = v \mathfrak{Y}$

Then, for $\delta \neq 0$:

$$\delta \mathfrak{X} \leq \delta \mathfrak{Y} = \delta (v \mathfrak{Y}) = (\delta v) \mathfrak{Y} = v \cdot \mathfrak{Y} = \mathfrak{Y}; \text{ hence}$$

$$\sup_{\delta \neq 0} (\delta \mathcal{X}) \leq \gamma \quad \text{and thus} \quad \mathcal{X}^* = \bigvee_{\delta \neq 0} \sup (\delta \mathcal{X}) \leq \bigvee \gamma = \gamma,$$

which completes the proof.

Corollary. For any pseudo-topological vector space E one has:

$$\mathcal{X} \downarrow E^* \iff \mathcal{X}^* \downarrow E.$$

(3) Proposition. Let E be any pseudo-topological vector space. Then

$$E \text{ equable} \iff (\mathcal{X} \downarrow E \text{ implies } \sup_{\delta \neq 0} \delta \mathcal{X} \downarrow E).$$

Proof. (a) Suppose E equable, and let $\mathcal{X} \downarrow E$. Then there exists γ with $\mathcal{X} \leq \gamma = \bigvee \gamma \downarrow E$. Since \mathcal{X}^* is coarser than \mathcal{X} and equable, $\delta \cdot \mathcal{X} = \delta \cdot \mathcal{X}^* = \mathcal{X}^*$ for $\delta \neq 0$; hence $\sup_{\delta \neq 0} (\delta \mathcal{X}) \leq \mathcal{X}^* \leq \gamma$, showing that $\sup_{\delta \neq 0} \delta \mathcal{X} \downarrow E$.

(b) Suppose the condition satisfied, and let $\mathcal{X} \downarrow E$. Then we have $\mathcal{X} \leq \mathcal{X}^* = \bigvee \mathcal{X}^* \downarrow E$, which proves that E is equable.

(4) Proposition. Let $E_i, i \in I$ be a projective system of equable and admissible vector spaces (cf.(2.3.5)(c)).

Then $E' = \text{proj.lim}_{i \in I} E_i$ is equable and admissible.

Proof. The admissibility is proved in (7.3.2). Let $\mathcal{X} \downarrow E$. Then

$i_k(\mathcal{X}) \downarrow E_k$ for each $k \in I$. But $\sup_{\delta \neq 0} \delta i_k(\mathcal{X}) = i_k(\sup_{\delta \neq 0} \delta \mathcal{X})$, since one has $\bigcup_{\delta \neq 0} \delta i_k(\mathcal{X}_\delta) = \bigcup_{\delta \neq 0} i_k(\delta \mathcal{X}_\delta) = i_k(\bigcup_{\delta \neq 0} \delta \mathcal{X}_\delta)$ for each such union. Hence $\sup_{\delta \neq 0} \delta \mathcal{X} \downarrow E$ and thus E is equable by the preceding proposition.

(5) Proposition. For each projective system of admissible vector

$$\text{spaces, } \left(\text{proj. lim}_{i \in I} E_i \right)^* = \text{proj. lim}_{i \in I} E_i^* .$$

Proof. (a) Let $\mathfrak{X} \downarrow \left(\text{proj. lim}_{i \in I} E_i \right)^*$. Then there is \mathfrak{Y} with

$$\mathfrak{X} \ll \mathfrak{Y} = \mathfrak{V} \mathfrak{Y} \downarrow \text{proj. lim}_{i \in I} E_i . \text{ Since } i_k(\mathfrak{X}) \ll i_k(\mathfrak{Y}) = i_k(\mathfrak{V} \mathfrak{Y}) = \mathfrak{V} i_k(\mathfrak{Y}),$$

one gets $i_k(\mathfrak{X}) \downarrow E_k^*$ for each $k \in I$ and thus $\mathfrak{X} \downarrow \text{proj. lim}_{i \in I} E_i^*$.

(b) If $\mathfrak{X} \downarrow \text{proj. lim}_{i \in I} E_i^*$, then $i_k(\mathfrak{X}) \downarrow E_k$ for each $k \in I$ by

(2.6.3). But by (4) and (2.6.1) we can suppose that \mathfrak{X} is equable,

hence $\mathfrak{X} \downarrow \left(\text{proj. lim}_{i \in I} E_i \right)^*$.

NOTATIONS

$\mathcal{A}, \mathcal{F}, \mathcal{X}$	filters 1.1
\emptyset	empty set 1.1
$[B], [A], [a]$	generated filters 1.1
$\mathcal{X}_1 \leq \mathcal{X}_2$	comparison of filters (1.2.1)
$\mathcal{X} \downarrow_x E$	\mathcal{X} converges to x on E 2.1
$\mathcal{X} \downarrow E$	\mathcal{X} converges to zero on E 2.4
$\{\mathcal{X}_i\}_{i \in I}$	family of filters 1.2
$\sup_{i \in I} \mathcal{X}_i$	least upper bound of filters 1.2
$\mathcal{X}_1 \vee \mathcal{X}_2$	$\sup(\mathcal{X}_1, \mathcal{X}_2)$ 1.2
\mathcal{U}_x	$\sup_{\mathcal{X} \downarrow_x E} \mathcal{X}$ (2.4.3)
\mathcal{X}^0	2.7
$\hat{\mathcal{X}}, (\mathcal{X})^{\wedge}$	2.7
$\bar{\mathcal{X}}, (\mathcal{X})^{\bar{}}$	(5.3.3)
E, E_1, E_2	pseudo topological spaces
\underline{E}	underlying set
E^*	equable space associated to E 2.6
E^0	locally convex space associated to E 2.7
$E_1 \leq E_2$	comparison of structures 2.3
$E_1 \times E_2, \times_{i \in I} E_i$	direct product of pseudo-topological spaces 2.3

$\mathfrak{X}_1 \times \mathfrak{X}_2$	direct product of filters (on a direct product of two sets) 1.4
\mathbb{R}	the reals (with the natural topology)
\mathcal{V}	neighborhood filter of zero in \mathbb{R}
$[\alpha, \beta]$	closed interval in \mathbb{R}
I_δ	closed interval $[-\delta, \delta]$, $\delta > 0$
\mathbb{N}	$\{1, 2, 3, \dots\}$
\mathbb{N}^0	$\{0, 1, 2, \dots\}$
$\text{proj. lim}_{i \in I} E_i$	projective limit 2.3
$R(E_1; E_2)$	set of remainders (3.1.2)
$L_n(E_1; E_2)$	space of n-linear maps (6.1.6)
$C_k(E_1; E_2)$	space of C_k -mappings 10.2
$C_\infty(E_1; E_2)$	space of C_∞ -mappings 10.3
$L_n^*(E_1; E_2)$	instead of $(L_n(E_1; E_2))^*$ (6.1.7)
$C_k^*(E_1; E_2)$	instead of $(C_k(E_1; E_2))^*$ (6.1.7)
$C_\infty^*(E_1; E_2)$	instead of $(C_\infty(E_1; E_2))^*$
$f : E_1 \longrightarrow E_2$	mapping f of E_1 into E_2
$x \longmapsto y$	x is sent into y under the considered (anonymous) map
$[f_1, f_2], \prod_{i \in I} f_i$	1.3
$f_1 \times f_2, \times_{i \in I} f_i$	1.3

π_k	k-th projection	2.3
c	composition map	
e	evaluation map	
\tilde{u}	map associated to u	(11.1.1)
$u \cdot x$	instead of $u(x)$	
$\Delta f(x, y)$	abbreviation of $f(x+y) - f(x)$	
$\Theta f(\lambda, x)$	abbreviation of $\begin{cases} \frac{f(\lambda x) - f(x)}{\lambda} & \text{for } \lambda \neq 0 \\ 0 & \text{for } \lambda = 0 \end{cases}$	
$f^*(a)$	differential quotient	(4.3.2)
$f'(a), Df(a)$	derivative of f at the point a	(3.2.2)
$Rf(a)$	remainder belonging to a map f which is differentiable at a	(3.2.4)
$f^{(k)}(a)$	see	9.2
$D_1 f(a_1, a_2), D_2 f(a_1, a_2)$	partial derivatives	8.1
f^*	$f^*(g) = g \circ f$	
f_*	$f_*(g) = f \circ g$	
\implies	implies	
\iff	if and only if	
\simeq	linearly homeomorphic	

I N D E X

admissible	7.1
almost all	5.1
associated locally convex topological vector space	2.7
canonical isomorphisms	6.4
chain rule	3.3
coarser	2.3
compatible	2.4
composition map	6.3
continuous	2.2
continuous with respect to associated structures	2.9
C_k -mapping	10.1
derivative	3.2
diagonal map	1.3
differentiable at a point	3.2
differentiable map into a direct product	4.4
differential quotient	4.3
direct product	2.3, 4.4
equable continuity	2.8
equable filter	2.5
equable pseudo-topological vector space	2.6; appendix (3)
evaluation map	6.2
filter	1.1
filter-basis	1.1
finer	2.3
Fréchet condition	4.1
function spaces	6.1, 6.2
fundamental theorem of calculus	5.1

higher derivatives	9.1
higher order chain rule	9.2, 10.4
homeomorphism	2.2
images of filters under mappings	1.4
inclusion map	2.3
induced structures	2.3
infimum of filters	1.2
infinitely differentiable	9.2
local character	3.4
mappings into direct products	1.3
mean value theorem	5.2
natural	6.4
neighborhood, ϵ -neighborhood	3.4
open, ϵ -open	3.4
partial mapping	8.1
partial derivatives	8.1
projective limit	2.3
pseudo-topological space	2.1
pseudo-topological vector space	2.4
pseudo-topology	2.1
quasi-bounded filter	2.5
quasi-bounded map	2.8
remainder	3.1
separated	3.1
subspace	2.3
supremum of filters	1.2

symmetry of $f''(x)$	9.1
underlying space	2.1
uniform convergence on bounded sets	6.1

R E F E R E N C E S

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