A

Flatness

The algebraic notion of flatness, introduced for the first time in [164], is the basic technical tool for the study of families of algebraic varieties and schemes. In this appendix we will overview the main algebraic results needed. For the properties of flat morphisms between schemes we refer to [84]. See also § 4.2.

A module M over a ring A is *A-flat* (or *flat over A*, or simply *flat*) if the functor $N \mapsto M \otimes_A N$ from the category of A-modules into itself is exact. Since this functor is always right exact, the flatness means that it takes monomorphisms into monomorphisms. An *A-algebra B* is *flat over A* if B is flat as an A-module.

The A-module M is said to be *faithfully flat* if for every sequence of A-modules $N' \to N \to N''$ the sequence

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$$

is exact if and only if the original sequence is exact. Obviously, if M is faithfully flat then it is flat. In a similar way we give the notion of faithfully flat A-algebra. It is straightforward to check that if $A \to B$ is a local homomorphism of local rings, then a B-module of finite type is faithfully A-flat if and only if it is A-flat and nonzero.

Recall that the flatness of an A-module M is equivalent to any of the following conditions:

(1) $\mathrm{Tor}_i^A(M, N) = (0)$ for all $i > 0$ and for every A-module N.
(2) $\mathrm{Tor}_1^A(M, N) = (0)$ for every A-module N.
(3) $\mathrm{Tor}_1^A(M, N) = (0)$ for every finitely generated A-module N.
(4) $\mathrm{Tor}_1^A(M, A/I) = (0)$ for every ideal $I \subset A$.
(5) $I \otimes_A M \to M$ is injective for every ideal $I \subset A$.
(6) $I \otimes_A M \to IM$ is an isomorphism for every ideal $I \subset A$.

Example A.1. Let k be a ring, u, v indeterminates and $f : k[u, uv] \to k[u, v]$ the inclusion. Then

$$\frac{k[u, uv]}{(uv)} = k[u] \xrightarrow{u} k[u] = \frac{k[u, uv]}{(uv)}$$

is injective. Tensoring by $\otimes_{k[u,uv]} k[u,v]$ we obtain:

$$\frac{k[u,v]}{(uv)} \xrightarrow{u} \frac{k[u,v]}{(uv)}$$

which is not injective. Therefore f is not flat.

We list without proof a few *basic properties of flat modules*:

Proposition A.2. *(I) M is A-flat if and only if M_p is A_p-flat for every prime ideal p.*
(II) Every projective module is flat.
(III) Assume M is finitely generated. Then M is flat if and only if it is projective; if A is local then M is flat if and only if it is free.
(IV) If $S \subset A$ is a multiplicative subset then A_S is A-flat.
(V) A direct sum $M = \oplus_{i \in I} M_i$ is flat if and only if all M_i's are flat.
(VI) Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-modules with M'' flat. Then M is flat if and only if M' is flat.
(VII) Base change: if M is A-flat and $f : A \to B$ is a ring homomorphism, then $M \otimes_A B$ is B-flat.
(VIII) Transitivity: if B is a flat A-algebra and N is a flat B-module, then N is A-flat.
(IX) If A is a noetherian ring and I is an ideal, the I-adic completion $\hat{A}$ is a flat A-algebra. If I is contained in the Jacobson radical of A then $\hat{A}$ is a faithfully flat A-algebra.
(X) If B is an A-algebra and if there exists a B-module M which is faithfully flat, then the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.
(XI) If $X_1, \ldots, X_r$ are indeterminates, then $A[X_1, \ldots, X_r]$ and $A[[X_1, \ldots, X_r]]$ are A-flat.

The following result is frequently used:

Proposition A.3. *If A is an artinian local ring with residue field k the following are equivalent for an A-module M:*

(i) M is free
(ii) M is flat
(iii) $\mathrm{Tor}_1^A(M,k) = (0)$

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear.
$(iii) \Rightarrow (ii)$. Let N be a finitely generated A-module and let

$$N = N_0 \supset \cdots \supset N_n = (0)$$

be a composition series for N such that

$$N_i / N_{i+1} \cong k$$

for $i = 0, \ldots, n-1$. Using the Tor exact sequences from hypothesis (iii) we deduce that $\mathrm{Tor}_1(M, N) = (0)$ and the flatness of M follows from (3).

Let's now prove $(ii) \Rightarrow (i)$. Let $\{e_j\}_{j \in J}$ be a system of elements of M which induces a basis of $M \otimes_A k$ over k. The system $\{e_j\}$ defines a homomorphism $f : A^J \to M$ which induces an isomorphism $k^J \to M \otimes_A k$. From the following lemma we find that f is an isomorphism, and therefore M is free. □

Lemma A.4. *Let R be a ring, I an ideal and $f : F \to G$ a homomorphism of R-modules with G flat. Assume that one of the following conditions is satisfied:*

(a) I is nilpotent.
(b) R is noetherian, I is contained in the Jacobson radical of R and F and G are finitely generated.

If the induced homomorphism $F/IF \to G/IG$ is an isomorphism, then f is an isomorphism.

Proof. Let $K = \mathrm{coker}(f)$. Tensoring the exact sequence

$$F \to G \to K \to 0$$

with R/I we get $K/IK = 0$: from Nakayama's lemma (which holds in either of hypotheses (a) and (b)) it follows that $K = 0$, and therefore F is surjective. Letting $H = \ker(f)$ we deduce an exact sequence

$$0 \to H/IH \to F/IF \to G/IG \to 0$$

using the flatness of G. By Nakayama again we deduce $H = 0$ and the conclusion follows. □

The following is a basic criterion of flatness.

Theorem A.5 (Local criterion of flatness). *Suppose that $\varphi : A \to B$ is a local homomorphism of local noetherian rings, and let $k = A/m_A$ be the residue field of A. If M is a finitely generated B-module, then the following conditions are equivalent:*

(i) M is A-flat.
(ii) $\mathrm{Tor}_1^A(M, k) = 0$.
(iii) $M \otimes_A (A/m_A^n)$ is flat over A/m_A^n for every integer $n \geq 1$.
(iv) $M \otimes_A (A/m_A^n)$ is free over A/m_A^n for every integer $n \geq 1$.

Proof. $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ see [48], Th. 6.8, p. 167.

$(i) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$ It suffices to show that for every inclusion $N' \to N$ of A-modules of finite type we have an inclusion $M \otimes_A N' \to M \otimes_A N$. For this purpose it suffices to show that the kernel of this last map is contained in

$$K_n := \ker[M \otimes_A N' \to M \otimes_A (N'/N' \cap m_A^n N)]$$

for all n, because $\bigcap_n K_n = (0)$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & K_n & \to & M \otimes_A N' & \to & M \otimes_A (N'/N' \cap m_A^n N) & \to 0 \\ & & & \downarrow & & \downarrow & \\ & & & M \otimes_A N & \to & M \otimes_A (N/m_A^n N) & \to 0 \end{array}$$

The last vertical arrow coincides with the map obtained from the injection

$$N'/N' \cap m_A^n N \to N/m_A^n N$$

after tensoring over A/m_A^n with the A/m_A^n-flat module $M \otimes_A (A/m_A^n)$, and therefore it is injective. The conclusion follows from the above diagram.

$(iii) \Leftrightarrow (iv)$ follows from Proposition A.3 because A/m_A^n is artinian. □

For a more general version of the local criterion we refer to [3], exp. IV, Théorème 5.6. Note that A.3 is a special case of A.5.

Corollary A.6. *Suppose that $\varphi : A \to B$ is a local homomorphism of local noetherian rings, let $k = A/m_A$ be the residue field of A, M, N two finitely generated B-modules, and suppose that N is A-flat. Let $u : M \to N$ be a B-homomorphism. Then the following are equivalent:*

(i) u is injective and coker(u) is A-flat.
(ii) $u \otimes 1 : M \otimes k \to N \otimes k$ is injective.

Proof. (i) $\Rightarrow$ (ii). Let $G = \mathrm{coker}(u)$. Tensoring by k the exact sequence

$$0 \to M \xrightarrow{u} N \to G \to 0$$

by k we obtain the exact sequence:

$$\mathrm{Tor}_1^A(G, k) \to M \otimes_A k \xrightarrow{u \otimes 1} N \otimes_A k \to G \otimes_A k \to 0$$

Since G is A-flat we have $\mathrm{Tor}_1^A(G, k) = 0$, and it follows that $u \otimes 1$ is injective.

(ii) $\Rightarrow$ (i). Factor $u \otimes 1$ as

$$M \otimes_A k \xrightarrow{\alpha} \mathrm{Im}(u) \otimes_A k \xrightarrow{\beta} N \otimes_A k$$

Then α is an isomorphism and β is injective. Tensoring by k the exact sequence

$$0 \to \mathrm{Im}(u) \to N \to G \to 0 \tag{A.1}$$

we obtain the exact sequence:

$$\mathrm{Tor}_1^A(N, k) \to \mathrm{Tor}_1^A(G, k) \to \mathrm{Im}(u) \otimes_A k \xrightarrow{\beta} N \otimes_A k \to G \otimes_A k \to 0$$

Since N is A-flat we have $\mathrm{Tor}_1^A(N, k) = 0$; from the injectivity of β we deduce $\mathrm{Tor}_1^A(G, k) = 0$ and from A.5 it follows that G is A-flat. Applying (VI) to the exact sequence (A.1) we deduce that $\mathrm{Im}(u)$ is A-flat as well. Consider the exact sequence:

$$0 \to \ker(u) \to M \to \operatorname{Im}(u) \to 0$$

and tensor by k. We obtain the exact sequence:

$$0 \to \ker(u) \otimes_A k \to M \otimes_A k \xrightarrow{\alpha} \operatorname{Im}(u) \otimes_A k \to 0$$

Since α is an isomorphism we deduce that $\ker(u) \otimes_A k = 0$, and therefore $\ker(u) = 0$ by Nakayama's lemma. □

A related result is the following:

Lemma A.7. *Let B be a local ring with residue field K, and let $d : G \to F$ be a homomorphism of finitely generated B-modules, with F free. Then d is split injective if and only if $d \otimes_B K : G \otimes_B K \to F \otimes_B K$ is injective. In such a case G is also free.*

Proof. d is split injective if and only if $\operatorname{coker}(d)$ is free and d is injective. If this last condition is satisfied then clearly $d \otimes_B K$ is injective.

Conversely, assume that $d \otimes_B K$ is injective, and factor d as

$$G \to \operatorname{Im}(d) \to F$$

We see that

$$\begin{array}{ccll} G \otimes_B K & \to & \operatorname{Im}(d) \otimes_B K & \text{is bijective} \\ \operatorname{Im}(d) \otimes_B K & \to & F \otimes_B K & \text{is injective} \end{array}$$

From the exact sequence

$$0 \to \operatorname{Im}(d) \to F \to \operatorname{coker}(d) \to 0$$

we get

$$0 \to \operatorname{Tor}_1(\operatorname{coker}(d), K) \to \operatorname{Im}(d) \otimes_B K \to F \otimes_B K$$

so $\operatorname{Tor}_1(\operatorname{coker}(d), K) = (0)$ and this implies that $\operatorname{coker}(d)$ is free. From the above exact sequence we deduce that $\operatorname{Im}(d)$ is free as well, so that

$$0 \to \ker(d) \to G \to \operatorname{Im}(d) \to 0$$

is split exact. Recalling that $G \otimes_B K \cong \operatorname{Im}(d) \otimes_B K$ we deduce that $\ker(d) \otimes_B K = (0)$, hence $\ker(d) = (0)$ by Nakayama. □

For the reader's convenience we include the proof of the following well-known lemma:

Lemma A.8. *Let (B, m) be a noetherian local integral domain, with residue field K and quotient field L. If M is a finitely generated B-module and if*

$$\dim_K(M \otimes_B K) = \dim_L(M \otimes_B L) = r$$

then M is free of rank r.

Proof. Let $m_1, \ldots, m_r \in M$ be such that their images in $M \otimes_B K = M/mM$ form a basis. Then they define a homomorphism $\varphi : B^r \to M$ and we have an exact sequence:

$$0 \to N \to B^r \xrightarrow{\varphi} M \to Q \to 0$$

where N and Q are kernel and cokernel of φ. Since tensoring with K we get

$$K^r \xrightarrow{\bar{\varphi}} M/mM \to Q/mQ \to 0$$

and $\bar{\varphi}$ is surjective, we get $Q/mQ = (0)$ and from Nakayama's lemma it follows that $Q = (0)$: hence φ is surjective. Now we tensor the above exact sequence with L, which is flat over B (by (IV)), and we obtain the exact sequence:

$$0 \to N \otimes_B L \to L^r \xrightarrow{\tilde{\varphi}} M \otimes_B L \to 0$$

Since $M \otimes_B L \cong L^r$ and $\tilde{\varphi}$ is surjective, it follows that $N \otimes_B L = \ker(\tilde{\varphi}) = (0)$. Therefore N is a torsion module. But $N \subset B^r$ and therefore $N = (0)$. □

We have the following useful criterion:

Lemma A.9. *Let $A \to A'$ be a small extension in $\mathcal{A}$, and let $g : A \to R$ be a homomorphism of* **k***-algebras. Let $R_0 = R \otimes_A \mathbf{k}$. Then g is flat if and only if*

$$\ker(R \to R \otimes_A A') \cong R_0$$

and the homomorphism $g' : A' \to R \otimes_A A'$ induced by g is flat.

Proof. Assume that g is flat. Then since $R \otimes_A (\epsilon) \cong R \otimes_A \mathbf{k} = R_0$ and $\mathrm{Tor}_1^A(R, A') = 0$, from the exact sequence

$$0 \to \mathrm{Tor}_1^A(R, A') \to R \otimes_A (\epsilon) \to R \to R \otimes_A A' \to 0 \qquad \text{(A.2)}$$

we deduce that the first condition is satisfied. The flatness of g' is obvious.

Assume conversely that the conditions of the statement are satisfied. Then the sequence (A.2) implies that $\mathrm{Tor}_1^A(R, A') = 0$. If $A' = \mathbf{k}$ the conclusion follows from A.3. If not, from the exact sequence

$$0 \to m_{A'} \to A' \to \mathbf{k} \to 0$$

one gets the exact sequence:

$$\begin{array}{ccccccccc} \mathrm{Tor}_1^A(R, A') & \to & \mathrm{Tor}_1^A(R, \mathbf{k}) & \xrightarrow{\partial} & R \otimes_A m_{A'} & \to & R' & \to & R \otimes_A \mathbf{k} & \to 0 \\ \| & & & & \| & & & & \| & \\ 0 & & & & R' \otimes_{A'} m_{A'} & & & & R' \otimes_{A'} \mathbf{k} & \end{array}$$

From the flatness of R' over A' we deduce that $\partial = 0$, hence $\mathrm{Tor}_1^A(R, \mathbf{k}) = 0$, and we conclude by A.3. □

* * * * * *

Flatness in terms of generators and relations

Let P be a noetherian $\mathbf{k}$-algebra, $J \subset P$ an ideal. Let A be in $\mathrm{ob}(\mathcal{A})$, $P_A = P \otimes_{\mathbf{k}} A$, and $\mathbf{J} \subset P_A$ an ideal such that $(P_A/\mathbf{J}) \otimes_A \mathbf{k} \cong P/J$. We want to find the conditions $\mathbf{J}$ has to satisfy so that $P_A/\mathbf{J}$ is A-flat.

We have the following:

Theorem A.10. *Let*

$$\Pi_0 : \quad P^n \to P^N \to P \to P/J \to 0$$

be a presentation of P/J as a P-module. Then the following conditions are equivalent for an ideal $\mathbf{J} \subset P_A$:

(i) $P_A/\mathbf{J}$ is A-flat and $(P_A/\mathbf{J}) \otimes_A \mathbf{k} \cong P/J$.
(ii) There is an exact sequence

$$\Pi : \quad P_A^n \to P_A^N \to P_A \to P_A/\mathbf{J} \to 0$$

such that $\Pi_0 = \Pi \otimes_A \mathbf{k} \quad (= \Pi/m_A\Pi)$.
(iii) There is a complex

$$\Pi : \quad P_A^n \xrightarrow{\varphi} P_A^N \to P_A \to P_A/\mathbf{J} \to 0$$

which is exact except possibly at P_A^N, such that $\quad \Pi_0 = \Pi \otimes_A \mathbf{k}$.

Proof. $(ii) \Rightarrow (i)$. We have:

$$\mathrm{Tor}_1^A(P_A/\mathbf{J}, \mathbf{k}) = H_1(\Pi \otimes \mathbf{k}) = H_1(\Pi_0) = (0)$$

From A.3 it follows that $P_A/\mathbf{J}$ is A-flat. Moreover, (ii) implies that $(P_A/\mathbf{J}) \otimes_A \mathbf{k} \cong P/J$.

$(i) \Rightarrow (ii)$. Choose a P_A-homomorphism $p : P_A^N \to \mathbf{J}$ which makes the following diagram commute:

$$\begin{array}{cccc} p: & P_A^N & \to & \mathbf{J} \\ & \downarrow & & \downarrow \\ p_0: & P^N & \to & J \end{array}$$

where p_0 is the surjective homomorphism defined by the presentation Π_0. From the flatness of $P_A/\mathbf{J}$ it follows that $\mathrm{Tor}_1^A(P_A/\mathbf{J}, \mathbf{k}) = (0)$; hence the exact sequence

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Tor}_1^A(P_A/\mathbf{J}, \mathbf{k}) & \to & \mathbf{J} \otimes \mathbf{k} & \to & P_A \otimes \mathbf{k} & \to & (P_A/\mathbf{J}) \otimes_A \mathbf{k} & \to 0 \\ & & & & & \| & & \| & \\ & & & & & P & & P/J & \end{array}$$

implies that $\mathbf{J} \otimes \mathbf{k} = J$. It follows that $p \otimes_A \mathbf{k} = p_0$ and therefore

$$\mathrm{coker}(p) \otimes_A \mathbf{k} = \mathrm{coker}(p_0) = (0)$$

so that $\mathrm{coker}(p) = (0)$ by Nakayama's lemma. Hence p is surjective.

Now consider the exact sequence

$$0 \to \ker(p) \to P_A^N \to \mathbf{J} \to 0$$

and the associated Tor sequence:

$$\mathrm{Tor}_1^A(\mathbf{J}, \mathbf{k}) \to \ker(p)/m_A \ker(p) \to P^N \to J \to 0 \tag{A.3}$$

From the flatness of $P_A/\mathbf{J}$ and from the exact sequence

$$0 \to \mathbf{J} \to P_A \to P_A/\mathbf{J} \to 0$$

we have $\mathrm{Tor}_1^A(\mathbf{J}, \mathbf{k}) = \mathrm{Tor}_2^A(P_A/\mathbf{J}, \mathbf{k}) = (0)$. Therefore from (A.3) we see that

$$\ker(p)/m_A \ker(p) \cong \ker(p_0)$$

Arguing as before we can find a surjective homomorphism $q : P_A^n \to \ker(p)$ which makes the following diagram commutative:

$$\begin{array}{ccc} P_A^n & \xrightarrow{q} & \ker(p) \\ \downarrow & & \downarrow \\ P^n & \to & \ker(p_0) \end{array}$$

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$ If Π is not exact at P_A^N then we can add finitely many generators of the kernel of $P_A^N \to P_A$ to obtain an exact sequence

$$\Pi' : \qquad P_A^{n'} \xrightarrow{\varphi'} P_A^N \to P_A \to P_A/\mathbf{J} \to 0$$

Then $\Pi' \otimes_A \mathbf{k}$ has the form:

$$P^{n'} \xrightarrow{\varphi' \otimes \mathbf{k}} P^N \to P \to P/J \to 0$$

Since

$$\mathrm{Im}(\varphi \otimes \mathbf{k}) \subset \mathrm{Im}(\varphi' \otimes \mathbf{k}) \subset \ker[P^N \to P]$$

we see that $\mathrm{Im}(\varphi' \otimes \mathbf{k}) = \ker[P^N \to P]$ and therefore $\Pi' \otimes_A \mathbf{k}$ is exact. Now (i) follows from A.3. □

Corollary A.11. *Assume that* $J = (f_1, \ldots, f_N) \subset P$ *and that*

$$\mathbf{J} = (F_1, \ldots, F_N) \subset P_A$$

with $f_j = F_j$ *(mod* $m_A P_A$*),* $j = 1, \ldots, N$*. Then every relation among* $f_1, \ldots, f_N$ *lifts to a relation among* $F_1, \ldots, F_N$ *if and only if* $P_A/\mathbf{J}$ *is A-flat and* $(P_A/\mathbf{J}) \otimes_A \mathbf{k} \cong P/J$*.*

Proof. The condition that the F_j's reduce to the f_j's modulo $m_A P_A$ implies that the exact sequence

$$P_A^N \xrightarrow{\mathbf{F}} P_A \to P_A/\mathbf{J} \to 0$$

reduces to

$$P^N \xrightarrow{\mathbf{f}} P \to P/J \to 0 \tag{A.4}$$

when tensored by $\otimes_A \mathbf{k}$. Complete (A.4) to a presentation Π_0 of P/J. The condition that every relation among $f_1, \dots, f_N$ lifts to a relation among $F_1, \dots, F_N$ is a restatement of condition (iii) of A.10. Therefore the conclusion follows from Theorem A.10. □

Example A.12. Let A be in ob($\mathcal{A}$). Suppose that $f_1, \dots, f_N \in P$ form a regular sequence, and let $F_1, \dots, F_N \in P_A$ be any liftings of $f_1, \dots, f_N$, i.e. such that $f_j = F_j \pmod{mP_A}$, $j = 1, \dots, N$. Then $\mathbf{J} = (F_1, \dots, F_N) \subset P_A$ defines a flat family of deformations of $X = \mathrm{Spec}(P/J)$, where $J = (f_1, \dots, f_N)$.

In fact, every relation among $f_1, \dots, f_N$ is a linear combination of the trivial ones

$$r_{ij} = (0, \dots, f_j, \dots, -f_i, \dots, 0) \qquad 1 \leq i < j \leq N$$

and these can be lifted to the corresponding trivial relations

$$R_{ij} = (0, \dots, F_j, \dots, -F_i, \dots, 0)$$

among $F_1, \dots, F_N$. Applying Corollary A.11 it is easy to show that $F_1, \dots, F_N$ form a regular sequence.

NOTES

1. In the proof of Theorem A.10 the condition that A is artinian has only been used in the proof of $(i) \Rightarrow (ii)$ in order to apply Nakayama's lemma. In particular, the implications $(ii) \Rightarrow (i)$, $(iii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$ hold for any $A \in$ ob($\mathcal{A}^*$). Using the local criterion of flatness it is easy to verify that the implication $(i) \Rightarrow (ii)$ (and therefore the equivalence of the three conditions) holds as well if A is in ob($\hat{\mathcal{A}}$).

B

Differentials

Let $A \to B$ be a ring homomorphism. As usual, we will denote by $\Omega_{B/A}$ the *module of differentials of B over A*, and by $d_{B/A} : B \to \Omega_{B/A}$ the canonical A-derivation. Recall that

$$\Omega_{B/A} := I/I^2$$

where $I = \ker(B \otimes_A B \xrightarrow{\mu} B)$ is the natural map, and for each $b \in B$

$$d_{B/A}(b) = b \otimes 1 - 1 \otimes b$$

is called the *differential of b*. We have a natural isomorphism of B-modules

$$\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_B(\Omega_{B/A}, M)$$

Note that the exact sequence

$$0 \to \Omega_{B/A} \to (B \otimes_A B)/I^2 \xrightarrow{\mu'} B \to 0 \tag{B.1}$$

where μ' is induced by μ, is an A-extension of B. The ring

$$P_{B/A} := (B \otimes_A B)/I^2$$

is called the *algebra of principal parts* of B over A. The A-extension (B.1) is trivial because we have splittings:

$$\lambda_1, \lambda_2 : B \to P_{B/A}$$

defined by $\lambda_1(b) = \overline{b \otimes 1}$, $\lambda_2(b) = \overline{1 \otimes b}$; note that $d_{B/A} = \lambda_1 - \lambda_2$. We will consider $P_{B/A}$ as a B-algebra via λ_1.

The following are some fundamental properties of the modules of differentials:

Proposition B.1. *(i) If*

$$\begin{array}{ccc} B & & \\ \uparrow & & \\ A & \longrightarrow & A' \end{array}$$

are ring homomorphisms, then:

$$\Omega_{B/A} \otimes_A A' \cong \Omega_{B\otimes_A A'/A'}$$

(ii) If $A \to B$ is a ring homomorphism and $\Delta \subset B$ is a multiplicative system, then:

$$\Omega_{\Delta^{-1}B/A} \cong \Delta^{-1}\Omega_{B/A}$$

(iii) Let $K \to L$ be a finitely generated extension of fields. Then

$$\dim_L(\Omega_{L/K}) \geq \text{trdeg}(L/K)$$

and equality holds if and only if L is separably generated over K. In particular, $\Omega_{L/K} = (0)$ if and only if $K \subset L$ is a finite algebraic separable extension.

Proof. See [48]. □

We now introduce two standard exact sequences.

Theorem B.2 (Relative cotangent sequence). *Given ring homomorphisms*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

there is an exact sequence of C-modules:

$$\Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} \to 0 \tag{B.2}$$

where the maps are given by:

$$\alpha(d_{B/A}(b) \otimes c) = c d_{C/A}(g(b)); \quad \beta(d_{C/A}(r)) = d_{C/B}(r) \quad b \in B, \ \ c \in C$$

Proof. See [48], prop. 16.2. □

When $B \to C$ is surjective we have $\Omega_{C/B} = (0)$ and the next theorem describes $\ker(\alpha)$.

Theorem B.3 (Conormal sequence). *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be ring homomorphisms with g surjective, and let $J = \ker(g)$, so that $C = B/J$. Then:

(i) We have an exact sequence

$$J/J^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \to 0 \tag{B.3}$$

where δ is the C-linear map defined by $\delta(\bar{x}) = d_{B/A}(x) \otimes 1$.

(ii) There is an isomorphism

$$\Omega_{(B/J^2)/A} \otimes_{(B/J^2)} C \cong \Omega_{B/A} \otimes_B C$$

In other words the conormal sequence (B.3) depends only on the first infinitesimal neighbourhood of Spec(C) *in* Spec(B).

(iii) The map δ is a split injection if and only if there is a map of A-algebras $C \to B/J^2$ splitting the projection $B/J^2 \to C$.

Proof. (i) see e.g. [48], prop. 16.3.

(ii) Comparing the exact sequence (B.3) with the analogous sequence associated to $A \to B/J^2 \to C$ we get a commutative diagram:

$$\begin{array}{ccccccc} J/J^2 & \to & \Omega_{B/A} \otimes_B C & \to & \Omega_{C/A} & \to 0 \\ \| & & \downarrow & & \| & \\ J/J^2 & \to & \Omega_{(B/J^2)/A} \otimes_{(B/J^2)} C & \to & \Omega_{C/A} & \to 0 \end{array}$$

and the vertical arrow, which is induced by $B \to B/J^2$, must be an isomorphism.

(iii) By (ii) we may assume that $J^2 = 0$, i.e. that $0 \to J \to B \to C \to 0$ is an A-extension. Assume that $\delta : J \to \Omega_{B/A} \otimes_B C$ is a split injection, and let $\sigma : \Omega_{B/A} \otimes_B C \to J$ be a splitting. Then the composition

$$B \xrightarrow{\bar{d}} \Omega_{B/A} \otimes_B C \xrightarrow{\sigma} J$$

is an A-derivation. It follows that $1 - \sigma\bar{d} : B \to B$ is an A-homomorphism such that $(1 - \sigma\bar{d})(J) = 0$ and therefore it induces an A-homomorphism $C \to B$ which splits g.

Conversely, assume that $g : B \to C$ has a section $\tau : C \to B$. Then we have a derivation

$$D : B \to J \oplus \Omega_{C/A}$$

given by $D(b) = (b - (\tau g)(b), d_{C/A}(g(b)))$. One easily checks that D induces an isomorphism $\Omega_{B/A} \otimes_B C \cong J \oplus \Omega_{C/A}$, thus proving the assertion. □

As an application we have the following:

Proposition B.4. *Let K be a field and (B, m) a local K-algebra with residue field $B/m = K'$. Then the map*

$$\delta : m/m^2 \to \Omega_{B/K} \otimes_B K'$$

in the exact sequence (B.3) relative to $K \to B \to K'$ is injective if and only if $K \subset K'$ is a separable field extension.
In particular, if $B/m = K$ then

$$\delta : m/m^2 \to \Omega_{B/K} \otimes_B K$$

is an isomorphism. Therefore

$$\dim(B) \le \dim_K(\Omega_{B/K} \otimes_B K)$$

Proof. See [48], cor. 16.13. The last assertion follows from the conormal sequence relative to $K \to B \to K$. □

The following theorem describes the module of differentials for regular local rings.

Theorem B.5. *Assume that K is a field and B is a local noetherian K-algebra with residue field $B/m = K$. If $\Omega_{B/K}$ is a free B-module of rank equal to* $\dim(B)$ *then B is a regular local ring. If K is perfect (e.g. algebraically closed) and B is e.f.t. over K then the converse is also true.*

Proof. Assume first that $\Omega_{B/K}$ is free of rank equal to $\dim(B)$. Then

$$\dim_K(m/m^2) = \dim(B)$$

by B.4, so B is a regular local ring.

Assume conversely that K is perfect and that B is a regular local ring, e.f.t. over K. Then we have

$$\dim_K(\Omega_{B/K} \otimes_B K) = \dim_K(m/m^2) = \dim(B)$$

Let L be the quotient field of B. Then, by B.1(3), we have

$$\Omega_{B/K} \otimes_B L = \Omega_{L/K}$$

and

$$\dim_L(\Omega_{L/K}) = \text{trdeg}(L/K) = \dim(B)$$

because L is separably algebraic over K, since K is perfect. Therefore we have

$$\dim_K(\Omega_{B/K} \otimes_B K) = \dim(B) = \dim_L \Omega_{B/K} \otimes_B L$$

Since B is e.f.t. over K, $\Omega_{B/K}$ is a finitely generated B-module, and from Lemma A.8 it follows that it is free of rank equal to $\dim(B)$. □

In particular, we have the following:

Corollary B.6. *Let k be an algebraically closed field, and let B be an integral k-algebra of finite type. Then B is a regular ring if and only if $\Omega_{B/k}$ is a projective B-module of rank equal to* $\dim(B)$.

Proof. Both conditions are satisfied if and only if they are satisfied after localizing at the maximal ideals of B. For every maximal ideal $m \subset B$ the local ring B_m is a k-algebra e.f.t. with residue field k. By B.5, B_m is a regular local ring if and only if $\Omega_{B_m/k} = (\Omega_{B/k})_m$ is free of rank equal to $\dim(B)$. The conclusion follows. □

Proposition B.7. *If the ring homomorphism $A \to B$ is e.f.t. then $\Omega_{B/A}$ is a B-module of finite type.*
If, in particular, $B = S^{-1}A[X_1, \ldots, X_n]$ for some multiplicative system S, then $\Omega_{B/A}$ is a free B-module of rank n with basis $\{d_{B/A}(X_1), \ldots, d_{B/A}(X_n)\}$.

Proof. The last assertion is elementary (see [48]). To prove the first, let $B = (S^{-1}P)/J$, where $P = A[X_1, \dots, X_n]$ and $S \subset P$ is a multiplicative system. Then $\Omega_{B/A}$ is a quotient of $\Omega_{S^{-1}P/A} \otimes_{S^{-1}P} B$, by the conormal sequence. □

Remark B.8. If A and B are only assumed to be noetherian then $\Omega_{B/A}$ is not necessarily a B-module of finite type even if A is a field. An example is given by $\Omega_{\mathbf{Q}[[X]]/\mathbf{Q}}$ (see [1] ch. $\mathbf{0}_{IV}$, n. 20.7.16).

Examples B.9. (i) Assume that $B = S^{-1}A[X_1, \dots, X_n]$ for some multiplicative system S. Then $\mathrm{Der}_A(B, B) = \mathrm{Hom}_B(\Omega_{B/A}, B)$ is a free module of rank n with basis

$$\left\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right\}$$

which is the dual of the basis

$$\{d_{B/A}(X_1), \dots, d_{B/A}(X_n)\}$$

of $\Omega_{B/A}$, and where $\frac{\partial}{\partial X_j} : B \to B$ is the partial A-derivation with respect to X_j.

Let $Y_1, \dots, Y_n \in B$ be such that the jacobian determinant

$$\det\left(\frac{\partial Y_i}{\partial X_j}\right)$$

is a unit in B. Then

$$\{d_{B/A}(Y_1), \dots, d_{B/A}(Y_n)\}$$

is another basis of $\Omega_{B/A}$ and we have:

$$d_{B/A}(X_j) = \frac{\partial X_j}{\partial Y_1} d_{B/A}(Y_1) + \cdots + \frac{\partial X_j}{\partial Y_n} d_{B/A}(Y_n)$$

Dually:

$$\frac{\partial}{\partial X_j} = \frac{\partial Y_1}{\partial X_j}\frac{\partial}{\partial Y_1} + \cdots + \frac{\partial Y_n}{\partial X_j}\frac{\partial}{\partial Y_n} \tag{B.4}$$

The proof of these statements is straightforward.

(ii) Let k be a field and let $B = k[X, Y]/(XY)$, where X, Y are indeterminates. Then, since $\Omega_{k[X,Y]/K} \otimes B \cong BdX \oplus BdY$, using the conormal sequence we deduce that

$$\Omega_{B/k} \cong \frac{BdX \oplus BdY}{(YdX \oplus XdY)}$$

It follows that the element $YdX = -XdY$ is killed by the maximal ideal (X, Y) and therefore it generates a torsion submodule

$$T := (YdX) \cong k \subset \Omega_{B/k}$$

The quotient is

$$\frac{\Omega_{B/k}}{T} = \frac{BdX \oplus BdY}{(YdX, XdY)} \cong k[X]dX \oplus k[Y]dY \cong (X, Y) \subset B$$

where the last isomorphism is given by

$$f(X)dX \oplus g(Y)dY \mapsto f(X)X + g(Y)Y$$

Therefore we have an exact sequence:

$$0 \to T \to \Omega_{B/k} \to B \to k \to 0$$

(iii) Let k be a field and let $B = k[t, X, Y]/(f)$ where t, X, Y are indeterminates and $f = XY + t$. Then arguing as before we see that

$$\Omega_{B/k[t]} \cong \frac{BdX \oplus BdY}{(YdX \oplus XdY)}$$

The element $YdX = -XdY$ is not killed by any $b \in B$; therefore $\Omega_{B/k[t]}$ is torsion free of rank one. The homomorphism

$$\Omega_{B/k[t]} \to B$$

sending $f(t, X)dX \oplus g(t, Y)dY \mapsto f(t, X)X + g(t, Y)Y$ is bijective onto the maximal ideal (t, X, Y) so that we have an exact sequence:

$$0 \to \Omega_{B/k[t]} \to B \to k \to 0$$

(iv) Let k be a field and let $k[\epsilon] := k[t]/(t^2)$, where we have denoted by ϵ the class of t mod (t^2). Then the conormal sequence of $k \to k[t] \to k[\epsilon]$ is

$$(t^2)/(t^4) \xrightarrow{\delta} \Omega_{k[t]/k} \otimes_{k[t]} k[\epsilon] \to \Omega_{k[\epsilon]/k} \to 0$$

and the middle term is isomorphic to $k[\epsilon]$. The map δ acts as

$$\begin{array}{rcl} \bar{t}^2 & \mapsto & 2\epsilon \\ \bar{t}^3 & \mapsto & 0 \end{array}$$

In particular, we see that δ is not injective. Therefore

$$\Omega_{k[\epsilon]/k} = \begin{cases} kd\epsilon & \text{if } \mathrm{char}(k) \neq 2; \\ k[\epsilon]d\epsilon & \text{if } \mathrm{char}(k) = 2 \end{cases}$$

and $d : k[\epsilon] \to \Omega_{k[\epsilon]/k}$ acts as $d(\alpha + \epsilon\beta) = \beta d\epsilon$.

(v) An obvious generalization of the above computation shows that if $A = k[t]/(t^n)$, $n \geq 2$ and $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > n$ then

$$\Omega_{A/k} = A/(\bar{t}^{n-1})$$

(vi) If $B \in \mathrm{ob}(\mathcal{A}^*)$ then $t_B^\vee := m_B/m_B^2$ and $t_B := (m_B/m_B^2)^\vee$ are the (Zariski) *cotangent space*, respectively *tangent space* of B. We have $m_B/m_B^2 \cong \Omega_{B/\mathbf{k}} \otimes_B \mathbf{k}$ by Prop. B.4, and therefore

$$\mathrm{Der}_{\mathbf{k}}(B, \mathbf{k}) = \mathrm{Hom}_B(\Omega_{B/\mathbf{k}}, \mathbf{k}) = \mathrm{Hom}_{\mathbf{k}}(\Omega_{B/\mathbf{k}} \otimes_B \mathbf{k}, \mathbf{k}) = (m_B/m_B^2)^\vee$$

Moreover, there is a natural identification

$$\mathrm{Der}_{\mathbf{k}}(B, \mathbf{k}) = \mathrm{Hom}_{\mathbf{k}-alg}(B, \mathbf{k}[\epsilon])$$

which we leave to the reader to verify.

If $\mu : \Lambda \to B$ is a homomorphism in $\mathcal{A}^*$, the induced homomorphism

$$d\mu^\vee : m_\Lambda/m_\Lambda^2 \to m_B/m_B^2$$

is the *codifferential* of μ, while its transpose

$$d\mu : t_B \to t_\Lambda$$

is the *differential* of μ. We define the *relative cotangent space of B over Λ* to be

$$t_{B/\Lambda}^\vee := \mathrm{coker}(d\mu^\vee) = m_B/(m_B^2 + m_\Lambda B)$$

and the *relative tangent space of B over Λ* as its dual:

$$t_{B/\Lambda} = \ker(d\mu) = \left[m_B/(m_B^2 + m_\Lambda B)\right]^\vee$$

From the exact sequence

$$\Omega_{\Lambda/\mathbf{k}} \otimes_\Lambda B \to \Omega_{B/\mathbf{k}} \to \Omega_{B/\Lambda} \to 0$$

tensored by $\mathbf{k}$ we deduce an identification $t_{B/\Lambda}^\vee = \Omega_{B/\Lambda} \otimes_B \mathbf{k}$ and therefore

$$t_{B/\Lambda} = \mathrm{Hom}_B(\Omega_{B/\Lambda}, \mathbf{k}) = \mathrm{Der}_\Lambda(B, \mathbf{k}) = \mathrm{Hom}_{\Lambda-alg}(B, \mathbf{k}[\epsilon])$$

where the Λ-algebra structure on $\mathbf{k}[\epsilon]$ is defined by the composition $\Lambda \to \mathbf{k} \to \mathbf{k}[\epsilon]$ (the last equality is straightforward to verify).

The following lemma describes a situation where the conormal sequence is exact.

Lemma B.10. *Assume char*$(\mathbf{k}) = 0$. *Let*

$$e : \quad 0 \to (t) \to R' \to R \to 0$$

be a small extension in $\mathcal{A}$. Then the conormal sequence

$$\eta : 0 \to (t) \xrightarrow{\delta} \Omega_{R'/\mathbf{k}} \otimes_{R'} R \to \Omega_{R/\mathbf{k}} \to 0$$

is exact also on the left.

Proof. Assume first that e is trivial, so that $R' = R\tilde{\oplus}\mathbf{k}$. Then the codifferential

$$\begin{array}{ccc} \Omega_{R'/\mathbf{k}} \otimes_{R'} \mathbf{k} & \to & \Omega_{R/\mathbf{k}} \otimes_R \mathbf{k} \\ \| & & \| \\ m_{R'}/m_{R'}^2 & & m_R/m_R^2 \end{array}$$

has a nontrivial kernel (Example 1.1.2) so that a fortiori

$$\Omega_{R'/\mathbf{k}} \otimes_{R'} R \to \Omega_{R/\mathbf{k}}$$

has a nontrivial kernel.

Assume now that e is not trivial. Then, letting $m = \dim(t_{R'}) = \dim(t_R)$, we can write

$$R' = P/J', \quad R = P/J$$

where $P = \mathbf{k}[X_1, \ldots, X_m]$, a polynomial algebra, and $J', J \subset (\underline{X})^2 \subset P$ ideals such that $J' \subset J$ and $J/J' \cong (t)$. Let $T \in J$ be such that

$$t = T + J'$$

Since e is small we have $(\underline{X})J \subset J'$ and therefore $T \notin (\underline{X})J$.

Claim: We can choose T so that

$$\frac{\partial T}{\partial X_i} \notin J \quad \text{for some } i$$

If $\frac{\partial T}{\partial X_i} \in J$ then $X_i \frac{\partial T}{\partial X_i} \in J'$ so that we can replace T by

$$T_1 := T - X_i \frac{\partial T}{\partial X_i}$$

If

$$-X_i \frac{\partial^2 T}{\partial X_i^2} = \frac{\partial T_1}{\partial X_i} \notin J$$

we are done, otherwise we replace T_1 by

$$T_2 := T_1 - \frac{X_i}{2} \frac{\partial T_1}{\partial X_i} = T - X_i \frac{\partial T}{\partial X_i} + \frac{X_i^2}{2} \frac{\partial^2 T}{\partial X_i^2}$$

and we apply the same argument. After ν steps of this process we obtain

$$T_\nu := T_{\nu-1} - \frac{X_i}{\nu} \frac{\partial T_{\nu-1}}{\partial X_i} = T + \sum_{s \geq 1}^{\nu} (-1)^s \frac{X_i^s}{s!} \frac{\partial^s T}{\partial X_i^s}$$

Since $\frac{\partial T_\nu}{\partial X_i} = 0$ for $\nu \gg 0$ we see that either $\frac{\partial T_\nu}{\partial X_i} \notin J$ for some ν and we replace T by the first T_ν with this property, or we can replace T by a T_ν which is constant with respect to X_i. Repeating this process for every index i we will end up by replacing

T by a $\bar{T}$ having the required property or otherwise constant with respect to every variable, which is clearly a contradiction. The claim is proved.

From the claim we deduce that

$$dT = \sum_i \frac{\partial T}{\partial X_i} dX_i \notin J\Omega_{P/\mathbf{k}} \tag{B.5}$$

where $d = d_{P/\mathbf{k}} : P \to \Omega_{P/\mathbf{k}}$ is the universal derivation. But we have:

$$\Omega_{R'/\mathbf{k}} = \Omega_{P/\mathbf{k}} / \big(J'\Omega_{P/\mathbf{k}} + (dg')_{g' \in J'}\big)$$

so that

$$\Omega_{R'/\mathbf{k}} \otimes_{R'} R = \Omega_{P/\mathbf{k}} / \big(J\Omega_{P/\mathbf{k}} + (dg')_{g' \in J'}\big)$$

But since clearly $dt \neq dg'$ for all $g' \in J'$, (B.5) implies that

$$dt \neq 0 \in \Omega_{R'/\mathbf{k}} \otimes_{R'} R$$

□

* * * * * *

If $f : X \to Y$ is a morphism of schemes, we denote by $\Omega^1_{X/Y}$ the *sheaf of relative differentials*, or the *relative cotangent sheaf*, on X. It satisfies

$$\Omega^1_{X/Y,x} = \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$$

for all $x \in X$. If $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes then

$$\Omega^1_{\mathrm{Spec}(B)/\mathrm{Spec}(A)} = (\Omega_{B/A})^{\sim}$$

We denote by

$$T_{X/Y} := Hom(\Omega^1_{X/Y}, \mathcal{O}_X)$$

the *sheaf of relative derivations*, or the *relative tangent sheaf* of f.

We will write Ω^1_X and T_X instead of $\Omega^1_{X/\mathrm{Spec}(\mathbf{k})}$ and $T_{X/\mathrm{Spec}(\mathbf{k})}$ respectively; they are the *cotangent sheaf* and the *tangent sheaf* of X, respectively (cotangent and tangent *bundles* if locally free).

If X is algebraic and $x \in X$ is closed then, by B.4:

$$\Omega^1_{X,x} \otimes \mathbf{k}(x) = \frac{m_{X,x}}{m^2_{X,x}}$$

is the cotangent space of X at x, and

$$T_x X := T_{X,x} \otimes \mathbf{k}(x) = \left(\frac{m_{X,x}}{m^2_{X,x}}\right)^{\vee} \cong \mathrm{Der}_{\mathbf{k}}(\mathcal{O}_{X,x}, \mathbf{k})$$

is the Zariski tangent space of X at x.

Let S be a scheme and

$$X \xrightarrow{g} Y$$

a morphism of S-schemes. The induced homomorphism of sheaves on X:

$$g^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$$

is called the *relative codifferential* of g. The dual homomorphism:

$$T_{X/S} \to Hom(g^*\Omega^1_{Y/S}, \mathcal{O}_X)$$

is the *relative differential* of g. When $S = \text{Spec}(\mathbf{k})$ we have $g^*\Omega^1_Y \to \Omega^1_X$, which is the *codifferential* of g, while its dual

$$dg : T_X \to Hom(g^*\Omega^1_Y, \mathcal{O}_X)$$

is the *differential* of g. Note that if $\Omega^1_{Y/S}$ is locally free then

$$Hom(g^*\Omega^1_{Y/S}, \mathcal{O}_X) = g^*Hom(\Omega^1_{Y/S}, \mathcal{O}_Y) = g^*T_{Y/S}$$

but in general the first and the second sheaves are different. The *relative cotangent sequence* is

$$g^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \tag{B.6}$$

Conditions for the injectivity of the first map in this sequence are given in Theorem C.15, page 302 and Theorem D.2.8, page 310.

If $X \subset Y$ is an embedding of schemes and $\mathcal{I} = \mathcal{I}_{X/Y} \subset \mathcal{O}_Y$ is the ideal sheaf of X in Y, then $\mathcal{I}/\mathcal{I}^2$ is a sheaf of $\mathcal{O}_X$-modules in a natural way, called the *conormal sheaf* of X in Y. Its dual

$$N_{X/Y} := Hom_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = Hom_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_X)$$

is called the *normal sheaf* of X in Y. $N_{X/Y}$ (resp. $\mathcal{I}/\mathcal{I}^2$) is called the *normal bundle* (resp. the *conormal bundle*) of X in Y if it is locally free. Given a closed embedding of S-schemes $i : X \subset Y$, we have an exact sequence of sheaves on X:

$$\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0 \tag{B.7}$$

where $\mathcal{I} \subset \mathcal{O}_Y$ is the ideal sheaf of X in Y. (B.7) is called the *relative conormal sequence*. When $S = \text{Spec}(\mathbf{k})$ we obtain the *conormal sequence*

$$\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_Y \to \Omega^1_X \to 0$$

Conditions for the injectivity of the first map in these sequences are given in Proposition D.1.4, page 306, and Theorem D.2.7, page 310.

Examples B.11. In the following examples we will describe the global vector fields on the given schemes by exhibiting their restrictions to an affine open set. All will be done by explicit computation.

(i) $H^0(T_{I\!P^1})$ can be described explicitly as follows. Consider $I\!P^1 = U_0 \cup U_1$ where $U_0 = \mathrm{Spec}(\mathbf{k}[\xi])$ and $U_1 = \mathrm{Spec}(\mathbf{k}[\eta])$ with $\eta = \xi^{-1}$ on $U_0 \cap U_1$. We have

$$\frac{\partial}{\partial \eta} = \frac{\partial \xi}{\partial \eta}\frac{\partial}{\partial \xi} = -\frac{1}{\eta^2}\frac{\partial}{\partial \xi} = -\xi^2 \frac{\partial}{\partial \xi}$$

on $U_0 \cap U_1$. Let $\theta \in H^0(T_{I\!P^1})$; then

$$\theta_{|U_0} = g(\xi)\frac{\partial}{\partial \xi} \qquad g(\xi) \in \mathbf{k}[\xi]$$

and

$$\theta_{|U_1} = h(\eta)\frac{\partial}{\partial \eta}, \quad h(\eta) \in \mathbf{k}[\eta]$$

On $U_0 \cap U_1$ we have

$$g(\xi)\frac{\partial}{\partial \xi} = h(\eta)\frac{\partial}{\partial \eta} = -h(\xi^{-1})\xi^2 \frac{\partial}{\partial \xi}$$

and therefore $g(\xi) = -h(\xi^{-1})\xi^2$. It follows that $g(\xi) = a_0 + a_1\xi + a_2\xi^2$ and $h(\eta) = -(a_0\eta^2 + a_1\eta + a_2)$, with $a_0, a_1, a_2 \in \mathbf{k}$. In particular, $H^0(T_{I\!P^1}) \cong \mathbf{k}^3$.

Moreover, $H^i(T_{I\!P^1}) = 0$ if $i \geq 1$. For $i \geq 2$ it is obvious. Let $\theta \in H^1(T_{I\!P^1})$ be represented by a Čech 1-cocycle defined by $\theta_{01} \in \Gamma(U_0 \cap U_1, T_{I\!P^1})$. It can be written as

$$\theta_{01} = \sum_{i=-m}^{n} a_i \xi^i$$

Letting $\theta_1 = \sum_{i=-m}^{-1} a_i \eta^{-i}$ and $\theta_0 = -\sum_{i=0}^{n} a_i \xi^i$ we obtain:

$$\theta_{01} = \theta_1 - \theta_0$$

so (θ_{01}) is a coboundary.

(ii) We want to describe $H^0(T_{\mathbf{A}^1 \times I\!P^1})$. Let $\mathbf{A}^1 \times I\!P^1 = V_0 \cup V_1$ where

$$V_0 = \mathbf{A}^1 \times U_0 = \mathrm{Spec}[z, \xi])$$

$$V_1 = \mathbf{A}^1 \times U_1 = \mathrm{Spec}[z, \eta])$$

and $\eta = \xi^{-1}$ on $V_0 \cap V_1 = \mathrm{Spec}(\mathbf{k}[z, \xi, \xi^{-1}])$. We have

$$\frac{\partial}{\partial \eta} = \frac{\partial \xi}{\partial \eta}\frac{\partial}{\partial \xi} = -\frac{1}{\eta^2}\frac{\partial}{\partial \xi} = -\xi^2 \frac{\partial}{\partial \xi}$$

on $V_0 \cap V_1$. Let $\theta \in H^0(T_{\mathcal{A}^1 \times I\!P^1})$; then

$$\theta_{|V_0} = g(z, \xi)\frac{\partial}{\partial z} + h(z, \xi)\frac{\partial}{\partial \xi} \qquad g(z, \xi), h(z, \xi) \in \mathbf{k}[z, \xi]$$

$$\theta_{|V_1} = \gamma(z, \eta)\frac{\partial}{\partial z} + \chi(z, \eta)\frac{\partial}{\partial \eta} \qquad \gamma(z, \eta), \chi(z, \eta) \in \mathbf{k}[z, \eta]$$

On $V_0 \cap V_1$ we have:

$$g(z,\xi) = \gamma(z,\xi^{-1})$$

and therefore $g(z,\xi) = g(z)$ is constant with respect to ξ. Moreover,

$$h(z,\xi)\frac{\partial}{\partial\xi} = \chi(z,\eta)\frac{\partial}{\partial\eta} = -\chi(z,\xi^{-1})\xi^2\frac{\partial}{\partial\xi}$$

and therefore

$$h(z,\xi) = -\chi(z,\xi^{-1})\xi^2$$

It follows that $h(z,\xi) = a(z) + b(z)\xi + c(z)\xi^2$, with $a(z), b(z), c(z) \in \mathbf{k}[z]$. In conclusion, every $\theta \in H^0(T_{\mathbf{A}^1\times I\!P^1})$ restricts to V_0 as a vector field of the form

$$\theta_{|V_0} = g(z)\frac{\partial}{\partial z} + (a(z) + b(z)\xi + c(z)\xi^2)\frac{\partial}{\partial\xi} \tag{B.8}$$

with $g(z), a(z), b(z), c(z) \in \mathbf{k}[z]$, and conversely, every such vector field is the restriction of a global section of $T_{\mathbf{A}^1\times I\!P^1}$. As in example (i) we also deduce that $H^i(T_{\mathbf{A}^1\times I\!P^1}) = 0$ if $i \geq 1$.
In a similar way, one describes $H^0(T_{(\mathbf{A}^1\setminus\{0\})\times I\!P^1})$ by showing that the image of the restriction

$$H^0(T_{(\mathbf{A}^1\setminus\{0\})\times I\!P^1}) \to H^0(T_{(\mathbf{A}^1\setminus\{0\})\times U_0})$$

consists of the vector fields of the form (B.8) with $g(z), a(z), b(z), c(z) \in \mathbf{k}[z, z^{-1}]$.

(iii) We now consider, for a given integer $m \geq 0$, the rational ruled surface

$$F_m = I\!P(\mathcal{O}_{I\!P^1}(m) \oplus \mathcal{O}_{I\!P^1})$$

Let $\pi : F_m \to I\!P^1$ be the projection. Then F_m can be represented as

$$F_m = \pi^{-1}(U) \cup \pi^{-1}(U') = (U \times I\!P^1) \cup (U' \times I\!P^1)$$

where $U = \mathrm{Spec}(\mathbf{k}[z])$, $U' = \mathrm{Spec}(\mathbf{k}[z'])$ and $z' = z^{-1}$ on $U \cap U'$. We consider the affine open sets

$$V_0 = \mathrm{Spec}(\mathbf{k}[z,\xi]) \subset U \times I\!P^1$$

$$V_0' = \mathrm{Spec}(\mathbf{k}[z',\xi']) \subset U' \times I\!P^1$$

where on $V_0 \cap V_0' = \mathrm{Spec}(\mathbf{k}[z, z^{-1}, \xi]) = \mathrm{Spec}(\mathbf{k}[z', z'^{-1}, \xi'])$ we have:

$$z' = z^{-1}, \quad \xi' = z^m\xi$$

Therefore we have:

$$\begin{aligned} \frac{\partial}{\partial z'} &= -z^2\frac{\partial}{\partial z} + mz\xi\frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\xi'} &= z^{-m}\frac{\partial}{\partial\xi} \end{aligned} \tag{B.9}$$

We will describe a typical element $\theta \in H^0(T_{F_m})$ by describing its restriction to the open sets V_0 and V_0'. We have, by example (ii) above:

$$\theta_{|V_0} = g(z)\frac{\partial}{\partial z} + (a(z) + b(z)\xi + c(z)\xi^2)\frac{\partial}{\partial \xi}$$

with $g(z), a(z), b(z), c(z) \in \mathbf{k}[z]$ and similarly

$$\theta_{|V_0'} = \rho(z')\frac{\partial}{\partial z'} + (\alpha(z') + \beta(z')\xi' + \gamma(z')\xi'^2)\frac{\partial}{\partial \xi'}$$

with $\rho(z'), \alpha(z'), \beta(z'), \gamma(z') \in \mathbf{k}[z']$. Imposing their equality on $V_0 \cap V_0'$ and using (B.9) we obtain the following conditions:

$$\begin{aligned} g(z) = &\ -\rho(z^{-1})z^2 \\ a(z) = &\ \alpha(z^{-1})z^{-m} \\ b(z) = &\ \beta(z^{-1}) + \rho(z^{-1})mz \\ c(z) = &\ \gamma(z^{-1})z^m \end{aligned} \tag{B.10}$$

We distinguish the cases $m = 0$ and $m > 0$. If $m = 0$ (B.10) give:

$$\begin{aligned} g(z) = &\ g_0 + g_1 z + g_2 z^2 \\ a(z) = &\ a \\ b(z) = &\ b \\ c(z) = &\ c \end{aligned} \qquad g_0, g_1, g_2, a, b, c \in \mathbf{k}$$

In the case $m > 0$ we have:

$$\begin{aligned} g(z) = &\ g_0 + g_1 z + g_2 z^2 \\ a(z) = &\ 0 \\ b(z) = &\ b - mz(g_1 + g_2 z) \\ c(z) = &\ c_0 + c_1 z + \cdots + c_m z^m \end{aligned} \qquad g_0, g_1, g_2, b, c_0, \ldots, c_m \in \mathbf{k}$$

Since the restriction $H^0(T_{F_m}) \to H^0(T_{V_0})$ is injective and we have described its image, we can conclude:

$$\begin{aligned} H^0(T_{F_0}) &\cong \mathbf{k}^6 \\ H^0(T_{F_m}) &\cong \mathbf{k}^{m+5} \end{aligned}$$

In particular, *F_m and F_n are not isomorphic if $m \neq n$* (note that $F_0 \cong I\!P^1 \times I\!P^1$ is not isomorphic to $F_1 \cong Bl_{(1,0,0)} I\!P^2$).

Since, by the calculations of the previous example (ii)

$$h^i(T_{U \times I\!P^1}) = h^i(T_{U' \times I\!P^1}) = h^i(T_{(U \cap U') \times I\!P^1}) = 0, \quad i \geq 1 \tag{B.11}$$

we deduce that:

$$H^1(T_{F_m}) = H^0(T_{(U \cap U') \times I\!P^1})/H^0(T_{U \times I\!P^1}) + H^0(T_{U' \times I\!P^1})$$

An easy computation based on (B.10) shows that, for $m \geq 1$, $H^1(T_{F_m})$ consists of the classes, modulo $H^0(T_{U \times I\!P^1}) + H^0(T_{U' \times I\!P^1})$, of the vector fields

$$(b_1 z + \cdots + b_{m-1} z^{m-1})\frac{\partial}{\partial \xi}$$

In particular,

$$H^1(T_{F_m}) \cong \mathbf{k}^{m-1} \tag{B.12}$$

It also follows from (B.11) that

$$H^2(T_{F_m}) = (0) \tag{B.13}$$

NOTES

1. Let $X \to Y$ be a morphism of algebraic schemes. Prove that there is an exact sequence

$$0 \to \Omega^1_{X/Y} \to \mathcal{P}^1_{X/Y} \to \mathcal{O}_X \to 0 \tag{B.14}$$

which globalizes (B.1). $\mathcal{P}^1_{X/Y}$ is called the *sheaf of principal parts of X over Y*, denoted by $\mathcal{P}^1_X$ if $Y = \mathrm{Spec}(\mathbf{k})$.

Let $X = I\!P(V)$ for a finite-dimensional $\mathbf{k}$-vector space V. Then the exact sequence (B.14) is the dual of the Euler sequence; in particular,

$$\mathcal{P}^1_{I\!P(V)} \cong \mathcal{O}_{I\!P(V)}(-1) \otimes V^\vee$$

Therefore (B.14) is a generalization of the Euler sequence to any $X \to Y$.

2. Consider $I\!P = I\!P(V)$ for a finite-dimensional $\mathbf{k}$-vector space V and the *incidence relation*:

$$\mathbf{I} = \{(x, H) : x \in H\} \subset I\!P \times I\!P^\vee \tag{B.15}$$

Consider the twisted and dualized Euler sequence:

$$0 \to \Omega^1_{I\!P(V)}(1) \to \mathcal{O}_{I\!P(V)} \otimes V^\vee \to \mathcal{O}_{I\!P(V)}(1) \to 0$$

From its definition it follows that $\mathbf{I} = I\!P(\Omega^1_{I\!P(V)}(1))$ and $I\!P \times I\!P^\vee = I\!P(\mathcal{O}_{I\!P(V)} \otimes V^\vee)$ and the inclusion in (B.15) is induced by the first homomorphism in the above sequence.

C

Smoothness

The notion of “formal smoothness”, introduced in [1], Ch. IV § 17, is of crucial importance in deformation theory, and therefore plays a special role in this book. In this appendix we introduce this concept from scratch, and we show how it is related to the notion of “smooth morphism” as introduced in [3] Exposé II, and [84]. We will not give a systematic treatment of the properties of smooth morphisms in algebraic geometry: the reader is referred to the above quoted references for them. For more details on the approach taken in this section the reader can also consult [13] and [94].

Definition C.1. *A ring homomorphism $f : R \to B$ is called* formally smooth*, and B is called a* formally smooth R-algebra*, if for every exact sequence:*

$$0 \to I \to A \xrightarrow{\eta} A' \to 0 \tag{C.1}$$

where A and A' are local artinian R-algebras, each R-algebra homomorphism $B \to A'$ has a lifting $B \to A$; equivalently, if the map:

$$\mathrm{Hom}_{R-alg}(B, A) \to \mathrm{Hom}_{R-alg}(B, A') \tag{C.2}$$

is surjective.
f is called smooth *if it is formally smooth and essentially of finite type (shortly e.f.t.). If the map (C.2) is bijective (instead of only being surjective) for all exact sequences (C.1), then f is* formally etale*; f is* etale *if it is formally etale and e.f.t..*

Recall that e.f.t. means that B is a localization of an R-algebra of finite type (see e.g. [127]). It is easy to prove by induction that it suffices to check the above conditions only for the exact sequences (C.1) such that $I^2 = (0)$, i.e. for *extensions* of local artinian R-algebras.

Proposition C.2. *(i) If B is a ring and $\Delta \subset B$ is a multiplicative system, then $B \to \Delta^{-1}B$ is formally etale. In particular, B is a formally etale B-algebra.*
(ii) The composition of formally smooth (resp. formally etale) homomorphisms is formally smooth (resp. formally etale).

(iii) If $f : R \to B$ is formally smooth (resp. formally etale) and C is an R-algebra, then $C \to C \otimes_R B$ is formally smooth (resp. formally etale).
(iv) A finitely generated field extension $K \subset L$ is smooth if and only if L is separable over K.
(v) Let $R \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms, and assume that f is formally etale. Then gf is formally smooth (resp. formally etale) if and only if g is formally smooth (resp. formally etale).

Proof. (i) Given an exact sequence (C.1) and a commutative diagram

$$\begin{array}{ccc} B & \rightarrow & \Delta^{-1}B \\ \downarrow \varphi' & & \downarrow \varphi \\ A & \underset{p}{\longrightarrow} & A' \end{array}$$

we must find $\tilde{\varphi} : \Delta^{-1}B \to A$ which makes it commutative. For every $s \in \Delta$ choose $a_s \in A$ such that $\varphi(s)^{-1} = p(a_s)$. Since

$$p(\varphi'(s)a_s) = \varphi(s)\varphi(s)^{-1} = 1_{A'}$$

we have

$$\varphi'(s)a_s = 1_A + i_s \qquad i_s \in I$$

for every $s \in \Delta$. Therefore

$$\varphi'(s)a_s(1_A - i_s) = 1_A$$

Hence $\varphi'(s) \in A$ is invertible. Now define $\tilde{\varphi}(r/s) = \varphi'(r)\varphi'(s)^{-1}$.

Noting that $\tilde{\varphi}$ is uniquely determined by φ' we get the assertion.

(ii) and (iii) are straightforward.

(iv) Assume first that $K \subset L$ is separable. By (ii) it suffices to consider the cases $L = K(X)$ and $L = K[X]/(f(X))$ where f is irreducible and $f'(x) \neq 0$. The first case is left to the reader (see remark C.3(i)).
In the second case consider an extension $\bar{A} = A/I$ of local artinian K-algebras, where $I \subset A$ is an ideal with $I^2 = (0)$. Let $\varphi : K[X]/(f(X)) \to \bar{A}$ be a homomorphism, sending $\bar{X} \mapsto \bar{\alpha}$. Choose arbitrarily $\alpha \in A$ such that $\bar{\alpha} = \alpha \bmod I$. It will suffice to find $e \in I$ such that

$$f(\alpha + e) = 0$$

We have $f(\alpha + e) = f(\alpha) + f'(\alpha)e$. Since $f'(\alpha)$ is a unit mod I it is also a unit in A, and therefore we can take $e = -f(\alpha)/f'(\alpha)$.

Assume conversely that $K \subset L$ is smooth. Then $L = F[X]/J$ where F is a purely transcendental extension of K and J is a principal ideal. We have an exact sequence of finite-dimensional L-vector spaces:

$$J/J^2 \to \Omega_{F[X]/K} \otimes L \to \Omega_{L/K} \to 0$$

where J/J^2 is one-dimensional. By the first part of the proof F is smooth over K and by B.3(ii) the left map is injective because, by the smoothness of L over K, the surjection $F[X]/J^2 \to L$ splits. It follows that

$$\dim(\Omega_{L/K}) = \dim(\Omega_{F[X]/K} \otimes L) - 1 = \mathrm{trdeg}_K(F[X]) - 1$$

$$= \mathrm{trdeg}_K(F) = \mathrm{trdeg}_K(L)$$

From B.1(iii) it follows that $K \subset L$ is separable.

(v) "if" follows immediately from (ii); "only if" is left to the reader. □

Remarks C.3. (i) Any polynomial algebra $R[X_1, X_2 \ldots]$ is trivially a formally smooth R-algebra. From C.2(i) it follows that a localization of a polynomial R-algebra is also a formally smooth R-algebra.

More precisely, a localization $P = S^{-1}R[X_1, X_2, \ldots]$ of a polynomial algebra over a ring R satisfies the following condition, stronger than formal smoothness:

For every extension of R-algebras:

$$0 \to I \to A \to A' \to 0$$

where A and A' are R-algebras and $I^2 = 0$ the map

$$\mathrm{Hom}_{R-alg}(P, A) \to \mathrm{Hom}_{R-alg}(P, A')$$

is surjective.

Every R-algebra B is a quotient of a formally smooth R-algebra, because it is a quotient of a polynomial R-algebra. From C.2(i) it follows that every e.f.t. R-algebra is a quotient of a smooth R-algebra.

This is trivial for polynomial rings, and in the general case it can be proved adapting the proof of C.2(i) in an obvious way.

(ii) if R is in $\mathrm{ob}(\hat{\mathcal{A}})$ then every formal power series ring $R[[X_1, X_2, \ldots]]$ is a formally smooth R-algebra, because local artinian R-algebras are complete. More precisely, a formal power series ring $R[[X_1, X_2, \ldots]]$ satisfies the following condition, stronger than formal smoothness over R:

For every extension:

$$0 \to I \to A \to A' \to 0$$

of complete local R-algebras the map

$$\mathrm{Hom}_{R-alg}(P, A) \to \mathrm{Hom}_{R-alg}(P, A')$$

is surjective.

The proof is straightforward and is left to the reader.

The following result characterizes an important class of formally smooth algebras.

Theorem C.4. *Let k be a field and let (B, m) be a noetherian local k-algebra with residue field K. Suppose that K is finitely generated and separable over k. Then the following are equivalent:*

(i) B is regular.
(ii) $\hat{B} \cong K[[X_1, \ldots, X_d]]$, where $d = \dim(B)$.
(iii) B is a formally smooth k-algebra.

Proof. (i) $\Leftrightarrow$ (ii) is standard (see [48], prop. 10.16 and exercise 19.1).
(ii) $\Rightarrow$ (iii). It follows directly from the definition that B is formally smooth over k if and only if $\hat{B}$ is. Since $\hat{B}$ is formally smooth over K (remark C.3(ii)), and since K is smooth over k by C.2(iv), the conclusion follows by transitivity.
(iii) $\Rightarrow$ (i). Let $\{x_1, \ldots, x_d\}$ be a system of generators of m. Then, since B/m^2 is complete and K is separable over k, B/m^2 contains a coefficient field ([48], Theorem 7.8). Therefore there exists an isomorphism

$$\upsilon_1 : B/m^2 \cong K[X_1, \ldots, X_d]/M^2 \qquad\qquad M = (X_1, \ldots, X_d)$$

Let $\upsilon : B \to B/m^2 \xrightarrow{\upsilon_1} K[X_1, \ldots, X_d]/M^2$. By the formal smoothness of B and by induction we can find a lifting of υ:

$$\upsilon_n : B \to K[X_1, \ldots, X_d]/M^{n+1}$$

for every $n \geq 2$. Consider the elements

$$\upsilon_n(x_1), \ldots, \upsilon_n(x_d) \in M/M^{n+1}$$

Their classes generate M/M^2, hence they generate M/M^{n+1}, by Nakayama. Then we have:

$$\begin{aligned} K[X_1, \ldots, X_d]/M^{n+1} \quad & = \upsilon_n(B) + (M/M^{n+1}) \\ = \upsilon_n(B) + \textstyle\sum_i \upsilon_n(x_i)\big[\upsilon_n(B) + (M/M^{n+1})\big] & = \upsilon_n(B) + (M/M^{n+1})^2 = \cdots \\ = \upsilon_n(B) + (M/M^{n+1})^{n+1} \quad & = \upsilon_n(B) \end{aligned}$$

hence υ_n is surjective. Since $m^{n+1} \subset \ker(\upsilon_n)$ we have:

$$\ell(B/m^{n+1}) \geq \ell(K[X_1, \ldots, X_d]/M^{n+1}) = \binom{d+n}{d}$$

and this implies that $\dim(B) \geq d$. Since m is generated by d elements it follows that B is regular. □

For the reader's convenience we include the proof of the following well-known lemma:

Lemma C.5. *(i) A surjective endomorphism $f : A \to A$ of a noetherian ring is an automorphism.*

(ii) Let A be a complete noetherian local ring and $\psi : A \to A$ an endomorphism inducing an isomorphism $\psi_1 : A/m_A^2 \to A/m_A^2$. Then ψ is an automorphism.

Proof. (i) We have an ascending chain of ideals

$$\ker(f) \subseteq \ker(f^2) \subseteq \ker(f^3) \subseteq \cdots$$

Since A is noetherian we have $\ker(f^n) = \ker(f^{n+1}) = \ker(f^{n+2}) = \cdots$ for some n, and it suffices to prove that $\ker(f^n) = (0)$. After replacing f by f^n we may assume $\ker(f) = \ker(f^2)$. Let $a \in \ker(f)$; by assumption there exists $b \in A$ such that $a = f(b)$. Then $0 = f(a) = f^2(b)$ and therefore $b \in \ker(f^2) = \ker(f)$, i.e. $a = f(b) = 0$.

(ii) Let $gr(A) = A/m \oplus m/m^2 \oplus \cdots$ be the associated graded ring. Since $gr(A)$ is generated by m/m^2 over A/m the endomorphism $gr(\psi) : gr(A) \to gr(A)$ induced by ψ is surjective. It follows that ψ is also surjective. In fact, given $a \in A$ the surjectivity of $gr(\psi)$ implies that there are $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots \in A$ such that $a_i \in m^{i-1}$, $b_i \in m^i$, and

$$a = f(a_1) + b_1,\ b_1 = f(a_2) + b_2,\ b_2 = f(a_3) + b_3, \ldots$$

We obtain a convergent power series $\bar{a} = a_1 + a_2 + a_3 + \cdots$ such that

$$a - \psi(a_1 + a_2 + \cdots + a_n) = b_n \in m^{n+1}$$

On the limit we therefore get $a = \psi(\bar{a})$. The conclusion is now a consequence of (i). □

Proposition C.6. *Let $f : R \to B$ be a local homomorphism of noetherian local rings containing a field k isomorphic to their residue fields. Then the following conditions are equivalent:*

(i) f is formally smooth.
(ii) $\hat{B}$ is isomorphic to a formal power series ring over $\hat{R}$.
(iii) The homomorphism $\hat{f} : \hat{R} \to \hat{B}$ induced by f is formally smooth.

Proof. (i) ⇒ (ii). Let $m \subset B$ and $n \subset R$ be the maximal ideals. Choose elements $x_1, \ldots, x_d \in \hat{B}$ inducing a k-basis of $\hat{B}/(\hat{m}^2 + \hat{f}(\hat{n}))$, and let $F = \hat{R}[[X_1, \ldots, X_d]]$, where $X_1, \ldots, X_d$ are indeterminates. Denote by $M \subset F$ the maximal ideal.

The homomorphism

$$\begin{array}{rccc} u: & F & \to & \hat{B} \\ & X_i & \mapsto & x_i \end{array}$$

induces an isomorphism

$$u_1 : F/(M^2 + \hat{n}F) \to \hat{B}/(\hat{m}^2 + \hat{f}(\hat{n}))$$

By the formal smoothness of f the composition

$$v_1 : B \to \hat{B} \to \hat{B}/(\hat{m}^2 + \hat{f}(\hat{n})) \xrightarrow{u_1^{-1}} F/(M^2 + \hat{n}F)$$

can be lifted to an R-homomorphism

$$v_k : B \to F/M^k$$

for each $k \geq 2$. Therefore the sequence $\{v_k\}$ defines an $\hat{R}$-homomorphism

$$v : \hat{B} \to F$$

such that $vu : F \to F$ and $uv : \hat{B} \to \hat{B}$ induce isomorphisms $(vu)_1 : F/M^2 \to F/M^2$ and $(uv)_1 : \hat{B}/\hat{m}^2 \to \hat{B}/\hat{m}^2$ respectively. From Lemma C.5 it follows that u and v are isomorphisms.
(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (i) is left to the reader. □

Corollary C.7. *Let $f : R \to B$ be a local homomorphism of noetherian local rings containing a field k isomorphic to their residue fields. Then the following conditions are equivalent:*

(i) f is formally etale.
(ii) The homomorphism $\hat{f} : \hat{R} \to \hat{B}$ induced by f is an isomorphism.

Proof. Left to the reader.

Corollary C.8. *Let R be in* $\mathrm{ob}(\mathcal{A}^*)$. *The inclusion $f : R \to \hat{R}$ is formally etale.*

The proof is obvious. We now restrict our attention to smooth homomorphisms, i.e. we add the condition that the homomorphism is e.f.t.. In this case the module of differentials comes into play; moreover, the defining condition of Definition C.1 can be replaced by the more general condition (i) in the following statement.

Theorem C.9. *Let $f : R \to B$ be an e.f.t. ring homomorphism. Then the following conditions are equivalent:*

(i) For every extension of R-algebras:

$$0 \to I \to A \to A' \to 0 \tag{C.3}$$

the map

$$\mathrm{Hom}_{R-alg}(B, A) \to \mathrm{Hom}_{R-alg}(B, A')$$

is surjective.

(ii) If $B = P/J$, where $P = S^{-1}R[X_1, \ldots, X_d]$, $S \subset R[X_1, \ldots, X_d]$ is a multiplicative system and $J \subset P$ is an ideal, the conormal sequence

$$0 \to J/J^2 \xrightarrow{\delta} \Omega_{P/R} \otimes_P B \to \Omega_{B/R} \to 0$$

is split exact. In particular, J/J^2 and $\Omega_{B/R}$ are finitely generated projective B-modules.

(iii) B is a smooth R-algebra.
(iv) (Jacobian criterion of smoothness) If P and J are as in (ii) the map

$$(J/J^2) \otimes_B K(p) \overset{\delta\otimes_B K(p)}{\longrightarrow} \Omega_{P/R} \otimes_P K(p) \qquad \text{where } K(p) = B_p/m_{B_p}$$

is injective for every prime ideal $p \subset B$.

Proof. (i) ⇒ (ii). The hypothesis implies that the extension:

$$0 \to J/J^2 \to P/J^2 \to B \to 0$$

splits. Therefore the conormal sequence is split exact by B.3(iii) and it follows that J/J^2 and $\Omega_{B/R}$ are finitely generated projective because the module $\Omega_{P/R} \otimes_P B$ is free of finite rank.

(ii) ⇒ (i). Consider an exact sequence (C.3) and a homomorphism of R-algebras $f' : B \to A'$. By Remark C.3(i) there exists an R-homomorphism $g : P \to A$ making the following diagram commute:

$$\begin{array}{ccc} P & \to & B \\ \downarrow g & & \downarrow f' \\ A & \to & A' \end{array}$$

Since $g(J) \subset I$, we see that g factors through P/J^2, so that we have a commutative diagram:

$$\begin{array}{ccc} P/J^2 & \to & B \\ \downarrow \bar{g} & & \downarrow f' \\ A & \to & A' \end{array}$$

The hypothesis implies, via B.3(iii), that there exists $h : B \to P/J^2$ a splitting of $P/J^2 \to B$. The composition $f = \bar{g}h : B \to A$ gives a lifting of f'.

(i) ⇒ (iii) is obvious.

(iii) ⇒ (iv). We may assume B and P local with residue field K. To prove that $\delta \otimes_B K$ is injective, it suffices to show that for every K-vector space V the map induced by δ:

$$\begin{array}{ccc} \mathrm{Hom}_K(\Omega_{P/R} \otimes_P K, V) & \to & \mathrm{Hom}_K((J/J^2) \otimes_B K, V) \\ \| & & \| \\ \mathrm{Der}_R(P, V) & & \mathrm{Hom}_B(J/J^2, V) \end{array}$$

is surjective. Consider a homomorphism $g : J/J^2 \to V$, and the associated pushout diagram (see § 1.1 for the definition):

$$\begin{array}{ccccccccc} \Lambda : & 0 & \to & J/J^2 & \to & P/J^2 & \to & B & \to & 0 \\ & & & \downarrow g & & \downarrow & & \| & & \\ g_*(\Lambda) : & 0 & \to & V & \to & Q & \to & B & \to & 0 \end{array}$$

We can write $m_Q = V \oplus m'$, where $m' \subset Q$ is an ideal, because V is annihilated by m_Q. Therefore the previous diagram can be embedded in the following:

$$\begin{array}{lccccccccc}
 & & & & & P & & & & \\
 & & & & & \downarrow & \searrow & & & \\
\Lambda: & 0 & \to & J/J^2 & \to & P/J^2 & \to & B & \to & 0 \\
 & & & \downarrow g & & \downarrow & & \| & & \\
g_*(\Lambda): & 0 & \to & V & \to & Q & \to & B & \to & 0 \\
 & & & \| & & \downarrow & & \downarrow \bar{v} & & \\
\eta: & 0 & \to & V & \to & Q/m' & \to & K & \to & 0
\end{array}$$

where η is an extension of local artinian R-algebras. From the smoothness of B we deduce the existence of $v : B \to Q/m'$ lifting the projection $\bar{v} : B \to K$. Denoting by $r : P \to B$ the natural map, and by $w : P \to P/J^2 \to Q \to Q/m'$ the composition, consider the homomorphism:

$$d = w - vr : P \to V$$

It is easy to show that this is an R-derivation, which induces g.

(iv) $\Rightarrow$ (ii). From Nakayama's lemma it follows that $\ker(\delta) \otimes B_p = (0)$ for all prime ideals $p \subset B$ and therefore $\ker(\delta) = (0)$. Moreover, since $\Omega_{P/R} \otimes_B B_p$ is free and finitely generated it follows that $\mathrm{Tor}_1^{B_p}(\Omega_{B/R} \otimes_B B_p, K(p)) = 0$: it follows that $\Omega_{B/R} \otimes_B B_p$ is flat, and therefore, being finitely generated, it is free. Thus $\Omega_{B/R}$ is projective, δ has a splitting and J/J^2 is also projective and finitely generated. □

The following result follows easily from what we have seen so far.

Theorem C.10. *Let B be an integral* **k***-algebra of finite type and of dimension d. Then the following are equivalent:*

(i) B_p is smooth over **k** *for each prime ideal $p \in \mathrm{Spec}(B)$.*
(ii) B is a regular ring.
(iii) $\Omega_{B/\mathbf{k}}$ is projective of rank d.
(iv) B is smooth over **k**.

Proof. (ii) $\Leftrightarrow$ (iii) is Corollary B.6.

(i) $\Leftrightarrow$ (ii) follows from Theorem C.4.

(iv) $\Rightarrow$ (i). for each $p \in \mathrm{Spec}(B)$, B_p is smooth over B by Proposition C.2(i); from Proposition C.2(ii) it follows that B_p is smooth over **k**.

(i) $\Rightarrow$ (iv). (i) implies that condition (iv) of Theorem C.9 is satisfied for all $p \in \mathrm{Spec}(B)$, so that B is smooth by Theorem C.9. □

From now on we will freely replace the defining property for smooth homomorphisms given in Definition C.1 by condition (i) of Theorem C.9. Here is a first example.

Proposition C.11. *Let R be a ring, P an R-algebra and $B = P/J$ for an ideal $J \subset P$. If B is a smooth R-algebra the conormal sequence*

$$0 \to J/J^2 \to \Omega_{P/R} \otimes_P B \to \Omega_{B/R} \to 0$$

is split exact and $\Omega_{B/R}$ is projective and finitely generated. If, moreover, P is a smooth R-algebra then J/J^2 is finitely generated and projective as well.

Proof. Since B is smooth the R-algebra extension

$$0 \to J/J^2 \to P/J^2 \to B \to 0$$

splits. Therefore the conormal sequence splits by Theorem B.3(iii) and $\Omega_{B/R}$ is finitely generated and projective by Theorem C.9. If P is smooth then $\Omega_{P/R}$ is finitely generated and projective as well and so is J/J^2. □

Corollary C.12. *Let P be an e.f.t. $\mathbf{k}$-algebra and $B = P/J$ for an ideal $J \subset P$. Assume that B is reduced. Then in the conormal sequence*

$$J/J^2 \xrightarrow{\delta} \Omega_{P/\mathbf{k}} \otimes_P B \to \Omega_{B/\mathbf{k}} \to 0 \qquad \text{(C.4)}$$

$\ker(\delta)$ *is a torsion B-module whose support is contained in the singular locus of* $\mathrm{Spec}(B)$.

If J/J^2 is torsion free then δ is injective.

Proof. Since B is reduced there is a dense open subset $U \subset \mathrm{Spec}(B)$ such that B_p is a regular local ring for all $p \in U$. From Theorem C.4 it follows that B_p is a smooth $\mathbf{k}$-algebra for all such p and, by Propositions C.11 and B.1(ii), the conormal sequence (C.4) localized at p is split exact. It follows that $\ker(\delta)_p = (0)$ for all $p \in U$ and the conclusion follows. The last assertion is an obvious consequence of the first part. □

The next result explains the relation between smoothness and the relative cotangent sequence.

Theorem C.13. *Let $K \xrightarrow{f} R \xrightarrow{g} B$ be ring homomorphisms, with g smooth. Then the relative cotangent sequence:*

$$0 \to \Omega_{R/K} \otimes_R B \xrightarrow{\alpha} \Omega_{B/K} \to \Omega_{B/R} \to 0$$

is split exact.

Proof. By Theorem B.2 it suffices to prove that α is a split injection; this is equivalent to showing that, for any B-module M, the induced map:

$$\begin{array}{ccc} \mathrm{Hom}_B(\Omega_{B/K}, M) & \xrightarrow{\alpha^\vee} & \mathrm{Hom}_B(\Omega_{R/K} \otimes_R B, M) \\ \| & & \| \\ \mathrm{Der}_K(B, M) & & \mathrm{Der}_K(R, M) \\ & & \\ D' & \mapsto & D'g \end{array}$$

is split surjective. Let $D : R \to M$ be a K-derivation and consider the commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{1_B} & B \\ \uparrow g & & \uparrow \\ R & \xrightarrow{\gamma} & B\tilde{\oplus}M \end{array}$$

where $\gamma(r) = (g(r), D(r))$, $r \in R$. By the smoothness of g we can find a homomorphism of R-algebras $\psi : B \to B\tilde{\oplus}M$ making the diagram

$$\begin{array}{ccc} B & \xrightarrow{1_B} & B \\ \uparrow g & \searrow \psi & \uparrow \\ R & \underset{\gamma}{\longrightarrow} & B\tilde{\oplus}M \end{array}$$

commutative. The homomorphism ψ is necessarily of the form:

$$\psi(b) = (b, D'(b))$$

and $D' : B \to M$ is a K-derivation such that $D = D'g$. This proves the surjectivity of $\alpha^\vee$. Now take $M = \Omega_{R/K} \otimes_R B$ and $D = d_{R/K} \otimes g : R \to \Omega_{R/K} \otimes_R B$ and let

$$\alpha' : \Omega_{B/K} \to \Omega_{R/K} \otimes_R B$$

be the B-linear map corresponding to $D' : B \to \Omega_{R/K} \otimes_R B$. Then $\alpha'\alpha = 1_M$ and this proves that α is split injective. □

Corollary C.14. *Let* $K \xrightarrow{f} R \xrightarrow{g} B$ *be ring homomorphisms, with g etale. Then*

$$\Omega_{R/K} \otimes_R B \cong \Omega_{B/K}$$

and

$$\Omega_{B/R} = (0)$$

Proof. By the relative cotangent sequence the two assertions are equivalent. We will prove the first. Keeping the notations of the proof of C.13, the hypothesis that g is etale implies that the derivation D' is unique and consequently α is an isomorphism. □

* * * * * *

A morphism $\varphi : X \to Y$ of algebraic schemes is *smooth at a point* $x \in X$ if $\mathcal{O}_{X,x}$ is a smooth $\mathcal{O}_{Y,\varphi(x)}$-algebra; φ is *smooth* if it is smooth at every point. The definition of *etale morphism* is given similarly. This definition is equivalent to the definition of smooth (resp. etale) morphism as given in [3] and in [84]. The equivalence can be seen by means of the jacobian criterion of smoothness, proved in Theorem C.9, and using [3], Exposé II, Corollaire 5.9.

By translating into geometrical language the algebraic results proved above we deduce in particular the following.

Theorem C.15. *Let S be an algebraic scheme, and* $\varphi : X \to Y$ *a morphism of algebraic S-schemes. Then:*

(i) If φ is smooth at $x \in X$ then the relative cotangent sequence

$$0 \to \varphi^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

is split exact at x and $\Omega^1_{X/Y}$ is locally free at x. The rank of the free module $\Omega^1_{X/Y,x}$ is called the relative dimension of φ at x.

(ii) φ is etale at $x \in X$ if and only if it is smooth of relative dimension zero at x. In particular, $\Omega^1_{X/Y,x} = 0$ (i.e. φ is unramified at x) and therefore we have an isomorphism

$$\varphi^*\Omega^1_{Y/S,x} \cong \Omega^1_{X/S,x}$$

(iii) If X is smooth over S at x and φ is a closed embedding with ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ then the relative conormal sequence

$$O \to \mathcal{I}/\mathcal{I}^2 \to \varphi^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0$$

is exact at x and $\Omega^1_{X/S}$ is free at x; if, moreover, Y is also smooth over S at $\varphi(x)$ then $\mathcal{I}/\mathcal{I}^2$ is free at x as well.

The exactness of the relative conormal sequence in part (iii) holds under more general assumptions as well (see Theorem D.2.7).

D

Complete intersections

D.1 Regular embeddings

Definition D.1.1. *An embedding of schemes* $j : X \subset Y$ *is a* regular embedding of codimension n at the point $x \in X$ *if* $j(x)$ *has an affine open neighbourhood* $\mathrm{Spec}(R)$ *in* Y *such that the ideal of* $j(X) \cap \mathrm{Spec}(R)$ *in* R *can be generated by a regular sequence of length* n. *If this happens at every point of* X *we say that* j *is a* regular embedding of codimension n.

An open embedding is a regular embedding of codimension 0. If X and Y are both nonsingular then $X \subset Y$ is a regular embedding. The set of points of X where an embedding $j : X \subset Y$ is regular is open.

If $X \subset Y$ is a regular embedding of codimension n then $\mathcal{I}/\mathcal{I}^2$ and $N_{X/Y}$ are both locally free of rank n ([48], Exercise 17.12, p. 440). It follows from standard facts in commutative algebra (see [48], Exercise 17.16, p. 441) that $\mathcal{I}^k/\mathcal{I}^{k+1}$ is locally free as well for every $k \geq 2$.

A ring B is called a *complete intersection* if $\mathrm{Spec}(B)$ can be regularly embedded in $\mathrm{Spec}(R)$ where R is a regular ring.

A scheme X is a *local complete intersection* (l.c.i.) if every local ring $\mathcal{O}_{X,x}$ is a complete intersection ring.

A *nonsingular scheme* X, i.e. a scheme all of whose local rings are regular, is an example of an l.c.i. scheme. If $X \subset Y$ is a regular embedding and Y is an l.c.i. scheme, then X is an l.c.i. scheme.

Lemma D.1.2. *Let* $f : X \to Y$ *be a morphism of schemes and let* $Z \subset Y$ *be a regular embedding of codimension* n. *Then the induced embedding* $j : X \times_Y Z \subset X$ *has codimension* $\leq n$ *at every point and if equality holds at a point* $x \in X \times_Y Z$ *then* j *is regular at* x.

Proof. If $\mathcal{I}_Z \subset \mathcal{O}_Y$ is the ideal sheaf of Z in Y then the ideal sheaf $f^{-1}\mathcal{I}_Z$ of $X \times_Y Z \subset X$ is locally generated at a point x by the n images of the local generators of $\mathcal{I}_{Z,f(x)}$. The conclusion follows easily from this fact. □

If we have a flag of embeddings of schemes $X \subset Y \subset Z$ and $\mathcal{I}_Y \subset \mathcal{I}_X \subset \mathcal{O}_Z$ are the ideal sheaves of X and Y, we have the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{I}_X \to \mathcal{I}_{X/Y} \to 0 \tag{D.1}$$

where $\mathcal{I}_{X/Y} \subset \mathcal{O}_Y$ is the ideal sheaf of X in Y. After tensoring by $\otimes_{\mathcal{O}_Z}\mathcal{O}_X$ we obtain an exact sequence of coherent $\mathcal{O}_X$-modules:

$$\frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \otimes \mathcal{O}_X \xrightarrow{\alpha} \frac{\mathcal{I}_X}{\mathcal{I}_X^2} \to \frac{\mathcal{I}_{X/Y}}{\mathcal{I}_{X/Y}^2} \to 0 \tag{D.2}$$

Its dual is the sequence:

$$0 \to N_{X/Y} \to N_{X/Z} \to N_{Y/Z} \otimes \mathcal{O}_X \tag{D.3}$$

Lemma D.1.3. *(i) If $f : X \subset Y$ and $g : Y \subset Z$ are regular embeddings of codimensions m and n respectively, then $gf : X \to Z$ is a regular embedding of codimension $m+n$.*
(ii) If the embeddings f and g are both regular then we have exact sequences of locally free sheaves on X:

$$0 \to \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \otimes \mathcal{O}_X \xrightarrow{\alpha} \frac{\mathcal{I}_X}{\mathcal{I}_X^2} \to \frac{\mathcal{I}_{X/Y}}{\mathcal{I}_{X/Y}^2} \to 0 \tag{D.4}$$

$$0 \to N_{X/Y} \to N_{X/Z} \to N_{Y/Z} \otimes \mathcal{O}_X \to 0 \tag{D.5}$$

Proof. (i) Left to the reader.
(ii) All sheaves in (D.4) are locally free because they are conormal bundles of regular embeddings. Since $\mathrm{Im}(\alpha)$ is a torsion free sheaf of the same rank of $(\mathcal{I}_Y/\mathcal{I}_Y^2) \otimes \mathcal{O}_X$, it follows that α must be injective. The sequence (D.5) is exact because $Ext^1_{\mathcal{O}_X}(\mathcal{I}_{X/Y}/\mathcal{I}_{X/Y}^2, \mathcal{O}_X) = 0$. □

Proposition D.1.4. *Let $j : X \subset Y$ be an embedding of algebraic schemes, with X reduced and Y nonsingular. Consider the conormal sequence*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega^1_{Y|X} \to \Omega^1_X \to 0 \tag{D.6}$$

(where $\mathcal{I} \subset \mathcal{O}_Y$ is the ideal sheaf of X). Then:

(i) The homomorphism δ is injective on the open set where j is a regular embedding.
(ii) If X and Y are nonsingular then the dual sequence

$$0 \to T_X \to T_{Y|X} \to N_{X/Y} \to 0 \tag{D.7}$$

is exact.

Proof. (i) It suffices to show that δ is injective under the assumption that j is a regular embedding. In this case the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free of rank equal to the codimension of X. The sequence (D.6) is exact at every nonsingular point $x \in X$ by Theorem C.15(iii). Since X is reduced, this happens on a dense open subset so that ker(δ) is supported on a nowhere dense subset. But X has no embedded points because it is regularly embedded in Y: it follows that $\ker(\delta) = 0$.

(ii) Under the stated hypothesis, j is a regular embedding and Ω^1_X is locally free, so we have $Ext^1(\Omega^1_X, \mathcal{O}_X) = 0$ and the exactness of (D.7) follows. □

Remark D.1.5. If we don't assume X reduced, part (i) of the proposition is false in general. An example is given by the closed regular embedding of codimension 1:

$$\mathrm{Spec}(\mathbf{k}[\epsilon]) \subset \mathrm{Spec}(\mathbf{k}[t]) = \mathbf{A}^1$$

(see Example B.9(iv)).

A morphism $f : X \to Y$ of schemes will be called a *cover* (or a *covering*) if it is finite and surjective.

Recall that a morphism of schemes $f : X \to Y$ is called *unramified at a point* $x \in X$ if $\Omega^1_{X/Y,x} = 0$; f is *unramified* if it is unramified at every $x \in X$. After identifying X with the diagonal $\Delta \subset X \times_Y X$, we see that $\Omega^1_{X/Y}$ gets identified with the conormal sheaf of this embedding. It follows that f is unramified at x if and only if $\Delta \subset X \times_Y X$ is an open embedding at x, and that the locus of $x \in X$ such that f is unramified at x is open. Moreover, f is unramified if and only if Δ is both open and closed in $X \times_Y X$.

D.2 Relative complete intersection morphisms

We now introduce a natural class of morphisms which generalize smooth morphisms and behave well with respect to differentials and base change.

Definition D.2.1. *A flat morphism of finite type $f : X \to S$ is called a* relative complete intersection (r.c.i.) morphism at the point $x \in X$ *if there is an open neighbourhood U of x such that the restriction of f to U can be obtained as a composition*

$$U \xrightarrow{j} V \xrightarrow{g} S$$

where j is a regular embedding and g is smooth. If f is an r.c.i. morphism at every point we call it an r.c.i. morphism*, and we call X a* complete intersection over S.

This definition is equivalent to Def. 19.3.6 of Ch. IV of [1]; the equivalence is proved in [19], Prop. 1.4. Note that in case $S = \mathrm{Spec}(\mathbf{k})$ the morphism f is an r.c.i. if and only if X is an l.c.i. of finite type.

If $X \to S$ is a flat morphism of finite type of nonsingular varieties then f is an r.c.i. because it factors as

$$X \to X \times S \to S$$

where the first morphism is the graph of f.

Before discussing the main properties of this notion we need two lemmas.

Lemma D.2.2. *Let $A \to B$ be a ring homomorphism, M a B-module and $f_1, \dots, f_n$ an M-regular sequence of elements of B. Assume that for each $i = 1, \dots, n$ the module $M/(\sum_{j=1}^{i-1} f_j M)$ is A-flat. Then, for every ring homomorphism $A \to A'$, letting $B' = B \otimes_A A'$, $M' = M \otimes_A A'$, and $f_i' = f_i \otimes 1$ ($1 \leq i \leq n$), the sequence $f_1', \dots, f_n'$ of elements of B' is M'-regular and the modules $M'/(\sum_{j=1}^{i-1} f_j' M')$ are A'-flat.*

Proof. Consider the exact sequence:

$$0 \to M \xrightarrow{f_1} M \to M/f_1 M \to 0$$

Since $M/f_1 M$ is A-flat, the sequence:

$$0 \to M \otimes_A A' \xrightarrow{f_1 \otimes 1} M \otimes_A A' \to (M/f_1 M) \otimes_A A' \to 0$$

is exact, and therefore f_1' is not a zero-divisor for M'. Let $M_i = M/(\sum_{j=1}^{i} f_j M)$, $M_i' = M'/(\sum_{j=1}^{i} f_j' M')$; then we have $M_i' = M_i \otimes_A A'$, $M_{i+1} = M_i/f_{i+1} M_i$, $M_{i+1}' = M_i'/f_{i+1}' M_i'$. Replacing M and f_1 by M_i and f_{i+1} in the above argument, one deduces that f_{i+1}' is not a zero-divisor for M_i', thereby proving the first assertion by induction. The last assertion follows from A.2(VII). □

Lemma D.2.3. *Let $A \to B$ be a local homomorphism of noetherian local rings, M a B-module of finite type, flat over A, and $f_1, \dots, f_n \in m_B$. For $1 \leq i \leq n$ let g_i be the image of f_i in $B \otimes_A k$, where $k = A/m_A$ is the residue field of A. Then the following conditions are equivalent:*

(i) $f_1, \dots, f_n$ is an M-regular sequence, and $M_i = M/(\sum_{j=1}^{i} f_j M)$ is A-flat for all $1 \leq i \leq n$.
(ii) $g_1, \dots, g_n$ is an $(M \otimes_A k)$-regular sequence.

Proof. (i) ⇒ (ii) follows from D.2.2 applied to $A' = k$.

(ii) ⇒ (i) Applying Corollary A.6, from the injectivity of $g_1 : M \otimes_A k \to M \otimes_A k$ we deduce that $f_1 : M \to M$ is injective and that $M_1 = M/f_1 M$ is A-flat. Proceeding by induction on i, assume M_i flat over A. Since $g_{i+1} : M_i \otimes_A k \to M_i \otimes_A k$ is injective from A.6, again we deduce that $f_{i+1} : M_i \to M_i$ is injective and that M_{i+1} is A-flat. □

In the next proposition some general properties of r.c.i. morphisms are proved.

Proposition D.2.4. *(i) An open embedding is an r.c.i. morphism. A smooth morphism of finite type is an r.c.i. morphism.*

(ii) If $f : X \to S$ is an r.c.i. morphism and $h : S' \to S$ is a morphism, then the morphism $f' : X \times_S S' \to S'$ induced by f after base change is an r.c.i. morphism.

Proof. (i) is an immediate consequence of the definition and (ii) follows easily from Lemma D.2.2. □

From D.2.4(ii) it follows in particular that if $f : X \to S$ is an r.c.i. morphism then X_s is an l.c.i. for every **k**-rational point $s \in S$. So for example, a non-l.c.i. algebraic scheme cannot be the fibre of a flat morphism of algebraic nonsingular varieties.

The next result gives a useful characterization of r.c.i. morphisms.

Proposition D.2.5. *Let*

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ & \searrow f \quad \swarrow g & \\ & S & \end{array} \tag{D.8}$$

be a commutative diagram of morphisms of algebraic schemes, where f is flat, g is smooth and j is an embedding. Then the following conditions are equivalent for a **k***-rational point $x \in X$:*

(i) f is an r.c.i. morphism at x.
(ii) Letting $s = f(x)$, the fibre X_s is an l.c.i. at x.
(iii) j is a regular embedding at x.

Proof. (i) ⇒ (ii) follows from D.2.4(ii) and (iii) ⇒ (i) is obvious.

(ii) ⇒ (iii) From (ii) it follows that the embedding $j_s : X_s \subset Y_s$ is regular at x. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal sheaf of X. Tensoring the exact sequence

$$0 \to \mathcal{I} \to O_Y \to \mathcal{O}_X \to 0$$

by $- \otimes_{\mathcal{O}_S} \mathbf{k}$ we obtain the sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_S} \mathbf{k} \to \mathcal{O}_{Y_s} \to \mathcal{O}_{X_s} \to 0$$

which is exact because f is flat. Therefore $\mathcal{I} \otimes_{\mathcal{O}_S} \mathbf{k}$ is the ideal sheaf of $j(X_s)$ in Y_s. Consider a sequence $f_1, \ldots, f_n$ of sections of $\mathcal{I}$ in an open neighbourhood of $j(x)$ which induce a basis of $\mathcal{I}_{j(x)}/(m_s\mathcal{I}_{j(x)} + \mathcal{I}^2_{j(x)})$ as a $\mathcal{O}_{Y,j(x)}/(m_s\mathcal{O}_{Y,j(x)} + \mathcal{I}_{j(x)})$-module. Then the images $f_1 \otimes 1 = g_1, \ldots, f_n \otimes 1 = g_n$ are generating sections of $\mathcal{I} \otimes_{\mathcal{O}_S} \mathbf{k}$ in an open neighbourhood of $j(x)$ in Y_s which form a regular sequence in $j(x)$. From Nakayama's lemma it follows that $f_1, \ldots, f_n$ generate $\mathcal{I}$ in an open neighbourhood of $j(x)$ in Y. From Lemma D.2.3 it follows that $f_1, \ldots, f_n$ form a regular sequence in $j(x)$ and therefore (iii) holds. □

Corollary D.2.6. *Under the hypothesis of Proposition D.2.5, the locus of points $x \in X$ such that f is an r.c.i. at x is open. If f is proper then the locus of points $s \in S$ such that X_s is an l.c.i. is open.*

Proof. The last assertion follows from the first because a proper map is closed. The first assertion can be proved using characterization D.2.5(iii) of r.c.i. morphism and the fact that the locus where an embedding is regular is open. □

We conclude this section with two results about the relative conormal sequence and cotangent sequence for r.c.i. morphisms.

Theorem D.2.7. *Let*

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ & \searrow f & \downarrow g \\ & & S \end{array}$$

be a commutative diagram of morphisms of algebraic schemes, with f an r.c.i., j an immersion and g smooth. Let $\mathcal{J} \subset \mathcal{O}_Y$ be the ideal sheaf of $j(X)$. If f is smooth on a dense open subset intersecting every fibre then the relative conormal sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \xrightarrow{\delta} j^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0$$

is exact and $\mathcal{J}/\mathcal{J}^2$ is locally free.

Proof. From the equivalence $(i) \Leftrightarrow (iii)$ in Proposition D.2.5 it follows that j is a regular embedding and therefore $\mathcal{J}/\mathcal{J}^2$ is locally free. Moreover, the support of $\ker(\delta)$ does not contain any generic point of X nor any fibre of f because it is contained in the locus where f is not smooth. Since f is generically smooth and j is a regular embedding, X has no embedded components except possibly for some union of fibres. It follows that δ is injective. □

Theorem D.2.8. *Let $f : X \to S$ be an r.c.i morphism of algebraic schemes, and assume f smooth on a dense open subset intersecting every fibre. Then the relative cotangent sequence*

$$0 \to f^*\Omega^1_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0 \tag{D.9}$$

is exact.

Proof. We only have to prove the injectivity of the left homomorphism and the question is local on X. Since all schemes are algebraic, locally on X we can construct the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{j} & V & \xrightarrow{i} & U \\ & \searrow f & \downarrow \psi & & \downarrow \varphi \\ & & S & \xrightarrow{h} & W \end{array}$$

where W, U, ψ, φ are smooth, i, h are closed embeddings and j is a regular closed embedding. From the smooth morphism φ we deduce the exact sequence of locally free sheaves on U:

$$0 \to \varphi^*\Omega^1_W \to \Omega^1_U \to \Omega^1_{U/W} \to 0$$

which restricts on X to the exact sequence:

$$0 \to (hf)^*\Omega^1_W \to (ij)^*\Omega^1_U \to j^*\Omega^1_{V/S} \to 0$$

Let $\mathcal{J} \subset \mathcal{O}_V$ and $\mathcal{I} \subset \mathcal{O}_U$ be the ideal sheaves of the embeddings j and ij respectively, and $\mathcal{H} \subset \mathcal{O}_W$ the ideal sheaf of the embedding h. Then we have an exact and commutative diagram of coherent sheaves on X:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & f^*(\mathcal{H}/\mathcal{H}^2) & \to & \mathcal{I}/\mathcal{I}^2 & \to & \mathcal{J}/\mathcal{J}^2 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow \delta_j & \\
0 \to & (hf)^*\Omega^1_W & \to & (ij)^*\Omega^1_U & \to & j^*\Omega^1_{V/S} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & f^*\Omega^1_S & \xrightarrow{df^\vee} & \Omega^1_X & \to & \Omega^1_{X/S} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

where the second and the third columns are the relative conormal sequences of ij and of j respectively; the first column is the pullback to X of the conormal sequence of h; the first row is exact because $\psi^*(\mathcal{H}/\mathcal{H}^2)$ is the conormal sheaf of i; the map δ_j is injective by Theorem D.2.7 and by the assumptions made on X and f. A diagram chasing shows that the codifferential $df^\vee$ is injective and proves the theorem. □

NOTES

1. An algebraic scheme can have different embeddings in $I\!P^r$, i.e. by means of nonisomorphic invertible sheaves, but with same normal sheaf. An example is given by a projective nonsingular curve C of genus 1, and by the embeddings in $I\!P^3$ given by two nonisomorphic invertible sheaves $L_1$and L_2 of degree 4 such that $L_1^2 = L_2^2$. Then C is embedded as a nonsingular complete intersection of two quadrics by both sheaves, and the normal bundles are $L_1^2 \oplus L_1^2 = L_2^2 \oplus L_2^2$.

2. Let S be a scheme, and X, Y smooth over S. Prove that every closed S-embedding $X \subset Y$ is regular. In particular, every section of a smooth morphism $f : Y \to S$ is a regular embedding of codimension equal to the relative dimension of f.

3. Let $f : \mathcal{X} \to S$ be a morphism of finite type and $s \in S$ a **k**-rational point. Let $m_s \subset \mathcal{O}_{S,s}$ be the maximal ideal and $\mathcal{I} = \mathcal{I}_{\mathcal{X}(s)}$ the ideal sheaf of the fibre $\mathcal{X}(s)$ of f over s. Prove that we have a surjective homomorphism

$$\frac{m_s}{m_s^2} \otimes_{\mathbf{k}} \mathcal{O}_{\mathcal{X}(s)} \to \mathcal{I}/\mathcal{I}^2$$

and an injection:

$$N_{\mathcal{X}(s)/\mathcal{X}} \subset T_{S,s} \otimes_{\mathbf{k}} \mathcal{O}_{\mathcal{X}(s)}$$

If f is flat then they are isomorphisms; in particular, if f is flat then $N_{\mathcal{X}(s)/\mathcal{X}}$ is free.

E

Functorial language

Let $\mathcal{C}$ be a category. A covariant (resp. contravariant) functor F from $\mathcal{C}$ to (sets) is said to be *representable* if there is an object X in $\mathcal{C}$ such that F is isomorphic to the functor

$$Y \mapsto \text{Hom}(X, Y) \tag{E.1}$$

(resp. $Y \mapsto \text{Hom}(Y, X)$). We will denote by h_X a functor of the form (E.1). The representable functors are a full subcategory, isomorphic to $\mathcal{C}^\circ$ (resp. to $\mathcal{C}$ in the contravariant case), of the category $Funct(\mathcal{C}, \text{(sets)})$ of covariant functors (resp. $Funct(\mathcal{C}^\circ, \text{(sets)})$ of contravariant functors) from $\mathcal{C}$ to (sets).

To fix ideas let's consider covariant functors. In order to investigate conditions for the representability of a given functor F it is convenient to study functorial morphisms $h_X \to F$. Such morphisms turn out to be easy to describe, thanks to the following elementary lemma:

Lemma E.1 (Yoneda). *Let $F : \mathcal{C} \to$ (sets) be a covariant functor. For each object X in $\mathcal{C}$ there is a canonical bijection:*

$$\begin{array}{ccc} \text{Hom}(h_X, F) & \leftrightarrow & F(X) \\ \Phi & \mapsto & \Phi(X)(1_X) \end{array}$$

Let's mention, in passing, that functorial morphisms $F \to h_X$ are much harder to control. They are related to the notion of "coarse moduli scheme".

We may consider *couples* of the form (X, ξ), where X is an object of $\mathcal{C}$ and $\xi \in F(X)$. Yoneda's lemma implies that to give such a couple is equivalent to giving a morphism of functors $h_X \to F$; if this morphism is an isomorphism then (X, ξ) is called a *universal couple*, and ξ a *universal element*, for F. The existence of a universal couple is equivalent to the representability of F.

The couples for F are the objects of a category in which a *morphism* $(X, \xi) \to (Y, \eta)$ between two couples is by definition a morphism $f : X \to Y$ in $\mathcal{C}$

such that $F(f)(\xi) = \eta$. We denote this category by I_F. A morphism $f : (X, \xi) \to (Y, \eta)$ in I_F corresponds to a commutative diagram of morphisms of functors:

$$\begin{array}{ccc} h_X & \xrightarrow{\xi} & F \\ \uparrow f & \nearrow \eta & \\ h_Y & & \end{array}$$

We have an obvious "forgetful functor"

$$I_F \to \mathcal{C}$$

The fibres of this functor are precisely the sets $F(X)$, which are embedded as subcategories of I_F by $\xi \mapsto (X, \xi)$.
(Recall that, given a functor $G : \mathbf{C} \to \mathbf{D}$, the fibre $G^{-1}(D)$ of G over an object D of $\mathbf{D}$ is a subcategory of $\mathbf{C}$, consisting of all objects C such that $G(C) = D$ and of all morphisms f such that $G(f) = 1_D$. A set can be viewed as a category whose objects are its elements and the only morphisms are the identity morphisms.)

Lemma E.2. *The functor F is representable if and only if the category I_F has an initial object (X, ξ). If this is the case, (X, ξ) is a universal couple for F.*

The proof is immediate. Note that, since an initial object is unique up to isomorphism, it follows that a representable functor has a unique universal couple, up to isomorphism.

* * * * * *

Let I and $\mathcal{D}$ be two categories. Given an object A of $\mathcal{D}$, the constant functor $c_A : I \to \mathcal{D}$ is defined as $c_A(i) = A$ for each object i of I and $c_A(f) = 1_A$ for each morphism f in I. Note that c_A is both covariant and contravariant. Every morphism $\alpha : A \to B$ in $\mathcal{D}$ induces an obvious morphism of functors $c_\alpha : c_A \to c_B$. Consider a covariant functor $\Phi : I \to \mathcal{D}$. An *inductive limit* of Φ is an object A of $\mathcal{D}$ and a functorial morphism $\lambda : \Phi \to c_A$ such that for every other morphism $\mu : \Phi \to c_B$ there is a morphism $\alpha : A \to B$ such that $\mu = c_\alpha \lambda$.

$$\begin{array}{ccc} \Phi & \xrightarrow{\lambda} & c_A \\ & \searrow \mu & \downarrow c_\alpha \\ & & c_B \end{array}$$

From the definition it follows that an inductive limit of Φ, if it exists, is unique up to unique isomorphism, and is denoted by

$$\varinjlim \Phi$$

In practice an inductive limit is an object A of $\mathcal{D}$ such that there is a morphism $\Phi(i) \to A$ for each $i \in Ob(I)$ with the condition that the diagram

$$\begin{array}{ccc} \Phi(i) & \to & A \\ \downarrow \Phi(f) & \nearrow & \\ \Phi(j) & & \end{array}$$

is commutative for each morphism $f : i \to j$ in I; moreover, these data must satisfy a universal property.

Dually, one has the notion of *projective limit* of a covariant functor $\Phi : I \to \mathcal{D}$: it is an object A of $\mathcal{D}$ and a morphism $\pi : c_A \to \Phi$ such that for every other morphism $\rho : c_B \to \Phi$ there is a morphism $\beta : B \to A$ such that $\rho = \pi c_\beta$. The projective limit of Φ, if it exists, is denoted by

$$\varprojlim \Phi$$

The above notions can be defined without changes replacing the covariant functor Φ by a contravariant one. We will write Φ_i for $\Phi(i)$, for each object i of I, and sometimes

$$\varinjlim \Phi_i \quad (\text{resp. } \varprojlim \Phi_i) \quad \text{instead of} \quad \varinjlim \Phi \quad (\text{resp. } \varprojlim \Phi)$$

Example E.3. Let J be a partially ordered set. We define a category $Ord(J)$ as follows. The objects of $Ord(J)$ are the elements of J; for any $i, j \in J$ the set $\mathrm{Hom}_{Ord(J)}(i, j)$ consists of one element if $i \leq j$ and is $\emptyset$ otherwise. A covariant (resp. contravariant) functor $\Phi : Ord(J) \to \mathcal{D}$ is called an *inductive system* (resp. a *projective system*) in $\mathcal{D}$ indexed by J; in the case $\mathcal{D}$ =(sets), we obtain the usual notions of inductive (projective) system and of inductive (projective) limit.
If I is a set and $\Phi : I \to \mathcal{D}$ is a functor, where $\mathcal{D}$ is a category with arbitrary coproducts, then

$$\varinjlim \Phi = \coprod_i \Phi_i$$

Similarly, if $\mathcal{D}$ has products then

$$\varprojlim \Phi = \prod_i \Phi_i$$

Proposition E.4. *The inductive limit and projective limit exist for every functor Φ : $I \to$ (sets) from any category I.*

Proof. We take

$$\varinjlim \Phi = \coprod_i \Phi_i / R$$

where R is the equivalence relation generated by pairs (x, y), $x \in \Phi_i$ and $y \in \Phi_j$, such that there exists $\varphi : i \to j$ with $\Phi(x) = y$. Similarly for the projective limit. □

Example E.5. Let $F : \mathcal{C} \to$ (sets) be a covariant functor, and let I_F be the category of couples for F. Then we have a contravariant functor

$$\Phi : I_F \to Funct(\mathcal{C}, \text{(sets)})$$

which sends a couple (X, ξ) to the functor $h_X : \mathcal{C} \to$ (sets), and a morphism $f : (X, \xi) \to (Y, \eta)$ to the functorial morphism $h_f : h_Y \to h_X$ induced by f.

By construction there is a morphism $\Phi \to c_F$. This morphism makes F the inductive limit of the functor Φ (the proof is an easy exercise). We will write:

$$F = \varinjlim_{(X,\xi)} h_X$$

Definition E.6. *A category I is* filtered *if:*

(a) for every pair of objects i, j in I there exists an object k in I and morphisms:

$$\begin{array}{ccc} & & i \\ & & \downarrow \\ j & \to & k \end{array}$$

(b) each pair of morphisms $i \rightrightarrows j$ has a coequalizer $i \rightrightarrows j \to k$.

The category I is cofiltered *if the dual category I° is filtered.*

Assume from now on that $\mathcal{C}$ is a category with products and fibred products.

Definition E.7. *A covariant functor $F : \mathcal{C} \to$ (sets) is called* left exact *if $F(B \times C) = F(B) \times F(C)$ and $F(B \times_A C) = F(B) \times_{F(A)} F(C)$ for each diagram*

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ B & \to & A \end{array}$$

in $\mathcal{C}$ (i.e. F commutes with finite products and finite fibred products).

Every representable functor is left exact by definition of product and fibred product.

Lemma E.8. *Let I be a filtered category and $\Phi : I \to Funct(\mathcal{C}, \text{(sets)})$ a covariant functor. Then, for each diagram in $\mathcal{C}$:*

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ B & \to & A \end{array}$$

there is a bijection:

$$\varinjlim \Phi_i(B) \times_{\lim \Phi_i(A)} \varinjlim \Phi_i(C) \cong \varinjlim [\Phi_i(B) \times_{\Phi_i(A)} \phi_i(C)]$$

The proof of this lemma is straightforward and we omit it. The following result is a useful characterization of left-exact functors.

Proposition E.9. *A covariant functor $F : \mathcal{C} \to$ (sets) is left exact if and only if the category I_F is cofiltered.*

Proof. Assume that I_F is cofiltered. Applying Lemma E.8 to the functor Φ of Example E.5, we see that the inductive limit $F = \lim_{(X,\xi)} h_X$ is left exact because each functor h_X is left exact.

Conversely, assume that F is left exact. Let $(X, \xi), (Y, \eta) \in Ob(I_F)$; we must find

$$\begin{array}{ccc} (Z,\zeta) & \to & (X,\xi) \\ \downarrow & & \\ (Y,\eta) & & \end{array}$$

Take $(Z, \zeta) = (X \times Y, (\xi, \eta))$. Now consider $(X, \xi) \overset{\to}{\to} (Y, \eta)$ coming from $\phi, \psi : X \to Y$. We have

$$F(\phi)(\xi) = F(\psi)(\xi) = \eta$$

Consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_\phi} & X \times Y \\ \uparrow & & \uparrow \Gamma_\psi \\ K & \to & X \end{array}$$

where $\Gamma_\phi = (1_X, \phi)$ and $\Gamma_\psi = (1_X, \psi)$ and $K = X \times_{X \times Y} X$. Since F is left exact

$$F(K) = F(X) \times_{F(X \times Y)} F(X)$$

and there is $\chi \in F(K)$ corresponding to (ξ, ξ):

$$\begin{array}{ccc} \xi & \overset{F(\Gamma_\phi)}{\longmapsto} & (\xi, \eta) \\ \uparrow & & \uparrow F(\Gamma_\psi) \\ \chi & \longmapsto & \xi \end{array}$$

Then (K, χ) is the equalizer of ϕ and ψ. Therefore I_F is cofiltered. □

Let I be a category. A full subcategory J of I is *cofinal* if for each $i \in Ob(I)$ there is a morphism $f : i \to j$ for some $j \in Ob(J)$. It follows immediately from the definitions that if $\Phi : I \to \mathcal{D}$ is a covariant functor and $\Phi_J : J \to \mathcal{D}$ is its restriction, then

$$\lim_{\to} \Phi = \lim_{\to} \Phi_J$$

* * * * * *

Let Z be a scheme. In this subsection we will consider contravariant functors defined on (schemes/Z). All we will say holds, with obvious modifications, for functors defined on (algschemes/Z), the full subcategory of algebraic Z-schemes. A contravariant functor

$$F : (\text{schemes}/Z)^\circ \to (\text{sets})$$

defines on every Z-scheme S a presheaf of sets:

$$U \mapsto F(U)$$

for all open sets $U \subset S$. For this reason a functor as above is also called a *presheaf*. F is called a *sheaf* (more precisely a *sheaf in the Zariski topology*) if it defines a sheaf on every scheme; namely, if for all Z-schemes S and for all open coverings $\{U_i\}$ of S the following is an exact sequence of sets:

$$F(S) \to \prod_i F(U_i) \overset{\to}{\to} \prod_{i,j} F(U_i \cap U_j)$$

The most important sheaves are the *representable functors*, i.e. functors isomorphic to one of the form:

$$S \mapsto \mathrm{Hom}_Z(S, X)$$

for some Z-scheme X. Such a functor is called the *functor of points* of X/Z.

It is very important to have conditions, easy to verify in practice, for a contravariant functor $F : (\text{schemes}/Z)^\circ \to (\text{sets})$ to be representable. Certainly a necessary condition is that F is a sheaf. Another necessary condition is the following.

Recall that a subfunctor G of F is said to be an *open* (resp. *closed*) *subfunctor* if for every scheme S and for every morphism of functors

$$\mathrm{Hom}(-, S) \to F$$

the fibred product $\mathrm{Hom}(-, S) \times_F G$, which is a subfunctor of $\mathrm{Hom}(-, S)$, is represented by an open (resp. closed) subscheme of S. A family of open subfunctors $\{G_i\}$ of F is a *covering* of F if for every Z-scheme S and for every morphism of functors $\mathrm{Hom}(-, S) \to F$ the family $\{\mathrm{Hom}(-, S) \times_F G_i\}$ of subschemes of S is an open covering of S.

An obvious example is obtained by considering an open (resp. closed) subscheme X' of a Z-scheme X: correspondingly, we obtain an open (resp. closed) subfunctor $\mathrm{Hom}(-, X')$ of $\mathrm{Hom}(-, X)$. An open cover $\{X_i\}$ of X defines a cover of $\mathrm{Hom}(-, X)$ by open subfunctors.

Therefore a second obvious necessary condition for a functor F to be representable is that it can be covered by representable open subfunctors. We will now show that these two necessary conditions are also sufficient.

Proposition E.10. *Let*

$$F : (\text{schemes}/Z)^\circ \to (\text{sets})$$

be a contravariant functor. Suppose that:

(a) F is a sheaf;
(b) F admits a covering by representable open subfunctors F_i.

Then F is representable.

Proof. Letting $F_{ij} = F_i \times_F F_j$, by (b) the projections $F_{ij} \to F_i$ correspond to open embedding of schemes $X_{ij} \to X_i$. Therefore the F_i's patch together to form a representable functor $\mathrm{Hom}(-, X)$, where X is the scheme obtained by patching the X_i's together along the X_{ij}'s. By (a), F and $\mathrm{Hom}(-, X)$ are isomorphic. □

The following is an easy but important remark.

Lemma E.11. *If F is a sheaf then F is determined by its restriction to the category of affine schemes.*

Proof. In fact, if S is any Z-scheme we can consider an affine open cover $\{U_i\}$. For any i, j we take an affine open cover $\{V_{i,j,\alpha}\}$ of $U_i \cap U_j$; composing the map

$$F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

with the inclusions $F(U_i \cap U_j) \to \prod_\alpha F(V_{i,j,\alpha})$ we obtain the exact sequence:

$$F(S) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j,\alpha} F(V_{i,j,\alpha})$$

which shows that $F(S)$ is determined by its values on affine schemes. □

This lemma implies that the functor of points of a scheme X

$$F = \mathrm{Hom}_Z(-, X) : (\text{schemes})^\circ \to (\text{sets})$$

is determined by its restriction to the category of affine schemes, or equivalently, by its covariant version:

$$F : (\mathbf{k}\text{-algebras}) \to (\text{sets})$$

Since the category of schemes is isomorphic to the category of functors of points, this means that we can *define* schemes as certain types of functors on (**k**-algebras). Thanks to Proposition E.10, we can say that these functors are precisely the sheaves admitting an open cover by affine schemes, i.e. by representable functors. This point of view is very fruitful because it gives the possibility of generalizing the notion of scheme by considering more general functors. The notion of *algebraic space* is such a generalization (see Artin [13]).

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