# Part III

# APPENDIX

# A

# An Overview of Itô Stochastic Calculus

The appendix provides a brief account of Itô stochastic integration theory. It is not our intention to compete here with any full-scale textbook on stochastic analysis. We merely gathered here the most relevant results that are frequently referred to in previous chapters. Hence the choice of results is definitely subjective and their exposition as concise as possible.

We start by recalling some classic concepts from the theory of stochastic processes. Next, we define the standard Brownian motion and we introduce the notion of the Itô stochastic integral. Subsequently, we present the fundamental formula of stochastic calculus, the *Itô formula*, and its most important applications. Finally, the probability distributions of certain functionals of a Brownian motion are examined. For more details on Itô stochastic integration with respect to the Brownian motion (and, more generally, continuous semimartingales) we refer the reader, for instance, to monographs by Krylov (1995), Durrett (1996), Karatzas and Shreve (1998a), Mikosch (1999), Revuz and Yor (1999), Steele (2000), Øksendal (2003), and Jeanblanc et al. (2006).

## A.1 Conditional Expectation

Let us start by considering a finite decomposition of the underlying probability space. We say that a finite collection $\mathcal{D} = \{D_1, \ldots, D_k\}$ of non-empty subsets of $\Omega$ is a *decomposition* of $\Omega$ if the sets $D_1, \ldots, D_k$ are pairwise disjoint; that is, if $D_i \cap D_j = \emptyset$ for every $i \neq j$, and the equality $D_1 \cup D_2 \cup \ldots \cup D_k = \Omega$ holds. A random variable $\psi$ on $\Omega$ is called *simple* if it admits a representation

$$\psi(\omega) = \sum_{i=1}^{m} x_i \mathbb{1}_{D_i(\psi)}(\omega), \tag{A.1}$$

where $D_i(\psi) = \{\omega \in \Omega \mid \psi(\omega) = x_i\}$ and $x_i,\ i = 1, \ldots, m$ are real numbers satisfying $x_i \neq x_j$ for $i \neq j$. For a simple random variable $\psi$, we denote by $\mathcal{D}(\psi)$

the decomposition $\{D_1(\psi), \dots, D_m(\psi)\}$ generated by $\psi$. It is clear that if $\Omega$ is a finite set then any random variable $\psi : \Omega \to \mathbb{R}$ is simple.

**Definition A.1.1** For any decomposition $\mathcal{D}$ of $\Omega$ and any event $A \in \mathcal{F}$, the *conditional probability* of $A$ with respect to $\mathcal{D}$ is defined by the formula

$$\mathbb{P}(A \,|\, \mathcal{D}) = \sum_{j=1}^{k} \mathbb{P}(A \,|\, D_j) \mathbb{1}_{D_j}. \tag{A.2}$$

Moreover, if $\psi$ is a simple random variable with the representation (A.1), its *conditional expectation* given $\mathcal{D}$ equals

$$\mathbb{E}_{\mathbb{P}}(\psi \,|\, \mathcal{D}) = \sum_{i=1}^{m} x_i \, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{D_i(\psi)} \,|\, \mathcal{D}) = \sum_{i=1}^{m} \sum_{j=1}^{k} x_i \, \mathbb{P}(D_i(\psi) \,|\, D_j) \mathbb{1}_{D_j}. \tag{A.3}$$

Observe that the conditional expectation $\mathbb{E}_{\mathbb{P}}(\psi \,|\, \mathcal{D})$ is constant on each set $D_j$ from $\mathcal{D}$. Let $\eta$ be another simple random variable on $\Omega$, i.e.,

$$\eta(\omega) = \sum_{l=1}^{r} y_l \mathbb{1}_{D_l(\eta)}(\omega), \tag{A.4}$$

where $D_l(\eta) = \{\, \omega \in \Omega \,|\, \eta(\omega) = y_l \,\}$ and $y_i \neq y_j$ for $i \neq j$.

Suppose that $\eta$ and $\psi$ are simple random variables given from (A.1) and (A.4) respectively. Then, by the definition of conditional expectation, $\mathbb{E}_{\mathbb{P}}(\psi \,|\, \eta)$ coincides with $\mathbb{E}_{\mathbb{P}}(\psi \,|\, \mathcal{D}(\eta))$, and thus

$$\mathbb{E}_{\mathbb{P}}(\psi \,|\, \eta) = \mathbb{E}_{\mathbb{P}}(\psi \,|\, \mathcal{D}(\eta)) = \sum_{i=1}^{m} x_i \, \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{D_k(\psi)} \,|\, \mathcal{D}(\eta)), \tag{A.5}$$

where the second equality follows by (A.3). Consequently,

$$\mathbb{E}_{\mathbb{P}}(\psi \,|\, \eta) = \sum_{i=1}^{m} \sum_{l=1}^{r} x_i \, \mathbb{P}(D_i(\psi) \,|\, D_l(\eta)) \mathbb{1}_{D_l(\eta)} = \sum_{l=1}^{r} c_l \mathbb{1}_{D_l(\eta)}, \tag{A.6}$$

where $c_l = \sum_{i=1}^{m} x_i \, \mathbb{P}(D_i(\psi) \,|\, D_l(\eta))$ for $l = 1, \dots, r$. Note that $\mathbb{E}_{\mathbb{P}}(\psi \,|\, \eta)$ does not depend on the particular values of $\eta$. More precisely, if for two random variables $\eta_1$ and $\eta_2$ we have $\mathcal{D}(\eta_1) = \mathcal{D}(\eta_2)$ then $\mathbb{E}_{\mathbb{P}}(\psi \,|\, \eta_1) = \mathbb{E}_{\mathbb{P}}(\psi \,|\, \eta_2)$.

Note, however, that in view of (A.6) we also have that $\mathbb{E}_{\mathbb{P}}(\psi \,|\, \eta) = g(\eta)$, where the function $g : \{y_1, \dots, y_r\} \to \mathbb{R}$ is given by the formula $g(y_l) = c_l$ for $l = 1, 2, \dots, r$.

It is not difficult to check that the conditional expectation of a simple random variable with respect to a decomposition $\mathcal{D}$ has the following properties ($\psi$ and $\eta$ stand here for arbitrary simple random variables):

(i) if $\mathcal{D}(\psi) \subseteq \mathcal{D}$ then $\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{D}) = \psi$;
(ii) for arbitrary real numbers $c, d$ we have

$$\mathbb{E}_{\mathbb{P}}(c\psi + d\eta \mid \mathcal{D}) = c\,\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{D}) + d\,\mathbb{E}_{\mathbb{P}}(\eta \mid \mathcal{D});$$

(iii) if $\mathcal{D}_1 \subseteq \mathcal{D}_2$ then

$$\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{D}_1) \mid \mathcal{D}_2) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{D}_2) \mid \mathcal{D}_1) = \mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{D}_1);$$

(iv) if $\psi$ is independent of $\mathcal{D}$ – that is, if for every $A \in \mathcal{D}(\psi)$ and every $B \in \mathcal{D}$ we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ – then $\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{D}) = \mathbb{E}_{\mathbb{P}}\psi$;
(v) if $\psi$ is $\mathcal{D}$-measurable and $\eta$ is independent of $\mathcal{D}$ then for any function $h : \mathbb{R}^2 \to \mathbb{R}$ we have

$$\mathbb{E}_{\mathbb{P}}(h(\psi, \eta) \mid \mathcal{D}) = H(\psi),$$

where the function $H : \mathbb{R} \to \mathbb{R}$ is given by the formula $H(x) = \mathbb{E}_{\mathbb{P}} h(x, \eta)$.

We shall now define the conditional expectation of a random variable with respect to an arbitrary $\sigma$-field $\mathcal{G}$.

**Definition A.1.2** Let $\psi$ be an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ (i.e., $\mathbb{E}_{\mathbb{P}}(|\psi|) < \infty$). For an arbitrary $\sigma$-field $\mathcal{G}$ of subsets of $\Omega$ satisfying $\mathcal{G} \subseteq \mathcal{F}$ (i.e., a *sub-$\sigma$-field* of $\mathcal{F}$), the *conditional expectation* $\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G})$ of $\psi$ with respect to $\mathcal{G}$ is defined by the following conditions: (i) $\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G})$ is $\mathcal{G}$-measurable, i.e., $\{\omega \in \Omega \mid \mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G})(\omega) \leq a\} \in \mathcal{G}$ for any $a \in \mathbb{R}$; (ii) for an arbitrary event $A \in \mathcal{G}$, we have

$$\int_A \mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G})\, d\mathbb{P} = \int_A \psi\, d\mathbb{P}. \tag{A.7}$$

It is well known that the conditional expectation exists and is unique (up to the $\mathbb{P}$-a.s. equivalence of random variables).

*Remarks.* If $\mathcal{D}$ is a (finite) decomposition of $\Omega$, then the family of all unions of sets from $\mathcal{D}$, together with an empty set, forms a $\sigma$-field of subsets of $\Omega$. We denote it by $\sigma(\mathcal{D})$ and we call it the $\sigma$-field *generated* by the decomposition[1] $\mathcal{D}$. It can be easily checked that if $\mathcal{G} = \sigma(\mathcal{D})$ then $\mathbb{P}(A \mid \mathcal{G}) = \mathbb{P}(A \mid \mathcal{D})$, so that the conditional expectation with respect to the $\sigma$-field $\sigma(\mathcal{D})$ coincides with the conditional expectation with respect to the decomposition $\mathcal{D}$.

**Definition A.1.3** For an arbitrary simple random variable $\eta$, the $\sigma$-field $\mathcal{F}(\eta) = \sigma(\mathcal{D}(\eta))$ is called the *$\sigma$-field generated by* $\eta$, or briefly, the *natural $\sigma$-field of* $\eta$. More generally, for any random variable $\eta$, the *$\sigma$-field generated by* $\eta$ is the least $\sigma$-field of subsets of $\Omega$ with respect to which $\eta$ is measurable. It is denoted by either $\sigma(\eta)$ or $\mathcal{F}(\eta)$.

For a real-valued random variable $\eta$, it can be shown that the $\sigma$-field $\mathcal{F}(\eta)$ is the smallest $\sigma$-field that contains all events of the form $\{\omega \in \Omega \mid \eta(\omega) \leq x\}$, where $x$

[1] In the case of a finite $\Omega$, any $\sigma$-field $\mathcal{G}$ of subsets of $\Omega$ is of this form; that is, $\mathcal{G} = \sigma(\mathcal{D})$ for some decomposition $\mathcal{D}$ of $\Omega$.

is an arbitrary real number. More generally, if $\eta = (\eta^1, \ldots, \eta^d)$ is a $d$-dimensional random variable then

$$\mathcal{F}(\eta) = \sigma\big(\{\omega \in \Omega \mid \eta^1(\omega) \le x_1, \ldots, \eta^d(\omega) \le x_d\} \mid x_1, \ldots, x_d \in \mathbb{R}\big).$$

For any $\mathbb{P}$-integrable random variable $\psi$ and any random variable $\eta$, we define the conditional expectation of $\psi$ with respect to $\eta$ by setting $\mathbb{E}_{\mathbb{P}}(\psi \mid \eta) = \mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{F}(\eta))$. It is possible to show that for arbitrary real-valued random variables $\psi$ and $\eta$, there exists a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}_{\mathbb{P}}(\psi \mid \eta) = g(\eta)$.

**Definition A.1.4** We say that the $\sigma$-fields $\mathcal{G}$ and $\mathcal{H}$ are *independent* under $\mathbb{P}$ whenever $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for any $A \in \mathcal{G}$ and $B \in \mathcal{H}$.

Let us summarize the basic properties of a conditional expectation.

**Lemma A.1.1** *Let $\psi$ and $\eta$ be integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Also let $\mathcal{G}$ and $\mathcal{H}$ be some sub-$\sigma$-fields of $\mathcal{F}$. Then*
(i) *if $\psi$ is $\mathcal{G}$-measurable (or equivalently if $\mathcal{F}(\psi) \subseteq \mathcal{G}$) then $\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G}) = \psi$;*
(ii) *for arbitrary real numbers $c, d$ we have*

$$\mathbb{E}_{\mathbb{P}}(c\psi + d\eta \mid \mathcal{G}) = c\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G}) + d\mathbb{E}_{\mathbb{P}}(\eta \mid \mathcal{G});$$

(iii) *if $\mathcal{H} \subseteq \mathcal{G}$ then*

$$\mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G}) \mid \mathcal{H}\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{H}) \mid \mathcal{G}\big) = \mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{H});$$

*in particular, if $\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G})$ is $\mathcal{H}$-measurable then $\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G}) = \mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{H})$;*
(iv) *if $\psi$ is independent of $\mathcal{G}$, i.e., the $\sigma$-fields $\mathcal{F}(\psi)$ and $\mathcal{G}$ are independent under $\mathbb{P}$, then $\mathbb{E}_{\mathbb{P}}(\psi \mid \mathcal{G}) = \mathbb{E}_{\mathbb{P}}(\psi)$;*
(v) *if $\psi$ is $\mathcal{G}$-measurable and $\eta$ is independent of $\mathcal{G}$ then for any Borel measurable function $h : \mathbb{R}^2 \to \mathbb{R}$ we have $\mathbb{E}_{\mathbb{P}}(h(\psi, \eta) \mid \mathcal{G}) = H(\psi)$, where $H(x) = \mathbb{E}_{\mathbb{P}} h(x, \eta)$, provided that the inequality $\mathbb{E}_{\mathbb{P}}|h(\psi, \eta)| < \infty$ holds.*

The next result is the conditional form of Fatou's lemma.

**Lemma A.1.2** *Let $\xi_1, \xi_2 \ldots$ be integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that for every $n$ we have $\xi_n \ge \eta$, where $\eta$ is some random variable with $\mathbb{E}_{\mathbb{P}}\eta > -\infty$. Then for any sub-$\sigma$-field $\mathcal{G}$ of $\mathcal{F}$ we have*

$$\mathbb{E}_{\mathbb{P}}(\liminf_{n\to\infty} \xi_n \mid \mathcal{G}) \le \liminf_{n\to\infty} \mathbb{E}_{\mathbb{P}}(\xi_n \mid \mathcal{G}).$$

The following result, referred to as (conditional) *Jensen's inequality,* is also well known.

**Lemma A.1.3** *Let $h : \mathbb{R} \to \mathbb{R}$ be a convex function, and $\xi$ a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}_{\mathbb{P}}|g(\xi)| < \infty$. Then for any sub-$\sigma$-field $\mathcal{G}$ of $\mathcal{F}$, Jensen's inequality holds – that is, $g\big(\mathbb{E}_{\mathbb{P}}(\xi \mid \mathcal{G})\big) \le \mathbb{E}_{\mathbb{P}}(g(\xi) \mid \mathcal{G})$.*

The last result of this section refers to the situation where two mutually equivalent probability measures, $\mathbb{P}$ and $\mathbb{Q}$ say, are defined on a common measurable space $(\Omega, \mathcal{F})$. Suppose that the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals (cf. Sect. A.14)

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta, \quad \mathbb{P}\text{-a.s.} \tag{A.8}$$

Note that the random variable $\eta$ is strictly positive $\mathbb{P}$-a.s., moreover $\eta$ is $\mathbb{P}$-integrable with $\mathbb{E}_{\mathbb{P}}\eta = \mathbb{Q}(\Omega) = 1$. Finally, in view of (A.8), it is clear that equality $\mathbb{E}_{\mathbb{Q}}\psi = \mathbb{E}_{\mathbb{P}}(\psi\eta)$ holds for any $\mathbb{Q}$-integrable random variable $\psi$.

We are in the position prove the so-called *abstract Bayes formula*.

**Lemma A.1.4** *Let $\mathcal{G}$ be a sub-$\sigma$-field of the $\sigma$-field $\mathcal{F}$, and let $\psi$ be a random variable integrable with respect to $\mathbb{Q}$. Then the following version of Bayes's formula holds*

$$\mathbb{E}_{\mathbb{Q}}(\psi \,|\, \mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(\psi\eta \,|\, \mathcal{G})}{\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{G})}. \tag{A.9}$$

*Proof.* It can be easily checked that $\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{G})$ is strictly positive $\mathbb{P}$-a.s. so that the right-hand side of (A.9) is well-defined. By our assumption, the random variable $\xi = \psi\eta$ is $\mathbb{P}$-integrable, it is therefore enough to show that

$$\mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{G}) = \mathbb{E}_{\mathbb{Q}}(\psi \,|\, \mathcal{G})\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{G}).$$

Since the right-hand side of the last formula defines a $\mathcal{G}$-measurable random variable, we need to verify that for any set $A \in \mathcal{G}$, we have

$$\int_A \psi\eta \, d\mathbb{P} = \int_A \mathbb{E}_{\mathbb{Q}}(\psi \,|\, \mathcal{G})\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{G}) \, d\mathbb{P}.$$

But for every $A \in \mathcal{G}$, we get

$$\begin{aligned} \int_A \psi\eta \, d\mathbb{P} &= \int_A \psi \, d\mathbb{Q} = \int_A \mathbb{E}_{\mathbb{Q}}(\psi \,|\, \mathcal{G}) \, d\mathbb{Q} = \int_A \mathbb{E}_{\mathbb{Q}}(\psi \,|\, \mathcal{G})\eta \, d\mathbb{P} \\ &= \int_A \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{Q}}(\psi \,|\, \mathcal{G})\eta \,\big|\, \mathcal{G}\big) \, d\mathbb{P} = \int_A \mathbb{E}_{\mathbb{Q}}(\psi \,|\, \mathcal{G}) \, \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{G}) \, d\mathbb{P}. \quad \square \end{aligned}$$

## A.2 Filtrations and Adapted Processes

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$. A *filtration* is an increasing family of $\sigma$-fields, that is, we have that $\mathcal{F}_u \subseteq \mathcal{F}_t$ for any $0 \leq u \leq t \leq T$. For concreteness, we assume that $T$ is a finite strictly positive real number and we set $\mathcal{F} = \mathcal{F}_T$. Hence we shall write $(\Omega, \mathbb{F}, \mathbb{P})$, rather than $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, in what follows.

**Definition A.2.1** A real-valued stochastic process $X = (X_t)_{t \in [0,T]}$ defined on $(\Omega, \mathbb{F}, \mathbb{P})$ is $\mathbb{F}$-*adapted* if for any $t \in [0, T]$ the random variable $X_t$ is $\mathcal{F}_t$-measurable, i.e., for any $x \in \mathbb{R}$ the event $\{X_t \leq x\}$ belongs to the $\sigma$-field $\mathcal{F}_t$.

Similarly, an $\mathbb{R}^k$-valued stochastic process $X = (X^1, \ldots, X^k)$ defined on $(\Omega, \mathbb{F}, \mathbb{P})$ is said to be $\mathbb{F}$-*adapted* if for any $t \in [0, T]$ the $\mathbb{R}^k$-valued random variable $X_t = (X_t^1, \ldots, X_t^k)$ is $\mathcal{F}_t$-measurable, i.e., for any choice of real numbers $x_1, \ldots, x_k \in \mathbb{R}$ the event

$$\{X_t^1 \leq x_1, X_t^2 \leq x_2, \ldots, X_t^k \leq x_k\}$$

belongs to the $\sigma$-field $\mathcal{F}_t$.

We shall sometimes say that a process is adapted, rather than $\mathbb{F}$-adapted, if the filtration $\mathbb{F}$ is fixed so that no danger of confusion arises. Let us mention that any stochastic process $X$ is adapted to its *natural filtration* (i.e., the filtration *generated* by $X$), which is defined as: $\mathcal{F}_t^X = \sigma(X_u \mid u \leq t)$.

Any stochastic processes considered in the sequel is assumed to be a *càdlàg* process, that is, almost all sample paths are right-continuous functions with finite left-hand limits. We say the a stochastic process is *continuous* if almost all sample paths are continuous functions.

**Definition A.2.2** We say that the processes $X$ and $Y$ defined on a common probability space are *indistinguishable* if the event $\{X_t = Y_t \text{ for } t \in [0, T]\}$ has probability 1. We identify any two processes that are indistinguishable.

From now on, we assume that the filtration $\mathbb{F}$ satisfies the *usual conditions*:
(i) $\mathbb{F}$ is $\mathbb{P}$-complete, i.e., any event $A \in \mathcal{F}_T$ such that $\mathbb{P}(A) = 0$ belongs to the $\sigma$-field $\mathcal{F}_0$, and thus it belongs to the $\sigma$-field $\mathcal{F}_t$ for any $t \in [0, T]$,
(ii) $\mathbb{F}$ is right-continuous, i.e., $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$ for any $t \in [0, T)$.

If a given filtration $\mathbb{F}$ is not $\mathbb{P}$-complete and/or right-continuous, we need first to modify it as follows. First, we set $\tilde{\mathcal{F}}_t = \sigma(\mathcal{F}_t \cup \mathcal{N})$, where $\mathcal{N}$ is the class of all $\mathbb{P}$-null events from $\mathcal{F}_T$ (i.e, all events from $\mathcal{F}_T$ of probability zero). Second, for any $t \in [0, T)$ we define $\hat{\mathcal{F}}_t = \tilde{\mathcal{F}}_{t+}$, where $\tilde{\mathcal{F}}_{t+} = \bigcap_{\epsilon>0} \tilde{\mathcal{F}}_{t+\epsilon}$. The filtration $\hat{\mathbb{F}}$ defined in that way is $\mathbb{P}$-complete and right-continuous; it is referred to as the $\mathbb{P}$-*augmentation* of $\mathbb{F}$.

## A.3 Martingales

A martingale is an $\mathbb{F}$-adapted stochastic processes enjoying the property that the conditional expected value of its future value at time $u$, based on the information $\mathcal{F}_t$ available at time $t \leq u$ coincides with its current value. The origins of this concept are related to the idea of a *fair game*, that is, a game in which the expected payoff matches the stake. Intuitively, a martingale describes the current value of our wealth when we make any bets associated with a fair game.

As already mentioned, we shall assume that all processes are defined on the finite interval $[0, T]$. For many results provided in what follows this assumption is not required, but to preserve consistency we will maintain it throughout this appendix. The formal definition of a martingale and closely related concepts of a super- and submartingale thus read as follows.

**Definition A.3.1** A real-valued, $\mathbb{F}$-adapted process $M = (M_t)_{t\in[0,T]}$, defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, is called an $\mathbb{F}$-*martingale* with respect to the filtration $\mathbb{F}$ if the following conditions are satisfied:
(i) $M$ is integrable, that is, $\mathbb{E}_{\mathbb{P}}|M_t| < \infty$ for $t \in [0, T]$,
(ii) the following *martingale equality* holds, for any $0 \le t \le u \le T$,

$$\mathbb{E}_{\mathbb{P}}(M_u \,|\, \mathcal{F}_t) = M_t. \tag{A.10}$$

If (A.10) is replaced by the inequality $\mathbb{E}_{\mathbb{P}}(M_u \,|\, \mathcal{F}_t) \le M_t$ then $M$ is said to be an $\mathbb{F}$-*supermartingale*. Finally, $M$ is called an $\mathbb{F}$-*submartingale* if for any $0 \le t \le u \le T$ $\mathbb{E}_{\mathbb{P}}(M_u \,|\, \mathcal{F}_t) \ge M_t$. Note that the expected value of a martingale is constant, $\mathbb{E}_{\mathbb{P}} M_t = \mathbb{E}_{\mathbb{P}} M_0$ for $t \in [0, T]$. For a supermartingale, the expected value is a non-increasing function, that is, $\mathbb{E}_{\mathbb{P}} M_t \le \mathbb{E}_{\mathbb{P}} M_0$ for any $t \in [0, T]$. Finally, for a submartingale we have that $\mathbb{E}_{\mathbb{P}} M_t \ge \mathbb{E}_{\mathbb{P}} M_0$ for any $t \in [0, T]$.

The following result summarizes the basic properties of martingales. Its easy proof is left as an exercise.

**Proposition A.3.1** *The following properties are valid:*
(i) *Let $X$ be an $\mathcal{F}_T$-measurable and integrable random variable. Then the process $M_t = \mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_t),\ t \in [0, T]$, is a (uniformly integrable) $\mathbb{F}$-martingale.*
(ii) *Assume that $M$ is either a super- or a submartingale with respect to $\mathbb{F}$. Then $M$ is an $\mathbb{F}$-martingale if and only if the expected value of $M$ is constant, or equivalently, whenever $\mathbb{E}_{\mathbb{P}} M_0 = \mathbb{E}_{\mathbb{P}} M_T$.*
(iii) *Let $M$ be an $\mathbb{F}$-martingale and let $h : \mathbb{R} \to \mathbb{R}$ be a convex function. If the process $X_t = h(M_t),\ t \in [0, T]$, is integrable then it is an $\mathbb{F}$-submartingale.*

## A.4 Standard Brownian Motion

We start by stating the definition of the *standard Brownian motion* (also known as the *Wiener process*).

**Definition A.4.1** A real-valued, $\mathbb{F}$-adapted process $W = (W_t)_{t\in[0,T]}$, with $W_0 = 0$, defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, is called a *one-dimensional standard Brownian motion* with respect to the filtration $\mathbb{F}$ if:
(i) for any $0 \le t \le u \le T$, the increment $W_u - W_t$ is a random variable independent of the $\sigma$-field $\mathcal{F}_t$,
(ii) for any $0 \le t \le u \le T$, the probability distribution of $W_u - W_t$ is Gaussian with expected value 0 and variance $u - t$, that is, $W_u - W_t \stackrel{d}{=} N(0, u - t)$,
(iii) almost all sample paths of $W$ are continuous functions, that is, the process $W$ is sample-paths continuous.

If the filtration $\mathbb{F}$ is not given in advance, by a *standard Brownian motion* we mean a process $W$ satisfying Definition A.4.1 with $\mathbb{F}$ being the natural filtration of $W$. More precisely, we assume that $\mathbb{F} = \mathbb{F}^W$, meaning that $\mathbb{F}$ is the $\mathbb{P}$-augmentation of the natural filtration $\sigma\{W_u \,|\, u \le t\}$ of the process $W$. In that case, condition (ii) in

Definition A.4.1 can be replaced by the requirement that $W$ is a *process of independent increments*, that is, for any $n \in \mathbb{N}$ and any sequence $0 < t_1 < \ldots < t_n \leq T$ the random variables $W_{t_1} - W_{t_0},\ W_{t_2} - W_{t_1}, \ldots,\ W_{t_n} - W_{t_{n-1}}$ are independent.

We take here for granted the existence of a Brownian motion (see, for instance, Karatzas and Shreve (1998a)) and we focus on the most important properties of this process.

**Lemma A.4.1** (i) *A Brownian motion $W$ is a continuous $\mathbb{F}$-martingale.*
(ii) *A Brownian motion $W$ has the Markov property with respect to $\mathbb{F}$, meaning that for any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ and $0 \leq t \leq u \leq T$*

$$\mathbb{E}_{\mathbb{P}}(g(W_u) \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(g(W_u) \mid W_t). \tag{A.11}$$

*Proof.* Let us prove the first part. It is clear that $W$ is a continuous, $\mathbb{F}$-adapted process and $\mathbb{E}_{\mathbb{P}}|W_t| < \infty$ for any $t \in [0, T]$. Moreover the martingale property holds since we have, for any $0 \leq t \leq u \leq T$,

$$\mathbb{E}_{\mathbb{P}}(W_u \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(W_u - W_t \mid \mathcal{F}_t) + \mathbb{E}_{\mathbb{P}}(W_t \mid \mathcal{F}_t) = W_t. \tag{A.12}$$

The last equality in formula (A.12) holds since condition (ii) implies that $\mathbb{E}_{\mathbb{P}}(W_u - W_t \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(W_u - W_t) = 0$. In fact, $W$ is a *square-integrable martingale* as manifestly $\mathbb{E}_{\mathbb{P}}(W_t^2) = t < \infty$ for any $t \in [0, T]$.

For the second part, note that

$$\mathbb{E}_{\mathbb{P}}(g(W_u) \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(g(W_u - W_t + W_t) \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(g(\eta + \psi) \mid \mathcal{F}_t),$$

where $\psi = W_t$ is $\mathcal{F}_t$-measurable and $\eta = W_u - W_t$ is independent of $\mathcal{F}_t$. Hence, by part (v) in Lemma A.1.1, $\mathbb{E}_{\mathbb{P}}(g(W_u) \mid \mathcal{F}_t)$ is $\sigma(W_t)$-measurable. Consequently, by part (iii) in Lemma A.1.1, equality (A.11) holds. □

A Brownian motion is an example of a process of *finite quadratic variation,* as made explicit by the following result.

**Proposition A.4.1** *For every $0 \leq u < t \leq T$, and any sequence $(\pi_n)_{n \in \mathbb{N}}$ of finite partitions $\pi_n = \{u = t_0^n < t_1^n < \cdots < t_{m_n}^n = t\}$ of the interval $[u, t]$ satisfying $\lim_{n\to\infty} \delta(\pi_n) = 0$, where $\delta(\pi_n) \stackrel{\mathrm{def}}{=} \max_{k=1,\ldots,m_n} (t_k^n - t_{k-1}^n)$, we have*

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} (W_{t_k^n} - W_{t_{k-1}^n})^2 = t - u, \tag{A.13}$$

*where the convergence in (A.13) is in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.*

*Proof.* Let us fix $0 \leq u < t \leq T$ and let us denote $\Delta_k^n = t_k^n - t_{k-1}^n$. Note that necessarily $\lim_{n\to\infty} m_n = \infty$. It suffices to check that $\lim_{n\to\infty} I_n = 0$, where we write

$$I_n = \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\Big( \sum_{k=1}^{m_n} \big( (W_{t_k^n} - W_{t_{k-1}^n})^2 - \Delta_k^n \big) \Big)^2.$$

Since the increments $W_{t^n_{j+1}} - W_{t^n_j}$ and $W_{t^n_{i+1}} - W_{t^n_i}$ are independent for $i \neq j$ and $\mathbb{E}_{\mathbb{P}}(W_{t^n_k} - W_{t^n_{k-1}})^2 = \Delta^n_k$, we obtain

$$I_n = \sum_{k=1}^{m_n} \Big( \mathbb{E}_{\mathbb{P}}(W_{t^n_k} - W_{t^n_{k-1}})^4 - 2\Delta^n_k \, \mathbb{E}_{\mathbb{P}}(W_{t^n_k} - W_{t^n_{k-1}})^2 + (\Delta^n_k)^2 \Big)$$
$$= 2\sum_{k=1}^{m_n} (\Delta^n_k)^2 \leq 2(t-u) \max_{k=1,\dots,m_n} (t^n_k - t^n_{k-1}) = 2(t-u)\delta(\pi_n).$$

In the second equality we used the fact that the assumption that the increment $W_{t^n_k} - W_{t^n_{k-1}}$ has the Gaussian distribution $N(0, \Delta^n_k)$ yields

$$\mathbb{E}_{\mathbb{P}}(W_{t^n_k} - W_{t^n_{k-1}})^4 = 3(\Delta^n_k)^2 = 3(t^n_k - t^n_{k-1})^2.$$

It is now clear that $I_n$ tends to zero as $n$ tends to infinity (so that $\delta(\pi_n)$ tends to zero). □

It is worth mentioning that if the series $\sum_{n=1}^{\infty} \delta(\pi_n)$ is convergent then it can be deduced from the Borel-Cantelli lemma that the convergence in (A.13) is almost sure (that is, it holds with probability 1). Let $\langle W \rangle_t$ stand for the quadratic variation of $W$ on $[0, t]$, so that $\langle W \rangle_t = t$ for any $t \in [0, T]$.

**Lemma A.4.2** *The process $M = W^2 - \langle W \rangle$ is a continuous $\mathbb{F}$-martingale.*

*Proof.* It is easy to see that the process $M$ is integrable and $\mathbb{F}$-adapted. Note also that $\mathbb{E}_{\mathbb{P}}(W_t W_u \,|\, \mathcal{F}_t) = W_t \, \mathbb{E}_{\mathbb{P}}(W_u \,|\, \mathcal{F}_t) = W_t^2$ for $0 \leq t \leq u \leq T$. Consequently

$$\mathbb{E}_{\mathbb{P}}(W_u^2 - W_t^2 \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}((W_u - W_t)^2 \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(W_u - W_t)^2 = u - t,$$

and thus, for $0 \leq t \leq u \leq T$,

$$\mathbb{E}_{\mathbb{P}}(M_u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(W_u^2 - u \,|\, \mathcal{F}_t) = W_t^2 - t = M_t. \qquad \square$$

*Remarks.* In view of the last result, the quadratic variation $\langle W \rangle$ can also be seen as the unique increasing and continuous process arising in the *Doob-Meyer decomposition* (see Proposition A.7.2) of continuous submartingale $X = W^2$.

Before stating an important corollary to Proposition A.4.1, let us recall that for any function $f : [0, T] \to \mathbb{R}$ the *variation of $f$ on the interval* $[t, u]$, denoted as $\mathrm{var}_{[t,u]}(f)$, is defined as

$$\mathrm{var}_{[t,u]}(f) = \sup_{\pi \in \Pi_{[t,u]}} \sum_{k=1}^{n(\pi)} |f(t_k) - f(t_{k-1})|, \tag{A.14}$$

where $\Pi_{[t,u]}$ is the class of all finite partitions $\pi = \{t = t_0 < t_1 < \cdots < t_{n(\pi)} = u\}$ of the interval $[t, u]$. Note that $n(\pi)$ stands for the number of subintervals in a partition $\pi$.

**Corollary A.4.1** *Sample paths of a Brownian motion are almost surely functions of infinite variation on any interval.*

*Proof.* The proof hinges on the observation that for any partition of $[t,u]$ we have

$$\sum_{k=1}^{n(\pi)} (f(t_k)-f(t_{k-1}))^2 \le \max_{m=1,\dots,n(\pi)} \big(f(t_m)-f(t_{m-1})\big) \sum_{k=1}^{n(\pi)} |f(t_k)-f(t_{k-1})|$$
$$\le \max_{m=1,\dots,n(\pi)} \big(f(t_m)-f(t_{m-1})\big)\,\mathrm{var}_{[t,u]}(f).$$

Since any continuous function is uniformly continuous on the compact interval $[t,u]$, it is not difficult to show that the last inequality would contradict Proposition A.4.1 if $\mathbb{P}\{\mathrm{var}_{[t,u]}(W) < +\infty\} > 0$. We conclude that for any $t < u$ we have $\mathrm{var}_{[t,u]}(W) = +\infty$ almost surely. □

*Remarks.* Though in the proof of Proposition A.4.1 we used the fact that the increments of a Brownian motion have the Gaussian probability distribution, the property established in Corollary A.4.1 is by far more general, since it is enjoyed by any continuous local martingale. Specifically, any continuous local martingale with sample paths of finite variation is a constant process (see Proposition A.7.3).

The definition of a one-dimensional Brownian motion can be easily extended to the case of a $d$-dimensional process.

**Definition A.4.2** An $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted process $W = (W^1,\dots,W^d)$, with $W_0 = 0$, defined on a filtered probability space $(\Omega,\mathbb{F},\mathbb{P})$, is called a *$d$-dimensional standard Brownian motion* with respect to the filtration $\mathbb{F}$ if processes $W^1,\dots,W^d$ are independent one-dimensional standard Brownian motions.

**Lemma A.4.3** *Let $W = (W^1,\dots,W^d)$ be a $d$-dimensional standard Brownian motion with respect to the filtration $\mathbb{F}$. Then for any $i \ne j$ the process $M = W^iW^j = (W^i_tW^j_t)_{t\in[0,T]}$ is a continuous square-integrable martingale.*

*Proof.* Let us fix $i \ne j$. It is easily seen that $M$ is an $\mathbb{F}$-adapted and square-integrable process. Let us thus focus on the equality $\mathbb{E}_{\mathbb{P}}(M_u \,|\, \mathcal{F}_t) = M_t$ for any $0 \le t \le u \le T$. We have

$$\mathbb{E}_{\mathbb{P}}(M_u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(M_u - M_t \,|\, \mathcal{F}_t) + \mathbb{E}_{\mathbb{P}}(M_t \,|\, \mathcal{F}_t) = M_t \tag{A.15}$$

since

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(M_u - M_t \,|\, \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}(W^i_uW^j_u - W^i_tW^j_t \,|\, \mathcal{F}_t)\\ &= W^i_t\,\mathbb{E}_{\mathbb{P}}(W^j_u - W^j_t \,|\, \mathcal{F}_t) + W^j_t\,\mathbb{E}_{\mathbb{P}}(W^j_u - W^i_t \,|\, \mathcal{F}_t)\\ &\quad + \mathbb{E}_{\mathbb{P}}((W^i_u - W^i_t)(W^j_u - W^j_t) \,|\, \mathcal{F}_t)\\ &= \mathbb{E}_{\mathbb{P}}((W^i_u - W^i_t)(W^j_u - W^j_t)) = \mathbb{E}_{\mathbb{P}}(W^i_u - W^i_t)\,\mathbb{E}_{\mathbb{P}}(W^j_u - W^j_t) = 0,\end{aligned}$$

where we used, in particular, the independence of the increments $W^i_u - W^i_t$ and $W^j_u - W^j_t$ and their independence of $\mathcal{F}_t$. □

**Corollary A.4.2** *Assume that* $W = (W^1, \dots, W^d)$ *is a* $d$*-dimensional standard Brownian motion with respect to the filtration* $\mathbb{F}$*. Then for any* $i, j = 1, \dots, d$ *the real-valued process* $W_t^i W_t^j - \delta_{ij} t$ *is an* $\mathbb{F}$*-martingale, where* $\delta_{ij} = 0$ *for any* $i \neq j$ *and* $\delta_{ii} = 1$ *for* $i = 1, \dots, d$.

Let us define the quadratic cross-variation $\langle W^i, W^j \rangle$ by setting

$$\langle W^i, W^j \rangle_t = \tfrac{1}{2} \big( \langle W^i + W^j, W^i + W^j \rangle_t - \langle W^i, W^i \rangle_t - \langle W^j, W^j \rangle_t \big).$$

It can be checked that $\langle W^i, W^j \rangle_t = \delta_{ij} t$ for $t \in [0, T]$.

## A.5 Stopping Times and Martingales

The following concept will prove useful in what follows.

**Definition A.5.1** A random variable $\tau : (\Omega, \mathbb{F}, \mathbb{P}) \to [0, T]$ is called a *stopping time* with respect to the filtration $\mathbb{F}$ (or briefly, an $\mathbb{F}$-*stopping time*) if, for every $t \in [0, T]$, the event $\{\tau \leq t\}$ belongs to the $\sigma$-field $\mathcal{F}_t$.

Recall that a process $X$ is *progressively measurable* with respect to the filtration $\mathbb{F}$ if, for every $t$, the map $(u, \omega) \to X_u(\omega)$ from $[0, t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$-measurable. It is known that any $\mathbb{F}$-adapted process with right- or left-continuous sample paths is progressively measurable with respect to the filtration $\mathbb{F}$. For any progressively measurable process $X$ and any stopping time $\tau$ the *stopped process* $X^\tau$, which is defined by $X_t^\tau = X_{\tau \wedge t}$ for $t \in [0, T]$, is also progressively measurable. For any $\mathbb{F}$-martingale $M$ and any $\mathbb{F}$-stopping time $\tau$, the stopped process $M^\tau$ is known to be an $\mathbb{F}$-martingale. The following result furnishes a useful characterization of martingales.

**Lemma A.5.1** *Let* $M = (M_t)_{t \in [0,T]}$ *be an* $\mathbb{F}$*-adapted and integrable stochastic process. Then* $M$ *is an* $\mathbb{F}$*-martingale if and only if we have* $\mathbb{E}_{\mathbb{P}}(M_\tau) = \mathbb{E}_{\mathbb{P}}(M_0)$ *for any stopping time* $\tau$ *with values in* $[0, T]$.

*Proof.* It suffices to check that $M_t = \mathbb{E}_{\mathbb{P}}(M_T \mid \mathcal{F}_t)$ for every $t \in [0, T]$. By assumption, we have that $\mathbb{E}_{\mathbb{P}}(M_\tau) = \mathbb{E}_{\mathbb{P}}(M_T)$ for any stopping time $\tau$ with values in $[0, T]$ (this is true since $\tau = T$ is also a stopping time with values in $[0, T]$). Let us fix $t$ and let us consider an event $A$ belonging to $\mathcal{F}_t$. Then the random variable $\tau_A$ defined as follows

$$\tau_A = \begin{cases} t, & \text{if } \omega \in A, \\ T, & \text{if } \omega \notin A, \end{cases}$$

is a stopping time with values in $[0, T]$. The equality $\mathbb{E}_{\mathbb{P}}(M_{\tau_A}) = \mathbb{E}_{\mathbb{P}}(M_T)$ yields $\mathbb{E}_{\mathbb{P}}(\mathbb{1}_A M_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_A M_T)$. Since this equality holds for any event $A$ in $\mathcal{F}_t$, by virtue of definition of conditional expectation, we obtain the equality $M_t = \mathbb{E}_{\mathbb{P}}(M_T \mid \mathcal{F}_t)$, which in turn implies that $M$ is an $\mathbb{F}$-martingale. □

It appears that the martingale property is too restrictive if we wish define the Itô integral for a large class of processes. Hence the following definition.

**Definition A.5.2** A process $M$ is said to be a *local martingale* with respect to $\mathbb{F}$ if there exists an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times such that $\tau_n$ tends to $T$ almost surely, and for every $n$ the process $M^n$, given by the formula

$$M_t^n = \begin{cases} M_{t\wedge\tau_n(\omega)}(\omega), & \text{if } \tau_n(\omega) > 0, \\ 0, & \text{if } \tau_n(\omega) = 0, \end{cases}$$

is a uniformly integrable martingale. Any sequence $(\tau_n)_{n\in\mathbb{N}}$ with these properties is called the *reducing sequence* for a local martingale $M$.

Any martingale is also a local martingale. It is noteworthy that the class of (continuous) local martingales is essentially larger than the class of (continuous) martingales.

## A.6 Itô Stochastic Integral

Since almost all sample paths of a Brownian motion have infinite variation on every open interval, the classical Lebesgue-Stieltjes integration theory cannot be applied to define an integral of a stochastic process with respect to a Brownian motion. We shall now describe briefly the Itô integration theory, which is the basis of the Itô stochastic calculus. Let $W$ be a standard $d$-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. For simplicity, the horizon date $T > 0$ will be fixed throughout.

**Itô integral for elementary processes.** Let $\mathcal{K}$ stand for the space of $d$-dimensional *elementary processes,* that is, processes of the form

$$\gamma(t) = \gamma_{-1}\mathbb{1}_0 + \sum_{j=0}^{m-1} \gamma_j \mathbb{1}_{(t_j,t_{j+1}]}(t), \quad \forall\, t \in [0, T], \tag{A.16}$$

where $t_0 = 0 < t_1 < \cdots < t_m = T$, the $\mathbb{R}^d$-valued random variables $\gamma_j$, $j = 0, \dots, m-1$ are uniformly bounded and $\mathcal{F}_{t_j}$-measurable and the random variable $\gamma_{-1}$ is $\mathcal{F}_0$-measurable. For any process $\gamma \in \mathcal{K}$, the *Itô stochastic integral* $\hat{I}_T(\gamma)$ with respect to $W$ over the time interval $[0, T]$ is defined by the formula (the dot '$\cdot$' stands hereafter for the Euclidean inner product in $\mathbb{R}^d$)

$$\hat{I}_T(\gamma) = \int_0^T \gamma_u \cdot dW_u \stackrel{\text{def}}{=} \sum_{j=0}^{m-1} \gamma_j \cdot (W_{t_{j+1}} - W_{t_j}). \tag{A.17}$$

Similarly, the Itô stochastic integral of $\gamma$ with respect to $W$ over any interval $[0, t]$, where $t \le T$, is defined by setting

$$\hat{I}_t(\gamma) = \int_0^t \gamma_u \cdot dW_u \stackrel{\text{def}}{=} \hat{I}_T(\gamma \mathbb{1}_{[0,t]}) = \sum_{j=0}^{m-1} \gamma_j \cdot \big(W_{t_{j+1}\wedge t} - W_{t_j\wedge t}\big), \tag{A.18}$$

where $x \wedge y = \min\{x, y\}$. It is not difficult to check that for any process $\gamma \in \mathcal{K}$ the Itô integral $I_t(\gamma),\ t \in [0, T]$, is a continuous $\mathbb{F}$-martingale; in particular, the equality $\mathbb{E}_{\mathbb{P}}(I_u(\gamma)\,|\,\mathcal{F}_t) = I_t(\gamma)$ holds for any $t \le u \le T$, or more explicitly

$$\mathbb{E}_{\mathbb{P}}\Big(\int_0^u \gamma_s \cdot dW_s \,\Big|\, \mathcal{F}_t\Big) = \int_0^t \gamma_s \cdot dW_s.$$

Moreover, the following properties are valid, for $t \in [0, T]$,
(i) (*linearity*) for any $a, b \in \mathbb{R}$ and any processes $\gamma, \eta \in \mathcal{K}$

$$\int_0^t (a\gamma_u + b\eta_u) \cdot dW_u = a \int_0^t \gamma_u \cdot dW_u + b \int_0^t \eta_u \cdot dW_u,$$

(ii) (*local property*) for any $\mathbb{F}$-stopping time $\tau$ and any process $\gamma \in \mathcal{K}$ we have

$$\int_0^t \gamma_u \mathbb{1}_{[0,\tau]}(u) \cdot dW_u = \int_0^t \gamma_u^\tau \cdot dW_u^\tau = I_t^\tau(\gamma).$$

**Itô integral for processes from $\mathcal{L}^2_{\mathbb{P}}(W)$.** In the next step, the definition of the Itô integral will be extended from the class $\mathcal{K}$ to the class $\mathcal{L}^2_{\mathbb{P}}(W)$ of all $\mathbb{F}$-progressively measurable processes $\gamma$ defined on $(\Omega, \mathbb{F}, \mathbb{P})$ for which

$$\| \gamma \|_W^2 \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{P}}\Big(\int_0^T |\gamma_u|^2 \, du\Big) < \infty, \tag{A.19}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^d$.

**Lemma A.6.1** *The class $\mathcal{K}$ is a subset of $\mathcal{L}^2_{\mathbb{P}}(W)$ and for any $\gamma \in \mathcal{K}$*

$$\mathbb{E}_{\mathbb{P}}\Big(\int_0^T \gamma_u \cdot dW_u\Big)^2 = \| I_T(\gamma) \|_{L^2}^2 = \| \gamma \|_W^2 . \tag{A.20}$$

*The space $\mathcal{L}^2_{\mathbb{P}}(W)$ equipped with the norm $\|\cdot\|_W$ is a complete normed linear space – that is, a Banach space. Moreover, the class $\mathcal{K}$ of elementary stochastic processes is a dense linear subspace of $\mathcal{L}^2_{\mathbb{P}}(W)$.*

By virtue of Lemma A.6.1 and well known results from the classic analysis, the isometry $\hat{I}_T : (\mathcal{K}, \|\cdot\|_W) \to L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ can be extended to the isometry $I_T : (\mathcal{L}^2_{\mathbb{P}}(W), \|\cdot\|_W) \to L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. This leads to the following definition.

**Definition A.6.1** For any process $\gamma \in \mathcal{L}^2_{\mathbb{P}}(W)$, the random variable $I_T(\gamma)$ is called the *Itô stochastic integral* of $\gamma$ with respect to $W$ over $[0, T]$, and it is denoted by $\int_0^T \gamma_u \cdot dW_u$.

In other words, for any $\gamma \in \mathcal{L}^2_{\mathbb{P}}(W)$ we set $I_T(\gamma) = \lim_{n\to\infty} I_T(\gamma)$ (with the limit in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$) for any sequence of processes $(\gamma^n)_{n\in\mathbb{N}}$ such that $\gamma^n \in \mathcal{K}$ for $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\Big(\int_0^T |\gamma_u - \gamma_u^n|^2 \, du\Big) = \lim_{n\to\infty} \| \gamma - \gamma^n \|_W^2 = 0.$$

More generally, for every $\gamma \in \mathcal{L}^2_{\mathbb{P}}(W)$ and every $t \in [0, T]$, we set

$$I_t(\gamma) = \int_0^t \gamma_u \cdot dW_u \stackrel{\text{def}}{=} I_T(\gamma \mathbb{1}_{[0,t]}) \tag{A.21}$$

and thus we define the Itô stochastic integral $I(\gamma)$ as an $\mathbb{F}$-adapted stochastic process. The next result summarizes its most important properties.

**Proposition A.6.1** *For any process $\gamma \in \mathcal{L}^2_{\mathbb{P}}(W)$, the Itô stochastic integral $I(\gamma)$ is a square-integrable continuous martingale on $(\Omega, \mathbb{F}, \mathbb{P})$. Moreover, the process*

$$(I_t(\gamma))^2 - \langle I(\gamma)\rangle_t, \quad \forall\, t \in [0, T], \tag{A.22}$$

*is a continuous martingale on $(\Omega, \mathbb{F}, \mathbb{P})$, where*

$$\langle I(\gamma)\rangle_t \stackrel{\text{def}}{=} \int_0^t |\gamma_u|^2 \, du, \quad \forall\, t \in [0, T]. \tag{A.23}$$

It is worth mentioning that the Itô integral is a linear map on $\mathcal{L}^2_{\mathbb{P}}(W)$ and it enjoys the local property with respect to $\mathbb{F}$-stopping times, that is, $I(\gamma \mathbb{1}_{[0,\tau]}) = I^\tau(\gamma)$ for any $\mathbb{F}$-stopping time $\tau$ and any process $\gamma \in \mathcal{L}^2_{\mathbb{P}}(W)$.

**Itô integral for processes from $\mathcal{L}_{\mathbb{P}}(W)$.** By applying the optional stopping technique (also known as a *localization*), we shall extend the definition of Itô stochastic integral to the class $\mathcal{L}_{\mathbb{P}}(W)$ of all progressively measurable processes $\gamma$ for which

$$\mathbb{P}\Big\{ \int_0^T |\gamma_u|^2 \, du < \infty \Big\} = 1. \tag{A.24}$$

Let a process $\gamma$ belong to $\mathcal{L}_{\mathbb{P}}(W)$. In order to define the Itô integral $I_T(\gamma)$, we first introduce an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of $\mathbb{F}$-stopping times by setting

$$\tau_n = \inf \Big\{ t \in [0, T] \,|\, \int_0^t |\gamma_u|^2 \, du \geq n \Big\},$$

where by convention $\inf \emptyset = T$. Since the process $\gamma$ belongs to the class $\mathcal{L}_{\mathbb{P}}(W)$, it is not difficult to verify that $\lim_{n\to\infty} \tau_n = T$.

Furthermore, for any $n \in \mathbb{N}$ the process $\gamma^n$, which is given by the formula $\gamma^n_t = \gamma_t \mathbb{1}_{[0,\tau_n]}(t)$, $t \in [0, T]$, manifestly belongs to the class $\mathcal{L}^2_{\mathbb{P}}(W)$ and thus the Itô integral $I_T(\gamma^n)$ is well defined. Using the local property of the Itô integral, we obtain

$$I(\gamma^n) = I(\gamma \mathbb{1}_{[0,\tau_n]}) = \big(I(\gamma^n)\big)^{\tau_n}.$$

We may thus define the Itô integral $I_T(\gamma)$ by setting

$$I_T(\gamma) = \lim_{n\to\infty} I_T(\gamma^n).$$

We need, of course, to check that the right-hand side in the last equality is well defined and does not depend on the choice of the *localizing* (or *reducing*) sequence of $\mathbb{F}$-stopping times. The details of this construction are left to the reader (see, for instance, Karatzas and Shreve (1998a)).

Let us stress that the Itô integral of a process from the class $\mathcal{L}_{\mathbb{P}}(W)$ does not necessarily follow an $\mathbb{F}$-martingale. We have, however, the following result, which is a rather straightforward consequence of the definition of the Itô integral $I(\gamma)$ for a process $\gamma$ belonging to the class $\mathcal{L}_{\mathbb{P}}(W)$.

**Proposition A.6.2** *Assume that a stochastic process $\gamma$ belongs to $\mathcal{L}_{\mathbb{P}}(W)$. Then the Itô stochastic integral $I(\gamma)$ follows a continuous local martingale with respect to the filtration $\mathbb{F}$.*

*Remarks.* It is clear that the space $\mathcal{L}_{\mathbb{P}}(W)$ of stochastic processes is invariant with respect to an equivalent change of probability measure on $(\Omega, \mathcal{F}_T)$. Specifically, the equality $\mathcal{L}_{\mathbb{P}}(W) = \mathcal{L}_{\tilde{\mathbb{P}}}(\tilde{W})$ holds whenever $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are mutually equivalent probability measures on $(\Omega, \mathcal{F}_T)$ and the processes $W$ and $\tilde{W}$ are Brownian motions under $\mathbb{P}$ and $\tilde{\mathbb{P}}$ respectively. Since we restrict ourselves to equivalent changes of probability measures, on a given reference probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$, we shall write shortly $\mathcal{L}(W)$ instead of $\mathcal{L}_{\mathbb{P}}(W)$ in what follows. Hence a process $\gamma$ is called *integrable with respect to $W$* if it belongs to the class $\mathcal{L}(W)$.

## A.7 Continuous Local Martingales

It order to simplify the presentation, we assume that the $\sigma$-field $\mathcal{F}_0$ is trivial, that is, $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$ for any $A \in \mathcal{F}_0$. This implies that the initial value $M_0$ can be interpreted as a real number. The foregoing result deals with non-negative (not necessarily continuous) local martingales.

**Proposition A.7.1** *Any non-negative local martingale $M$ with respect to $\mathbb{F}$ is also an $\mathbb{F}$-supermartingale. If, in addition, $M$ the expected value of $M$ is constant then $M$ is an $\mathbb{F}$-martingale.*

*Proof.* Recall that if $M$ is a local martingale with respect to $\mathbb{F}$ then there exists an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times such that

$$\mathbb{P}\{\lim_{n\to\infty} \tau_n = T\} = 1$$

and for any $n \in \mathbb{N}$ the stopped process $M^{\tau_n}$ is an $\mathbb{F}$-martingale. This means that we have, for any fixed $n \in \mathbb{N}$ and arbitrary dates $0 \le t \le u \le T$,

$$M_t^{\tau_n} = \mathbb{E}_{\mathbb{P}}(M_u^{\tau_n} \,|\, \mathcal{F}_t).$$

Since the process $M$ is non-negative, using the conditional form of Fatou's lemma (see Lemma A.1.2), we obtain, for $0 \le t \le u \le T$,

$$\mathbb{E}_{\mathbb{P}}(M_u \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\liminf_{n\to\infty} M_u^{\tau_n} \mid \mathcal{F}_t) \le \liminf_{n\to\infty} \mathbb{E}_{\mathbb{P}}(M_u^{\tau_n} \mid \mathcal{F}_t)$$
$$= \liminf_{n\to\infty} M_t^{\tau_n} = M_t.$$

We conclude that $M$ is an $\mathbb{F}$-supermartingale. In particular, $M$ is an integrable process, that is, $\mathbb{E}_{\mathbb{P}} M_t \le \mathbb{E}_{\mathbb{P}} M_0 < \infty$ for $t \in [0, T]$.

Assume, in addition, that $\mathbb{E}_{\mathbb{P}} M_u = \mathbb{E}_{\mathbb{P}} M_t$. To show that $M$ is an $\mathbb{F}$-martingale under this additional assumption, we shall argue by contradiction. Assume that $M$ is not an $\mathbb{F}$-martingale, so that the supermartingale inequality is strict with positive probability, that is,

$$\mathbb{P}\{M_t > \mathbb{E}_{\mathbb{P}}(M_u \mid \mathcal{F}_t)\} > 0.$$

The last inequality implies that

$$\mathbb{E}_{\mathbb{P}} M_t > \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(M_u \mid \mathcal{F}_t)) = \mathbb{E}_{\mathbb{P}} M_u$$

and this clearly contradicts our assumption that the expected value of $M$ is constant. □

The assumption that $M$ is non-negative can be replaced by the assumption that $M_t \ge \eta$ for some random variable $\eta$ with $\mathbb{E}_{\mathbb{P}} \eta > -\infty$.

**Corollary A.7.1** *Let $M$ be a non-negative local martingale $M$ with respect to $\mathbb{F}$. If $M_0 = 0$ then $M_t = 0$ for every $t \in [0, T]$.*

*Proof.* Assume that $M_{t_0} \ne 0$ for some $t_0 \in [0, T]$. Then $\mathbb{E}_{\mathbb{P}} M_{t_0} > 0 = \mathbb{E}_{\mathbb{P}} M_0$, which contradicts the property that $M$ is a supermartingale. □

The following definition extends the concept of quadratic variation to the class of all continuous local martingales.

**Definition A.7.1** Let $M$ be a continuous local martingale with respect to $\mathbb{F}$. We denote by $\langle M \rangle$ the unique continuous, increasing and $\mathbb{F}$-adapted process with $\langle M \rangle_0 = 0$ such that the process $M^2 - \langle M \rangle$ is a continuous local martingale with respect to $\mathbb{F}$. The process $\langle M \rangle$ is termed the *quadratic variation* of $M$.

In view of Proposition A.6.1, it is clear that the notation introduced previously in formula (A.23) is consistent with this more general definition.

It is worth noting that the quadratic variation is invariant with respect to an $\mathcal{F}_0$-measurable shift of $M$; specifically, if $M$ is a continuous local martingale and a continuous local martingale $N$ satisfies $N = \psi + M$ for some $\mathcal{F}_0$-measurable random variable $\psi$ then $\langle N \rangle = \langle M \rangle$. In particular, the quadratic variations of $M$ and $M - M_0$ are identical, that is, $\langle M \rangle = \langle M - M_0 \rangle$.

The existence and uniqueness of the process $\langle M \rangle$ satisfying conditions of Definition A.7.1 is a non-trivial issue, but it can be deduced from the Doob-Meyer decomposition theorem. We state below without proof a suitable version of this classic result. Let us stress that the result given below is by no means the most general version of the Doob-Meyer decomposition theorem (see, for instance, Karatzas and Shreve (1998a)).

**Proposition A.7.2** *Let $X$ be a non-negative continuous submartingale. Then there exists a continuous increasing process $A$ with $A_0 = 0$ such that the process $N = X - A$ is a continuous martingale. The processes $N$ and $A$ in the Doob-Meyer decomposition $X = N + A$ are unique up to indistinguishability of stochastic processes.*

It is common to refer to the process $A$ as the *compensator* of a submartingale $X$.

In our case, we apply Proposition A.7.2 to the process $X = M^2$, which is (after localization) a bounded non-negative continuous submartingale. In that case, the compensator $A$ of $X = M^2$ satisfies the definition of the quadratic variation $\langle M \rangle$.

*Remarks.* It is known that the quadratic variation $\langle M \rangle$ of a continuous local martingale $M$ can also be interpreted as the following limit (cf. (A.13))

$$\langle M \rangle_t = \lim_{n \to \infty} \sum_{k=1}^{m_n} (M_{t_k^n} - M_{t_{k-1}^n})^2, \tag{A.25}$$

where the convergence in probability holds for any sequence $(\pi_n)_{n \in \mathbb{N}}$ of finite partitions $\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = t\}$ of the interval $[0, t]$ satisfying $\lim_{n \to \infty} \delta(\pi_n) = 0$. Hence the name *quadratic variation* attributed to the process $\langle M \rangle$ is indeed justified.

**Lemma A.7.1** *Let $M$ be a continuous local martingale such that $\langle M \rangle_t = 0$ for $t \in [0, T]$. Then $M_t = M_0$ for every $t \in [0, T]$.*

*Proof.* We may and do assume, without loss of generality, that $M$ is a continuous bounded martingale and $M_0 = 0$. Then $\mathbb{E}_{\mathbb{P}} M_t^2 = \mathbb{E}_{\mathbb{P}} \langle M \rangle_t = 0$ for any $t \in [0, T]$ and thus $M_t = 0$ for any $t \in [0, T]$. □

**Definition A.7.2** An $\mathbb{F}$-adapted stochastic process $X = (X_t)_{t \in [0,T]}$ is said to be of *finite variation* if almost all sample paths of $X$ are functions of finite variation on $[0, T]$.

More explicitly, a process $X$ is of finite variation whenever (cf. (A.14))

$$\mathrm{var}_{[0,T]}(X) \stackrel{\text{def}}{=} \sup_{\pi \in \Pi_{[0,T]}} \sum_{k=1}^{n(\pi)} |X_{t_k} - X_{t_{k-1}}| < \infty, \quad \mathbb{P}\text{-a.s.}, \tag{A.26}$$

where $\Pi_{[0,T]}$ is the class of all finite partitions $\pi = \{0 = t_0 < t_1 < \cdots < t_{n(\pi)} = T\}$ of the interval $[0, T]$.

**Proposition A.7.3** *Let $M$ be a continuous local martingale of finite variation. Then $M_t = M_0$ for every $t \in [0, T]$.*

*Proof.* It suffices to observe that the assumption that sample paths of $M$ are continuous functions of finite variation implies that the quadratic variation of $M$ vanishes. This follows from the observation that sample path of $\langle M \rangle$ can be obtained as the almost sure limit in (A.25) for some sequence of partitions. The proof is thus based on similar arguments as those already employed in the proof of Corollary A.4.1. □

The last result shows that continuous local martingales are never processes of finite variation. In particular,

**Corollary A.7.2** *If $M_0 = 0$ and $M$ is a continuous local martingales of finite variation then $M$ vanishes (more precisely, it is indistinguishable from the null process).*

As in the special case of a Brownian motion, the stochastic integral of the form

$$\int_0^t \gamma_u \, dM_u$$

cannot be interpreted as the pathwise Lebesgue-Stieltjes integral. The theory of Itô integration with respect to continuous local martingales parallels the theory developed for the Brownian motion case. The only major modification in the one-dimensional case is that in the definition of spaces $\mathcal{L}^2_{\mathbb{P}}(M)$ and $\mathcal{L}_{\mathbb{P}}(M)$ the norm $\| \gamma \|^2_W$ should be replaced by its generalization:

$$\| \gamma \|^2_M \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{P}}\Big( \int_0^T \gamma_u^2 \, d\langle M \rangle_u \Big).$$

## A.8 Continuous Semimartingales

The concept of a continuous semimartingale is a natural extension of the notion of a continuous local martingale.

**Definition A.8.1** A real-valued, continuous, $\mathbb{F}$-adapted process $X$ is called a (*real-valued*) *continuous semimartingale* if it admits a decomposition

$$X_t = X_0 + M_t + A_t, \quad \forall\, t \in [0, T], \tag{A.27}$$

where $X_0$, $M$ and $A$ satisfy:
(i) $X_0$ is an $\mathcal{F}_0$-measurable random variable,
(ii) $M$ is a continuous local martingale with $M_0 = 0$,
(iii) $A$ is a continuous process whose almost all sample paths are of finite variation on the interval $[0, T]$ with $A_0 = 0$.

We denote by $\mathcal{S}^c(\mathbb{P})$ the class of all real-valued continuous semimartingales on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$. It appears that the decomposition of $X$ with the properties given in Definition A.8.1 is unique, up to indistinguishability of stochastic processes, as the following result shows.

**Proposition A.8.1** *Let $X$ be a continuous semimartingale with the decomposition $X = X_0 + M + A$ so that, in particular, $M_0 = A_0 = 0$. If $X$ admits also a decomposition $X = X_0 + \tilde{M} + \tilde{A}$ for some continuous local martingale $\tilde{M}$ with $\tilde{M}_0 = 0$ and some continuous process $\tilde{A}$ of finite variation on $[0, T]$ with $\tilde{A}_0 = 0$ then $M_t = \tilde{M}_t$ and $A_t = \tilde{A}_t$ for $t \in [0, T]$.*

*Proof.* It is enough to observe that a difference of two continuous local martingales is also a continuous local martingales, and a difference of two continuous processes of finite variation also follows a continuous process of finite variation. In our case, we have $M - \tilde{M} = \tilde{A} - A$ and $M_0 - \tilde{M}_0 = \tilde{A}_0 - A_0 = 0$. Consequently, in view of Lemma A.7.3, we conclude that the equalities $M_t = \tilde{M}_t$ and $A_t = \tilde{A}_t$ hold for every $t \in [0, T]$. □

In view of the uniqueness of the decomposition $X = X_0 + M + A$ established in Proposition A.8.1, it is justified to call it the *canonical decomposition* of a continuous semimartingale $X$ under $\mathbb{P}$. Since the continuous local martingale $M$ in the canonical decomposition of a continuous semimartingale $X$, it is customary to denote it by $X^c$. The *quadratic variation* $\langle X \rangle$ of a continuous semimartingale $X$ is then defined as as the quadratic variation of $X^c$, that is, $\langle X \rangle = \langle X^c \rangle$. Note that

$$\langle X \rangle_t = \lim_{n\to\infty} \sum_{k=1}^{m_n} (X_{t_k^n} - X_{t_{k-1}^n})^2 = \lim_{n\to\infty} \sum_{k=1}^{m_n} (X^c_{t_k^n} - X^c_{t_{k-1}^n})^2$$

where the convergence in probability holds for any sequence $(\pi_n)_{n\in\mathbb{N}}$ of finite partitions $\pi_n = \{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = t\}$ of the interval $[0, t]$ satisfying $\lim_{n\to\infty} \delta(\pi_n) = 0$.

The next result is an immediate consequence of Corollary A.7.2.

**Corollary A.8.1** *A continuous semimartingale is a continuous local martingale if and only if the process $A$ it its canonical decomposition $X = X_0 + M + A$ vanishes, that is, is indistinguishable from the null process.*

If $\tilde{\mathbb{P}}$ is a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ then $X$ is also a continuous semimartingale under $\tilde{\mathbb{P}}$. However, its canonical decompositions under $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are distinct, in general (cf. Theorem A.15.2).

The extension of the notion of a continuous semimartingale to a multidimensional setting is rather obvious. Specifically, we say that an $\mathbb{R}^d$-valued process $X = (X^1, \ldots, X^d)$ is a *$d$-dimensional continuous semimartingale* if each component $X^1, \ldots, X^d$ follows a real-valued continuous semimartingale. A $d$-dimensional Brownian motion $W = (W^1, \ldots, W^d)$ is a simple example of a $d$-dimensional continuous semimartingale.

**Itô processes.** Let $W$ be a $d$-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We shall now introduce a particular class of continuous semimartingales, referred to as *Itô processes.*

**Definition A.8.2** An $\mathbb{F}$-adapted continuous process $X$ is called an *Itô process* if it admits a representation

$$X_t = X_0 + \int_0^t \alpha_u \, du + \int_0^t \beta_u \cdot dW_u, \quad \forall\, t \in [0, T], \tag{A.28}$$

for some $\mathbb{F}$-adapted processes $\alpha$ and $\beta$ that are defined on $(\Omega, \mathbb{F}, \mathbb{P})$ and satisfy suitable integrability conditions.

It is customary to represent the integral formula (A.28) using the differential notation as

$$dX_t = \alpha_t \, dt + \beta_t \cdot dW_t.$$

It is clear that the Itô process $X$ follows a continuous semimartingale and formula (A.28) gives the canonical decomposition of $X$. In the present set-up, decomposition (A.28) is unique in the following sense: if $X$ satisfies (A.28) and simultaneously we have that

$$X_t = X_0 + \int_0^t \tilde{\alpha}_u \, du + \int_0^t \tilde{\beta}_u \, dW_u, \quad \forall \, t \in [0, T],$$

for some $\mathbb{F}$-adapted processes $\tilde{\alpha}$ and $\tilde{\beta}$, then the following equalities hold, for every $t \in [0, T]$,

$$\int_0^t \tilde{\alpha}_u \, du = \int_0^t \alpha_u \, du, \quad \int_0^t \tilde{\beta}_u \, dW_u = \int_0^t \beta_u \, dW_u.$$

It follow immediately from Corollary A.8.1 that an Itô process $X$ given by (A.28) is a continuous local martingale whenever it can be represented as

$$X_t = X_0 + \int_0^t \beta_u \cdot dW_u, \quad \forall \, t \in [0, T].$$

## A.9 Itô's Lemma

In this section, we shall deal with the following problem: does the process $g(X_t)$ follow a semimartingale if $X$ is a continuous semimartingale and $g$ is a sufficiently regular function? It turns out that the class of continuous semimartingales is invariant with respect to compositions with $C^2$-functions (let us stress that more general results are also available).

**One-dimensional case.** Consider a real-valued function $g = g(x, t)$, where $x \in \mathbb{R}$ is the space variable and $t \in [0, T]$ is the time variable. It is evident that if $X$ is a continuous semimartingale, and $g : \mathbb{R} \times [0, T] \to \mathbb{R}$ is a jointly continuous function, then the process $Y_t = g(X_t, t)$ is $\mathcal{F}_t$-adapted and has almost all sample paths continuous.

The next result, which is a special case of *Itô's lemma,* states that $Y$ is a semimartingale provided that the function $g$ is sufficiently smooth.

**Proposition A.9.1** *Suppose that* $g : \mathbb{R} \times [0, T] \to \mathbb{R}$ *is a function of class* $C^{2,1}(\mathbb{R} \times [0, T], \mathbb{R})$. *Then for any Itô process* $X$*, the process* $Y_t = g(X_t, t)$, $t \in [0, T]$*, is an Itô process. Moreover, its canonical decomposition is given by the Itô formula*

$$dY_t = g_t(X_t, t) \, dt + g_x(X_t, t)\alpha_t \, dt + g_x(X_t, t)\beta_t \, dW_t + \tfrac{1}{2} g_{xx}(X_t, t)\beta_t^2 \, dt.$$

More generally, if $X = X_0 + M + A$ is a real-valued continuous semimartingale, and $g$ is a function of class $C^{2,1}(\mathbb{R} \times [0, T], \mathbb{R})$ then the process $Y_t = g(X_t, t)$ follows a continuous semimartingale with the following canonical decomposition

$$dY_t = g_t(X_t, t) \, dt + g_x(X_t, t) \, dX_t + \tfrac{1}{2} \, g_{xx}(X_t, t) \, d\langle M \rangle_t. \tag{A.29}$$

**Multidimensional case.** Recall that a process $W = (W^1, \ldots, W^d)$ defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ is called the *d-dimensional standard Brownian motion* if the components $W^1, \ldots, W^d$ are independent one-dimensional standard Brownian motions with respect to $\mathbb{F}$.

In this paragraph, $W$ is assumed to be a $d$-dimensional standard Brownian motion. Let $\gamma$ belong to $\mathcal{L}_W(\mathbb{P})$, that is, $\gamma$ is a $\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable process satisfying

$$\mathbb{P}\Big\{ \int_0^T |\gamma_u|^2 \, du < \infty \Big\} = 1,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^d$. Then the Itô stochastic integral of $\gamma$ with respect to $W$ is well defined and we have, for any $t \in [0, T]$,

$$I_t(\gamma) = \int_0^t \gamma_u \cdot dW_u = \sum_{i=1}^d \int_0^t \gamma_u^i \, dW_u^i.$$

Let $X = (X^1, \ldots, X^k)$ be a $k$-dimensional Itô process given as

$$X_t^i = X_0^i + \int_0^t \alpha_u^i \, du + \int_0^t \beta_u^i \cdot dW_u, \tag{A.30}$$

where $\alpha^i$ are real-valued processes and $\beta^i = (\beta^{i1}, \ldots, \beta^{id})$ are $\mathbb{R}^d$-valued processes for $i = 1, \ldots, k$. It is implicitly assumed in (A.30) that the processes $\alpha^i, \beta^i$, $i = 1, \ldots, k$ are integrable in a suitable sense.

Let us introduce notation for the *cross-variation* (or the *quadratic covariation*) of two continuous semimartingales. If $X^i = X_0^i + M^i + A^i$ are in $\mathcal{S}^c(\mathbb{P})$ for $i = 1, \ldots, k$ then we set $\langle X^i, X^j \rangle \stackrel{\text{def}}{=} \langle M^i, M^j \rangle$, where in turn $\langle M^i, M^j \rangle$ is given by the following *polarization equality*

$$\langle M^i, M^j \rangle \stackrel{\text{def}}{=} \tfrac{1}{2} \big( \langle M^i + M^j \rangle - \langle M^i \rangle - \langle M^j \rangle \big) = \tfrac{1}{4} \big( \langle M^i + M^j \rangle - \langle M^i - M^j \rangle \big).$$

It is easily seen that $\langle X^i, X^i \rangle = \langle X^i \rangle$. If $X^i$ and $X^j$ are the Itô processes given by (A.30) then it is easily seen that we have, for any $t \in [0, T]$,

$$\langle X^i, X^j \rangle_t = \int_0^t \beta_u^i \cdot \beta_u^j \, du = \int_0^t \sum_{l=1}^d \beta_u^{il} \beta_u^{jl} \, du$$

The following result extends Proposition A.9.1.

**Proposition A.9.2** *Suppose that $g$ is a function of class $C^2(\mathbb{R}^k, \mathbb{R})$. Then the following form of the Itô formula is valid*

$$dg(X_t) = \sum_{i=1}^k g_{x_i}(X_t) \alpha_t^i \, dt + \sum_{i=1}^k g_{x_i}(X_t) \beta_t^i \cdot dW_t + \frac{1}{2} \sum_{i,j=1}^k g_{x_i x_j}(X_t) \, \beta_t^i \cdot \beta_t^j \, dt.$$

*More generally, if processes $X^i$ are in $\mathcal{S}^c(\mathbb{P})$ for $i = 1, \dots, k$ then*

$$g(X_t) = g(X_0) + \sum_{i=1}^{k} \int_0^t g_{x_i}(X_u)\, dX_u^i + \frac{1}{2} \sum_{i,j=1}^{k} \int_0^t g_{x_i x_j}(X_u)\, d\langle X^i, X^j \rangle_u,$$

*or equivalently*

$$dg(X_t) = \sum_{i=1}^{k} g_{x_i}(X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j=1}^{k} g_{x_i x_j}(X_t)\, d\langle X^i, X^j \rangle_t.$$

The *Itô integration by parts formula* is obtained by taking $g(x_1, x_2) = x_1 x_2$.

**Corollary A.9.1** *Let $X^1$ and $X^2$ belong to $\mathcal{S}^c(\mathbb{P})$. Then*

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_u^1\, dX_u^2 + \int_0^t X_u^2\, dX_u^1 + \langle X^1, X^2 \rangle_t. \tag{A.31}$$

*In particular, for any $X \in \mathcal{S}^c(\mathbb{P})$*

$$X_t^2 = X_0^2 + 2 \int_0^t X_u\, dX_u + \langle X \rangle_t. \tag{A.32}$$

**Itô-Tanaka-Meyer formula.** The classic Itô formula is obtained under the assumption that the transformation is twice continuously differentiable in the space variable. It is noteworthy that in the one-dimensional case the preservation of the semimartingale property of $X$ holds under a much weaker assumption that a function $g : \mathbb{R} \to \mathbb{R}$ is a difference of two convex functions.

Let $X$ be a real-valued continuous semimartingale, with the canonical decomposition $X = X_0 + M + A$. For any fixed $a \in \mathbb{R}$, we denote by $L_t^a(X)$ the (right) semimartingale *local time* of $X$ at the level $a$, that is, the process given explicitly by the formula

$$L_t^a(X) = |X_t - a| - |X_0 - a| - \int_0^t \operatorname{sgn}(X_u - a)\, dX_u$$

for every $t \in [0, T]$, where $\operatorname{sgn}(x) = 1$ for $x > 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$. By convention, we set $\operatorname{sgn}(0) = -1$ so that the function $\operatorname{sgn}(x - a)$ represents the left-hand side derivative of the function $|x - a|$. It is well known that the local time $L^a(X)$ of a continuous semimartingale $X$ is an adapted process whose sample paths are almost all continuous, non-decreasing functions, and we have

$$L_t^a(X) = \int_0^t \mathbb{1}_{\{X_u = a\}}\, dL_u^a(X),$$

as well as

$$L_t^a(X) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|X_u - a| \le \epsilon\}}\, d\langle M \rangle_u.$$

In addition, for any bounded (or nonnegative) measurable function $g : \mathbb{R} \to \mathbb{R}$ the following *density of occupation time formula* holds:

$$\int_{\mathbb{R}} g(a) L_t^a(X)\, da = \int_0^t g(X_u)\, d\langle M\rangle_u.$$

For an arbitrary convex function $f : \mathbb{R} \to \mathbb{R}$ and a continuous semimartingale $X$, the following decomposition, referred to as the *Itô-Tanaka-Meyer formula*, is valid

$$f(X_t) = f(X_0) + \int_0^t f_l'(X_u)\, dX_u + \frac{1}{2}\int_{\mathbb{R}} L_t^a(X)\, \mu(da),$$

where $f_l'$ is the left-hand-side derivative[2] of $f$ and the measure $\mu = f''$ represents the second order derivative of $f$, in the sense of distributions. If the function $f$ belongs to the class $C^2(\mathbb{R}, \mathbb{R})$, then $\mu(da) = f''(a)\, da$ for a nonnegative function $f''$, and the density of occupation time formula yields

$$\int_{\mathbb{R}} L_t^a(X)\, \mu(da) = \int_{\mathbb{R}} L_t^a(X)\, f''(a)\, da = \int_0^t f''(X_u)\, d\langle M\rangle_u.$$

Hence the Itô-Tanaka-Meyer formula reduces here to the classical Itô formula.

## A.10 Lévy's Characterization Theorem

In some instances, it would be convenient to have the possibility of checking whether a given process is a standard Brownian motion by establishing its martingale property and by computing its quadratic variation. To this end, we may use the *martingale characterization* of a Brownian motion, due to Paul Lévy.

**One-dimensional case.** Recall that if $W$ is a one-dimensional Brownian motion on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ then $W$ is a continuous $\mathbb{F}$-martingale with $W_0 = 0$ and the process $W_t^2 - t$ is also a continuous $\mathbb{F}$-martingale.

**Theorem A.10.1** *Let $M$ be a continuous local martingale on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ such that $M_0 = 0$ and the process $M_t^2 - t$ is a continuous local martingale. Then $M$ is a standard Brownian motion with respect to $\mathbb{F}$.*

The assumption that $M_t^2 - t$ is a continuous local martingale means that the quadratic variation of $M$ satisfies $\langle M\rangle_t = t$ for every $t \in [0, T]$. Under the stronger assumption that $M$ is a square-integrable continuous martingale, this property can be represented as follows

$$\mathbb{E}_{\mathbb{P}}(M_u^2 - M_t^2 \,|\, \mathcal{F}_t) = u - t, \quad \forall\, t \le u \le T, \tag{A.33}$$

or equivalently (in view of the martingale property of $M$)

$$\mathbb{E}_{\mathbb{P}}((M_u - M_t)^2 \,|\, \mathcal{F}_t) = u - t, \quad \forall\, t \le u \le T.$$

[2] Recall that the left-hand-side and right-hand-side derivatives of a convex function are nondecreasing functions.

*Remarks.* It should be stressed that the assumption of continuity of sample paths of $M$ is essential in Theorem A.10.1 and thus it cannot be dropped. Indeed, if we take as $M$ the *compensated Poisson process*, that is, the process $M = N_t - t$, where $N$ is the Poisson process with unit intensity, then it can be easily verified that the process $M_t^2 - t$ is a martingale as well.

The proof of Theorem A.10.1 is omitted. Let us only mention that is based on the following lemma, which in turn can be proved by directly checking conditions of Definition A.4.1.

**Lemma A.10.1** *A real-valued continuous $\mathbb{F}$-adapted process $X$ defined on $(\Omega, \mathbb{F}, \mathbb{P})$ is a standard Brownian motion with respect to $\mathbb{F}$ if and only if for any $\lambda \in \mathbb{R}$ the process*

$$M_t^\lambda = \exp\left(\lambda X_t - \tfrac{1}{2}\lambda^2 t\right), \quad \forall\, t \in [0, T], \tag{A.34}$$

*is a (local) martingale with respect to $\mathbb{F}$ and $M_0^\lambda = 1$.*

The process $M^\lambda$ is a special case of the *stochastic exponential* examined in some detail in Sect. A.13. It is easily seen that when $W$ is a one-dimensional Brownian motion with respect to $\mathbb{F}$ then we have

$$\mathbb{E}_{\mathbb{P}}\Big(e^{\lambda(W_u - W_t)}\,\Big|\,\mathcal{F}_t\Big) = e^{\frac{1}{2}\lambda^2(u-t)}, \quad \forall\, t \le u \le T.$$

This implies that the process $M^\lambda$ associated with $W$ is an $\mathbb{F}$-martingale.

**Multidimensional case.** Theorem A.10.1 can be extended to a multidimensional setting. Let $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

**Theorem A.10.2** *Let $M = (M^1, \dots, M^d)$ be a continuous $\mathbb{F}$-adapted process on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with the initial value $M_0 = (0, \dots, 0)$. Then $M$ is a $d$-dimensional standard Brownian motion if and only if the following conditions are satisfied:*
(i) *for any $i = 1, \dots, d$ the process $M^i$ is an $\mathbb{F}$-martingale,*
(ii) *for any $i, j = 1, \dots, d$ the process $M_t^i M_t^j - \delta_{ij} t$ is an $\mathbb{F}$-martingale.*

The following variant of the multidimensional Lévy's characterization theorem will prove useful.

**Theorem A.10.3** *Assume that $M = (M^1, \dots, M^d)$ is an $\mathbb{F}$-adapted process on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ such that $M_0 = (0, \dots, 0)$ and the components $M^1, \dots, M^d$ are continuous local martingales with $\langle M^i, M^j \rangle_t = \delta_{ij} t$. Then $M$ is a $d$-dimensional standard Brownian motion.*

## A.11 Martingale Representation Property

In this section we shall assume that the filtration $\mathbb{F} = \mathbb{F}^W$ is the standard augmentation of the natural filtration $\sigma\{W_u \,|\, u \le t\}$ of a $d$-dimensional Brownian motion $W$.

In other words, we assume here that the underlying probability space is $(\Omega, \mathbb{F}^W, \mathbb{P})$. We will now state two closely related versions of the so-called *predictable representation property* of the Brownian filtration.

The first theorem is known as the *Itô representation theorem*. It states that any square-integrable and $\mathcal{F}_T^W$-measurable random variables admits a representation as the Itô integral of some stochastic process.

**Theorem A.11.1** *For any random variable $X \in L^2(\Omega, \mathcal{F}_T^W, \mathbb{P})$, there exists a unique $\mathbb{F}$-predictable process $\gamma$ from the class $\mathcal{L}^2_{\mathbb{P}}(W)$ such that the following equality is valid*

$$X = \mathbb{E}_{\mathbb{P}} X + \int_0^T \gamma_u \cdot dW_u. \tag{A.35}$$

The condition that $\gamma$ belongs to $\mathcal{L}^2_{\mathbb{P}}(W)$ implies that the Itô integral $I(\gamma)$ is a square-integrable $\mathbb{F}$-martingale and thus we also have that, for every $t \in [0, T]$,

$$\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{F}_t^W) = \mathbb{E}_{\mathbb{P}} X + \int_0^t \gamma_u \cdot dW_u. \tag{A.36}$$

*Remarks.* In Theorem A.11.1 we assert that $\gamma$ is an $\mathbb{F}$-*predictable process* (for the definition see, for instance, Chapter 5 in Revuz and Yor (1999)). Let us only note that any left-continuous and $\mathbb{F}$-adapted process is $\mathbb{F}$-predictable. Also, if a process is $\mathbb{F}$-predictable then it is $\mathbb{F}$-progressively measurable, but the converse does not hold, in general.

The second result is commonly known as the *martingale representation theorem* for the Brownian filtration. Note that it is a rather straightforward consequence of Theorem A.11.1 (see, in particular, formula (A.36)).

**Theorem A.11.2** *Let a process $M = (M_t)_{t \in [0,T]}$ be an $\mathbb{F}^W$-martingale such that $\mathbb{E}_{\mathbb{P}} M_T^2 < \infty$. Then there exists a unique $\mathbb{F}$-predictable process $\gamma$ from the class $\mathcal{L}^2_{\mathbb{P}}(W)$ such that, for any $t \in [0, T]$,*

$$M_t = M_0 + \int_0^t \gamma_u \cdot dW_u. \tag{A.37}$$

Theorem A.11.2 can be extended to local martingales with respect to a Brownian filtration. As a consequence, we obtain the following notable result.

**Corollary A.11.1** *Any local martingale with respect to the Brownian filtration $\mathbb{F}^W$ is necessarily a continuous process.*

**Example A.11.1** Let $d = 1$ and let us first take $X = W_T^2$. Using the Itô formula of Proposition A.9.1 with $g(x, t) = x^2$ and $X = W$, we obtain

$$W_t^2 = \int_0^t 2W_u \, dW_u + t, \quad \forall \, t \in [0, T].$$

Since $\mathbb{E}_{\mathbb{P}} W_T^2 = T$, the equality (A.35) becomes

$$W_T^2 = \mathbb{E}_{\mathbb{P}} W_T^2 + \int_0^T 2W_u \, dW_u,$$

so that the process $\gamma_t = 2W_t,\ t \in [0, T]$, is independent of $T$.

A more interesting result is obtained for $X = W_T^3$. By applying the Itô formula to $g(x) = x^3$ and $X = W$, we obtain

$$W_t^3 = \int_0^t 3W_u^2 \, dW_u + \int_0^t 3W_u \, du, \quad \forall\, t \in [0, T].$$

Therefore

$$W_T^3 = \int_0^T 3(W_u^2 + (T-u)) \, dW_u. \tag{A.38}$$

since the Itô integration by parts formula (A.31) applied to $X_t^1 = W_t$ and $X_t^2 = T-t$ yields $\int_0^T W_u \, du = \int_0^T (T-u) \, dW_u$ (of course, here $\langle X^1, X^2 \rangle = 0$). Hence, for any fixed $T > 0$, we have $\gamma_t = 3W_t^2 + 3(T-t)$ for every $t \in [0, T]$.

Alternatively, by evaluating the conditional expectation, we obtain

$$\mathbb{E}_{\mathbb{P}}(W_T^3 \,|\, \mathcal{F}_t) = f(W_t, t), \quad \forall\, t \in [0, T],$$

where the function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is given by $f(x, t) = x^3 + 3x(T-t)$. Note that $f$ solves on $(0, T) \times \mathbb{R}$ the following partial differential equation

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0$$

with the terminal condition $f(x, T) = x^3$ for $x \in \mathbb{R}$. Hence, by applying the Itô formula to $g(x, t) = f(x, t)$ and $X = W$, we re-derive (A.38).

## A.12 Stochastic Differential Equations

The theory of stochastic differential equations is covered by numerous textbooks and monographs. In this short section, we mainly deal with the special case of linear stochastic differential equations.

Let $\mu : \mathbb{R} \times [0, T] \to \mathbb{R}$ and $\sigma : \mathbb{R} \times [0, T] \to \mathbb{R}^d$ be some given functions and let $W$ be a $d$-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

**Definition A.12.1** By a *solution* of the stochastic differential equation (SDE, for short)

$$dX_t = \mu(X_t, t) \, dt + \sigma(X_t, t) \cdot dW_t \tag{A.39}$$

with the initial condition $X_0$ given as an $\mathcal{F}_0$-measurable random variable, we mean an $\mathbb{R}^k$-valued, $\mathbb{F}$-adapted stochastic process $X$ defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and such that, for every $t \in [0, T]$,

$$X_t = X_0 + \int_0^t \mu(X_u, u) \, du + \int_0^t \sigma(X_u, u) \cdot dW_u. \tag{A.40}$$

It is clear from the definition above that a stochastic differential equation, usually represented in its differential form (A.39), is nothing else than the integral equation (A.40), in which the second integral is defined as the Itô stochastic integral. Definition A.12.1 implicitly assumes that both integrals in the right-hand side of (A.40) are well defined, that is,

$$\mathbb{P}\Big(\int_0^T |\mu(X_u, u)|\, du < \infty\Big) = 1$$

and the process $\gamma$ given as $\gamma_t = \sigma(X_t, t),\ t \in [0, T]$, belongs to the class $\mathcal{L}_{\mathbb{P}}(W)$. Note that any solution $X$ to the SDE (A.39) is an Itô process, in the sense of Definition A.8.2.

The concept of uniqueness of solutions to an SDE can be defined in several alternative ways. For our purposes, the most convenient form is the so-called *pathwise uniqueness* of solutions. If $X$ is a solution to (A.39), we shall informally say that $X$ is driven by $W$.

**Definition A.12.2** We say that the *pathwise uniqueness* of solutions to (A.39) holds is for any filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, any $d$-dimensional standard Brownian motions $W$ and $\tilde{W}$ defined on $(\Omega, \mathbb{F}, \mathbb{P})$, and any two solutions $X$ and $\tilde{X}$ driven by $W$ and $\tilde{W}$ respectively, the following implication is true

$$\mathbb{P}\{W_t = \tilde{W}_t \,|\, \forall\, t \in [0, T]\} = 1 \quad \Longrightarrow \quad \mathbb{P}\{X_t = \tilde{X}_t \,|\, \forall\, t \in [0, T]\} = 1.$$

The pathwise uniqueness of solutions (A.39) is sometimes referred to as the *strong uniqueness*. This is due to the fact that under pathwise uniqueness any solution to (A.39) is *strong*, meaning that it is adapted to the filtration generated by the driving Brownian motion $W$.

**Linear stochastic differential equation.** Let us first consider a time-homogeneous linear SDE of the form

$$dX_t = (aX_t + c)\, dt + (bX_t + f)\, dW_t, \tag{A.41}$$

where $W$ is a one-dimensional Brownian motion and $a, b, c$ and $f$ are real numbers. It is clear that $\mu(x, t) = ax + c$ and $\sigma(x, t) = bx + f$ so that $\mu$ and $\sigma$ are linear functions.

By setting $c = f = 0$, we obtain the SDE governing the stock price process in the Black-Scholes model (see Sect. 3.1). In the present notation, it becomes

$$dX_t = aX_t\, dt + bX_t\, dW_t = X_t(a\, dt + b\, dW_t). \tag{A.42}$$

**Proposition A.12.1** *The unique solution of the SDE (A.42) is given by the expression*

$$X_t = X_0 \exp\Big(bW_t + \big(a - \tfrac{1}{2}b^2\big)t\Big). \tag{A.43}$$

*Proof.* See the proof of Proposition 3.1.1 in Sect. 3.1. □

The following result is useful in the case of Vasicek's model of the short-term rate (see Sect. 10.1).

**Proposition A.12.2** *The unique solution of the SDE*

$$dX_t = (aX_t + c)\,dt + f\,dW_t \tag{A.44}$$

*is given by the formula*

$$X_t = X_0 e^{at} + \int_0^t c e^{a(t-u)}\,du + \int_0^t f e^{a(t-u)}\,dW_u. \tag{A.45}$$

*Proof.* See the proof of Proposition 10.1.2 in Sect. 10.1. □

The definition of an SDE can also be extended to the case where the coefficients $\mu$ and $\sigma$ explicitly depend on the time parameter $t$ and $\omega$. For instance, we may assume that the parameters (A.41) follow stochastic processes, so that it becomes

$$dX_t = \big(a_t X_t + c_t\big)\,dt + \big(b_t X_t + f_t\big)\cdot dW_t. \tag{A.46}$$

where $W$ is a $d$-dimensional Brownian motion and the $\mathbb{F}$-adapted stochastic processes $a, c$ are real-valued, whereas the processes $b, f$ are $\mathbb{R}^d$-valued. Then we have the following extension of Proposition A.12.1.

**Proposition A.12.3** *Let $a$ and $b$ be $\mathbb{F}$-adapted bounded processes. Then the unique solution of the SDE*

$$dX_t = X_t\big(a_t\,dt + b_t\cdot dW_t\big) \tag{A.47}$$

*is given by the following generalization of formula* (A.43)

$$X_t = X_0 \exp\Big(\int_0^t b_u\cdot dW_u + \int_0^t \big(a_u - \tfrac{1}{2}|b_u|^2\big)\,du\Big). \tag{A.48}$$

*Proof.* To check that the process $X$ given by (A.48) is a solution to (A.47), it suffices to differentiate the right-hand side in (A.48) using the Itô formula of Proposition A.9.2. The uniqueness of solutions to (A.47) can be proved directly or deduced from Theorem A.12.1. □

The following result extends Proposition A.12.2.

**Proposition A.12.4** *Let $a, c$ and $f$ be $\mathbb{F}$-adapted bounded processes. Then the unique solution of the linear SDE*

$$dX_t = \big(a_t X_t + c_t\big)\,dt + f_t\cdot dW_t \tag{A.49}$$

*is given by the formula*

$$X_t = \Phi_t\Big(X_0 + \int_0^t \Phi_u^{-1} c_u\,du + \int_0^t \Phi_u^{-1} f_u\cdot dW_u\Big) \tag{A.50}$$

*where*

$$\Phi_u = \exp\Big(\int_0^u a_s\,ds\Big).$$

**Itô's existence and uniqueness theorem for SDEs.** Let us end this section by stating the classic Itô's theorem, which provides sufficient conditions for the existence and uniqueness of solutions to the SDE (A.39).

**Theorem A.12.1** *Let* $\mu : \mathbb{R} \times [0, T] \to \mathbb{R}$ *and* $\sigma : \mathbb{R} \times [0, T] \to \mathbb{R}^d$ *satisfy the following conditions:*
(i) *the functions* $\mu$ *and* $\sigma$ *are Lipschitz continuous with respect to the variable* $x$*, that is, there exists a constant* $K_1 > 0$ *such that, for any* $x, y \in \mathbb{R}$ *and* $t \in \mathbb{R}_+$*,*

$$|\mu(t, x) - \mu(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \leq K_1 |x - y|^2, \tag{A.51}$$

(ii) *the functions* $\mu$ *and* $\sigma$ *satisfy the linear growth condition: there exists a constant* $K_2$ *such that, for any* $x \in \mathbb{R}$ *and* $t \in \mathbb{R}_+$*,*

$$|a(t, x)|^2 + |b(t, x)|^2 \leq K_2(1 + |x|^2). \tag{A.52}$$

*Then the SDE has the unique solution* $X$*.*

For notational simplicity, in Theorem A.12.1 the symbol $|\cdot|$ is used to denote the Euclidean norms in both $\mathbb{R}$ and $\mathbb{R}^d$, but this slight abuse of notation should not be confusing.

Let us finally mention that an analogous result is valid in the multidimensional case, that is, when the $k$-dimensional process $X = (X^1, \ldots, X^k)$ is given as

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t, \quad X_0 = x, \tag{A.53}$$

with the coefficients $\mu : \mathbb{R}^k \times [0, T] \to \mathbb{R}^k$ and $\sigma : \mathbb{R}^k \times [0, T] \to \mathbb{R}^{k \times d}$ satisfying the natural extensions of Itô's conditions (A.51) and (A.52).

## A.13 Stochastic Exponential

Let $W$ be a $d$-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. For an $\mathbb{R}^d$-valued process $\gamma$ belonging to $\mathcal{L}(W)$, we define the real-valued $\mathbb{F}$-adapted process $U$ by setting

$$U_t = I_t(\gamma) = \int_0^t \gamma_u \cdot dW_u, \quad \forall\, t \in [0, T]. \tag{A.54}$$

The process $U$ defined in this way is, of course, a continuous local martingale under $\mathbb{P}$ with respect to $\mathbb{F}$.

**Definition A.13.1** The *stochastic exponential* (also known as the *Doléans exponential*) of $U$ is given by the formula, for $t \in [0, T]$,

$$\mathcal{E}_t(U) = \mathcal{E}_t\Big(\int_0^{\cdot} \gamma_u \cdot dW_u\Big) = \exp\,\Big(\int_0^t \gamma_u \cdot dW_u - \frac{1}{2}\int_0^t |\gamma_u|^2\, du\Big),$$

that is, $\mathcal{E}_t(U) = \exp(U_t - \langle U \rangle_t / 2)$. More generally, for any continuous local martingale $M$ we set $\mathcal{E}_t(M) = \exp(M_t - \langle M \rangle_t / 2)$ for $t \in [0, T]$.

**Lemma A.13.1** *The stochastic exponential of $U$ is the unique solution $Z$ of the stochastic differential equation*

$$dZ_t = Z_t \gamma_t \cdot dW_t = Z_t \, dU_t \tag{A.55}$$

*with the initial condition $Z_0 = 1$.*

*Proof.* The assertion is a straightforward consequence of Proposition A.12.4. □

*Remarks.* (i) It follows immediately from Lemma A.13.1 that

$$d\mathcal{E}_t(U) = \mathcal{E}_t(U) \gamma_t \cdot dW_t = \mathcal{E}_t(U) \, dU_t, \tag{A.56}$$

Note that $\mathcal{E}(U)$ is a strictly positive continuous local martingale under $\mathbb{P}$ and thus, in view of Proposition A.7.1, it follows a supermartingale with respect to $\mathbb{F}$. If, in addition,

$$\mathbb{E}_{\mathbb{P}}(\mathcal{E}_T(U)) = \mathbb{E}_{\mathbb{P}}(\mathcal{E}_0(U)) = 1,$$

so that the expected value $\mathbb{E}_{\mathbb{P}}\mathcal{E}_t(U),\ t \in [0, T]$ is constant, then the process $\mathcal{E}(U)$ is a strictly positive continuous $\mathbb{F}$-martingale.
(ii) For any continuous local martingale $M$, the stochastic exponential $\mathcal{E}(M)$ is the unique solution $Z$ of the stochastic differential equation

$$dZ_t = Z_t \, dM_t \tag{A.57}$$

with the initial condition $Z_0 = 1$.

## A.14 Radon-Nikodým Density

Two probability measures $\tilde{\mathbb{P}}$ and $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ are said to be *equivalent* if, for any event $A \in \mathcal{F}_T$, the equality $\mathbb{P}(A) = 0$ holds if and only if $\tilde{\mathbb{P}}(A) = 0$. In other words, $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent on $(\Omega, \mathcal{F}_T)$ if they have the same set of null events in the $\sigma$-field $\mathcal{F}_T$. It is easily seen that if $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent on $\mathcal{F}_T$ then they also enjoy this property on any $\sigma$-field $\mathcal{G} \subseteq \mathcal{F}_T$; in particular, $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent on the $\sigma$-field $\mathcal{F}_t$, for any $t \in [0, T]$.

**Definition A.14.1** The *Radon-Nikodým density* of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is defined as the unique $\mathcal{F}_T$-measurable random variable $\eta_T$ such that we have, for any event $A \in \mathcal{F}_T$,

$$\tilde{\mathbb{P}}(A) = \int_A \eta_T \, d\mathbb{P} = \mathbb{E}_{\mathbb{P}}(\eta_T \mathbb{1}_A). \tag{A.58}$$

The existence and uniqueness of the random variable $\eta_T$ satisfying (A.58) is a consequence of the classic Radon-Nikodým theorem. Formula (A.58) implies that for any $\tilde{\mathbb{P}}$-integrable random variable $\psi$ we have that $\mathbb{E}_{\tilde{\mathbb{P}}}\psi = \mathbb{E}_{\mathbb{P}}(\psi \eta_T)$. Note also that $\psi$ is $\tilde{\mathbb{P}}$-integrable if and only if $\psi \eta_T$ is $\tilde{\mathbb{P}}$-integrable. Finally, it is easy to check that $\mathbb{P}\{\eta_T > 0\} = 1$ and $\mathbb{E}_{\mathbb{P}}\eta_T = \tilde{\mathbb{P}}(\Omega) = 1$.

Conversely, assume that $\eta_T$ is any $\mathcal{F}_T$-measurable random variable such that the last two conditions hold. Then formula (A.58) defines a probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$, which is then equivalent to $\mathbb{P}$.

To emphasize the role of $\eta_T$ as the link between the expectations with respect to $\tilde{\mathbb{P}}$ and $\mathbb{P}$, it is customary to use the short-hand notation

$$\eta_T = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}.$$

**Definition A.14.2** The *Radon-Nikodým density process* $\eta = (\eta_t)_{t\in[0,T]}$ of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ and a given filtration $\mathbb{F}$ is defined by setting, for $t \in [0, T]$,

$$\eta_t \stackrel{\text{def}}{=} \mathbb{E}_{\mathbb{P}}(\eta_T \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\Big(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\,\Big|\,\mathcal{F}_t\Big), \quad \forall\, t \in [0, T]. \tag{A.59}$$

It is clear that Radon-Nikodým density process $\eta$ is a strictly positive martingale under $\mathbb{P}$. It is also uniformly integrable since clearly $\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_T \,|\, \mathcal{F}_t)$. Finally, the random variable $\eta_t$ is the Radon-Nikodým density of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_t)$. The latter property is denoted as

$$\eta_t = \frac{d\tilde{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}_{|\mathcal{F}_t}} = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}. \tag{A.60}$$

**Lemma A.14.1** *A stochastic process $X$ is an $\mathbb{F}$-martingale under $\tilde{\mathbb{P}}$ if and only if the process $X\eta$ is an $\mathbb{F}$-martingale under $\mathbb{P}$.*

*Proof.* The proof relies on the application of the abstract Bayes formula (see Lemma A.1.4). Assume first that $X\eta$ is an $\mathbb{F}$-martingale under $\mathbb{P}$ so that equality $\mathbb{E}_{\mathbb{P}}(X_u\eta_u \,|\, \mathcal{F}_t) = X_t\eta_t$ holds for $0 \le t \le u \le T$. Then the Bayes formula yields, for $0 \le t \le u \le T$,

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{P}}}(X_u \,|\, \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(X_u\eta_T \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_T \,|\, \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(X_u\mathbb{E}_{\mathbb{P}}(\eta_T \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_T \,|\, \mathcal{F}_t)} \\ &= \frac{\mathbb{E}_{\mathbb{P}}(X_u\eta_u \,|\, \mathcal{F}_t)}{\eta_t} = \frac{X_t\eta_t}{\eta_t} = X_t.\end{aligned}$$

We conclude that $X$ is an $\mathbb{F}$-martingale under $\mathbb{P}$. The proof of the converse implication goes along the same lines. □

## A.15 Girsanov's Theorem

We are in the position to state the classic Girsanov theorem. It states that, under mild technical conditions, a Brownian motion with an absolutely continuous drift becomes a standard Brownian motion under an equivalent probability measure. Before proceeding to the general result, let us first consider a special case of a linear drift.

**Proposition A.15.1** *Let $W$ be a one-dimensional standard Brownian motion on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. For a real number $\gamma \in \mathbb{R}$, we define the process $X$ by setting $\tilde{W}_t = W_t - \gamma t$ for $t \in [0, T]$. Let the probability measure $\tilde{\mathbb{P}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, be defined through the formula*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\gamma W_T - \tfrac{1}{2}\gamma^2 T\right) = \eta_T, \quad \mathbb{P}\text{-a.s.}$$

*Then $X$ is a standard Brownian motion on the probability space $(\Omega, \mathbb{F}, \tilde{\mathbb{P}})$.*

*Proof.* To establish the proposition, it is enough to make use of Lemmas A.10.1 and A.14.1. In view of the former, it suffices to show that for any $\lambda \in \mathbb{R}$ the process

$$M_t^\lambda = \exp\left(\lambda \tilde{W}_t - \tfrac{1}{2}\lambda^2 t\right), \quad \forall\, t \in [0, T],$$

is an $\mathbb{F}$-martingale under $\mathbb{Q}$. Using the latter lemma, we see that it is enough to check that $M^\lambda \eta$ is an $\mathbb{F}$-martingale under $\mathbb{P}$. But clearly

$$M_t^\lambda \eta_t = \exp\left(\lambda(W_t - \gamma t) - \tfrac{1}{2}\lambda^2 t\right) \exp\left(\gamma W_t - \tfrac{1}{2}\gamma^2 t\right) = \exp\left(\alpha W_t - \tfrac{1}{2}\alpha^2 t\right)$$

where $\alpha = \lambda + \gamma$, and thus this process follows an $\mathbb{F}$-martingale under $\mathbb{P}$, by another application of Lemma A.10.1. We conclude that $\tilde{W}$ is a standard Brownian motion on $(\Omega, \mathbb{F}, \tilde{\mathbb{P}})$. □

Proposition A.15.1 can be extended to the case of a stochastic drift term, as the following result shows.

**Theorem A.15.1** *Let $W$ be a standard $d$-dimensional Brownian motion on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Suppose that $\gamma$ is an $\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable process such that*

$$\mathbb{E}_{\mathbb{P}}\left\{\mathcal{E}_T\left(\int_0^\cdot \gamma_u \cdot dW_u\right)\right\} = 1. \tag{A.61}$$

*Define a probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ by means of the Radon-Nikodým derivative*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T\left(\int_0^\cdot \gamma_u \cdot dW_u\right), \quad \mathbb{P}\text{-a.s.} \tag{A.62}$$

*Then the process $\tilde{W}$ given by the formula*

$$\tilde{W}_t = W_t - \int_0^t \gamma_u \, du, \quad \forall\, t \in [0, T], \tag{A.63}$$

*follows a standard $d$-dimensional Brownian motion on the space $(\Omega, \mathbb{F}, \tilde{\mathbb{P}})$.*

*Proof.* In view of Theorem A.10.2 and Lemma A.14.1, it suffices to check that the processes $\tilde{W}^i \eta$ and $(\tilde{W}^i_t \tilde{W}^j_t - \delta_{ij} t)\eta_t$ are continuous local martingales under $\mathbb{P}$. For the first process, an application of the Itô integration by parts formula gives

$$\begin{aligned} d(\tilde{W}^i_t \eta_t) &= \tilde{W}^i_t \, d\eta_t + \eta_t \, d\tilde{W}^i_t + d\langle \tilde{W}^i, \eta \rangle_t \\ &= \tilde{W}^i_t \eta_t \gamma_t \cdot dW_t + \eta_t \, dW^i_t - \eta_t \gamma^i_t \, dt + d\langle \tilde{W}^i, \eta \rangle_t \\ &= \tilde{W}^i_t \eta_t \gamma_t \cdot dW_t + \eta_t \, dW^i_t \end{aligned}$$

since $d\eta_t = \eta_t \gamma_t \cdot dW_t$ and thus $d\langle \tilde{W}^i, \eta \rangle_t = \gamma^i_t \eta_t \, dt$. We thus see that $\tilde{W}^i \eta$ is a continuous local martingale under $\mathbb{P}$.

Let us denote $\tilde{M}^{ij}_t = \tilde{W}^i_t \tilde{W}^j_t - \delta_{ij} t$. Then we have that (recall that $\langle W^i, W^j \rangle_t = 0$ for $i \neq j$ and $\langle W^i, W^i \rangle_t = t$ for $t \in [0, T]$)

$$d\tilde{M}^{ij}_t = \tilde{W}^i_t \, d\tilde{W}^j_t + \tilde{W}^j_t \, d\tilde{W}^i_t,$$

and this in turn implies that

$$d\langle \tilde{M}^{ij}, \eta \rangle_t = \eta_t (\tilde{W}^i_t \gamma^j_t + \tilde{W}^j_t \gamma^i_t) \, dt.$$

By applying once again the Itô integration by parts formula, we obtain

$$\begin{aligned} d(\tilde{M}^{ij}_t \eta_t) &= \tilde{M}^{ij}_t \, d\eta_t + \eta_t \, d\tilde{M}^{ij}_t + d\langle \tilde{M}^{ij}, \eta \rangle_t \\ &= \tilde{M}^{ij}_t \, d\eta_t + \eta_t \, (\tilde{W}^i_t \, d\tilde{W}^j_t + \tilde{W}^j_t \, d\tilde{W}^i_t) + \eta_t (\tilde{W}^i_t \gamma^j_t + \tilde{W}^j_t \gamma^i_t) \, dt \\ &= \tilde{M}^{ij}_t \eta_t \gamma_t \cdot dW_t + \eta_t (\tilde{W}^i_t \, dW^j_t + \tilde{W}^j_t \, dW^i_t) \end{aligned}$$

since $\tilde{W}^i_t = W^i_t - \gamma^i_t \, dt$. We conclude that the process $(\tilde{W}^i_t \tilde{W}^j_t - \delta_{ij} t)\eta_t$ follows a continuous local martingale under $\mathbb{P}$. □

In order to make use of Girsanov's theorem for a particular process, we need, of course, to be able to verify condition (A.61). The next result furnishes a sufficient condition for the equality (A.61) to hold. The inequality (A.64) is commonly referred to as *Novikov's condition*.

**Proposition A.15.2** *Assume that*

$$\mathbb{E}_{\mathbb{P}} \Big\{ \exp \Big( \frac{1}{2} \int_0^T |\gamma_u|^2 \, du \Big) \Big\} < \infty. \tag{A.64}$$

*Then* $\mathbb{E}_{\mathbb{P}}(\mathcal{E}_T(U)) = 1$, *where* $U$ *is given by* (A.54). *Consequently, the process* $\mathcal{E}(U)$ *is a strictly positive continuous* $\mathbb{F}$*-martingale.*

In particular, if the process $\gamma$ is uniformly bounded (that is, there exists a constant $K$ such that $|\gamma_t| \leq K$ for $t \in [0, T]$) then Novikov's condition is satisfied and thus $\mathbb{E}_{\mathbb{P}}(\mathcal{E}_T(U)) = 1$.

A weaker, but also sufficient for the validity of (A.61), is the *Kazamaki condition*

$$\mathbb{E}_{\mathbb{P}}\Big\{\exp\Big(\frac{1}{2}\int_0^t \gamma_u \cdot dW_u\Big)\Big\} < \infty, \quad \forall\, t \in [0, T]. \tag{A.65}$$

Let us comment briefly on relevant filtrations. Obviously, we always have that $\mathcal{F}_t^{\tilde{W}} \subseteq \mathcal{F}_t$ for any $t \in [0, T]$. The filtrations generated by $W$ and $\tilde{W}$ do not coincide, in general. In the special case where the underlying filtration $\mathbb{F}$ is the $\mathbb{P}$-augmentation of the natural filtration of $W$, the inclusion $\mathcal{F}_t^{\tilde{W}} \subseteq \mathcal{F}_t^W$ holds for any $t \in [0, T]$, but the filtration generated by $\tilde{W}$ can be essentially smaller than $\mathbb{F}^W$.

The next well-known result shows that, if the underlying filtration is generated by a $d$-dimensional Brownian motion, the Radon-Nikodým density process of any probability measure equivalent to $\mathbb{P}$ has necessarily the form of the stochastic exponential (A.62) for some process $\gamma$.

**Proposition A.15.3** *Assume that the filtration $\mathbb{F}$ is the usual augmentation of the natural filtration of $W$, that is, $\mathbb{F} = \mathbb{F}^W$. Then for any probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ there exists an $\mathbb{F}^W$-progressively measurable, $\mathbb{R}^d$-valued process $\gamma$ such that the Radon-Nikodým density of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ equals*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T\Big(\int_0^{\cdot} \gamma_u \cdot dW_u\Big), \quad \mathbb{P}\text{-a.s.} \tag{A.66}$$

*Proof.* Let $\eta_T$ be the Radon-Nikodým density of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ (see (A.60)). It is clear that the process $\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_T \,|\, \mathcal{F}_t)$ is an $\mathbb{F}$-martingale and $\eta_0 = 1$. Since the underlying filtration is a Brownian filtration, from remarks after martingale representation theorem A.11.2 we can deduce the existence of a process $\tilde{\gamma} \in \mathcal{L}_{\mathbb{P}}(W)$ such that

$$\eta_t = 1 + \int_0^t \tilde{\gamma}_u \cdot dW_u, \quad \forall\, t \in [0, T]. \tag{A.67}$$

Since $\mathbb{P}\{\eta_T > 0\} = 1$, we also have that $\mathbb{P}\{\eta_t > 0\} = 1$ for any $t \in [0, T]$, and thus, in view of continuity of $\eta$, which is apparent from representations (A.67), we obtain $\mathbb{P}\{\eta_t > 0 \,|\, \forall\, t \in [0, T]\} = 1$. Therefore, the process $\gamma_t = \tilde{\gamma}_t \eta_t^{-1}$ is well defined and we manifestly have

$$\eta_t = 1 + \int_0^t \tilde{\gamma}_u \cdot dW_u = 1 + \int_0^t \eta_u \gamma_u \cdot dW_u.$$

We conclude that the Radon-Nikodým density process is the unique solution to the SDE (cf. formula (A.55))

$$d\eta_t = \eta_t \gamma_t \cdot dW_t, \tag{A.68}$$

and thus, in view of Lemma A.13.1, it satisfies, for any $t \in [0, T]$,

$$\eta_t = \mathcal{E}_t\Big(\int_0^{\cdot} \gamma_u \cdot dW_u\Big).$$

This completes the proof of the proposition. □

**Case of continuous semimartingales.** We end this section by stating a generalization of Girsanov's theorem to the case of continuous semimartingales. Let $\tilde{\mathbb{P}}$ and $\mathbb{P}$ be two equivalent probability measures on a common filtered probability space. We assume, in addition, that the Radon-Nikodým density process $\eta$ given by (A.59) is continuous (we know already that this holds if $\mathbb{F} = \mathbb{F}^W$ for some Brownian motion $W$).

Let us set, for any $t \in [0, T]$,

$$U_t = \int_0^t \eta_u^{-1} \, d\eta_u$$

so that $U$ is a continuous local martingale and $\eta_t = \mathcal{E}_t(U)$ for $t \in [0, T]$.

**Theorem A.15.2** *Let $\tilde{\mathbb{P}}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ and such that the Radon-Nikodým density process $\eta$ given by (A.59) is continuous. Then any continuous real-valued $\mathbb{P}$-semimartingale $X$ is a continuous $\tilde{\mathbb{P}}$-semimartingale. If the canonical decomposition of $X$ under $\mathbb{P}$ is $X = X_0 + M + A$ then its canonical decomposition under $\tilde{\mathbb{P}}$ is $X = X_0 + \tilde{M} + \tilde{A}$ where*

$$\tilde{M}_t = M_t - \int_0^t \eta_u^{-1} \, d\langle \eta, M \rangle_u = M_t - \langle U, M \rangle_t$$

*and*

$$\tilde{A}_t = A_t + \int_0^t \eta_u^{-1} d\langle \eta, M \rangle_u = A_t + \langle U, M \rangle_t.$$

*In particular, $X$ follows a local martingale under $\tilde{\mathbb{P}}$ if and only if the process $A + \langle U, M \rangle$ vanishes identically, that is, $A_t + \langle U, M \rangle_t = 0$ for every $t \in [0, T]$.*

More generally, if $\eta = \mathcal{E}(U)$, where $U$ is a local martingale under $\mathbb{P}$ (not necessarily with continuous sample paths), the last theorem remains valid under the assumption that the cross-variation $\langle U, M \rangle$ exists.

## A.16 Martingale Measures

In this text, the main application of Girsanov's theorem is related to the issue of existence and uniqueness of a martingale measure for a given process.

**Definition A.16.1** Let $X = (X_t)_{t \in [0,T]}$ be an $\mathbb{F}$-adapted process on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We say that a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ is a *martingale measure* for $X$ whenever $X$ is a local martingale under $\tilde{\mathbb{P}}$ with respect to $\mathbb{F}$.

Some authors prefer to make a clear terminological distinction between the concepts of a *martingale measure* and a *local martingale measure*. We decided to use only one term, and thus our *martingale measure* corresponds to what is sometimes called a *local martingale measure*.

Let us examine in detail the existence and uniqueness of a martingale measure for the unique solution $X$ of the SDE (cf. equation (A.42))

$$dX_t = X_t(a\,dt + b\,dW_t) \tag{A.69}$$

where $a$ and $b \neq 0$ are real numbers and $W$ is a one-dimensional standard Brownian motion. Note that $X$ is an $\mathbb{F}$-martingale under $\mathbb{P}$ whenever $a = 0$.

From Proposition A.15.1, we know that for any real number $\gamma$ the process $\tilde{W}_t = W_t - \gamma t$ is a one-dimensional standard Brownian motion on a probability space $(\Omega, \mathbb{F}, \tilde{\mathbb{P}})$, where $\tilde{\mathbb{P}}$ satisfies

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\gamma W_T - \tfrac{1}{2}\gamma^2 T\right), \quad \mathbb{P}\text{-a.s.} \tag{A.70}$$

By inserting the differential $dW_t = d\tilde{W}_t + \gamma\,dt$ into (A.69), we see that $X$ satisfies under $\tilde{\mathbb{P}}$ the following SDE

$$dX_t = X_t((a + b\gamma)\,dt + b\,d\tilde{W}_t). \tag{A.71}$$

Hence $X$ is an $\mathbb{F}$-martingale under $\tilde{\mathbb{P}}$ provided that $\gamma$ is chosen in such a way that the drift term in under (A.71) vanishes, that is, when $a + b\gamma = 0$. Of course, this means that $\gamma = -a/b$. This lead to the following result.

**Proposition A.16.1** *Let $W$ be a one-dimensional standard Brownian motion and let $\mathbb{F} = \mathbb{F}^W$ be generated by $W$. Then the unique martingale measure $\tilde{\mathbb{P}}$ for the solution $X$ to the SDE* (A.69) *is given by* (A.70) *with $\gamma = -a/b$.*

*Proof.* We have already checked that $\tilde{\mathbb{P}}$ is a martingale measure for $X$. The uniqueness can be easily deduced from Proposition A.15.3. □

The uniqueness in Proposition A.16.1 is no longer true when $W$ is a $d$-dimensional standard Brownian motion for some $d > 1$, so that $b \in \mathbb{R}^d$. By applying Theorem A.15.1 and Proposition A.15.3, we conclude that a martingale measure for $X$ corresponds to any $\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable process $\gamma$ satisfying the equation $a + b \cdot \gamma_t = 0,\ t \in [0, T]$, and such that (A.61) is satisfied for $\gamma$. It is rather clear that there are infinitely many such processes, and thus uniqueness of a martingale measure for $X$ fails to hold.

## A.17 Feynman-Kac Formula

Let us first consider a particular example of a general set-up covered by Proposition A.17.2. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded function and $g : \mathbb{R} \to \mathbb{R}$ is a function bounded from below then the unique bounded solution $v(t, x)$ of the partial differential equation (PDE)

$$\frac{\partial v}{\partial t}(t, x) = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t, x) - g(x)v(t, x) \tag{A.72}$$

with the initial condition $v(0, x) = f(x)$ can be represented by the following version of the *Feynman-Kac formula*

$$v(t, x) = \mathbb{E}_{\mathbb{P}}\Big(f(x + W_t) \exp\Big(-\int_0^t g(x + W_u)\, du\Big)\Big). \tag{A.73}$$

It is clear that this correspondence between expected values of certain functionals of a Brownian motion and solutions to partial differential equations allows us to compute some probabilistic quantities using the PDE approach and, conversely, to solve initial value problems for certain PDEs through purely probabilistic techniques (for instance, the Monte Carlo simulation of sample paths of a stochastic process).

**One-dimensional case.** Let us first examine the one-dimensional version of the Feynman-Kac formula.

**Proposition A.17.1** *Let $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition (A.51) and the linear growth condition (A.52). Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function and $g : \mathbb{R} \to \mathbb{R}$ be a function bounded from below. Assume that $v : \mathbb{R} \times [0, T] \to \mathbb{R}$ is the unique bounded solution to the PDE*

$$\frac{\partial v}{\partial t}(t, x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 v}{\partial x^2}(t, x) + \mu(x)\frac{\partial v}{\partial x}(t, x) - g(x)u(t, x) \tag{A.74}$$

*with the initial condition $v(0, x) = f(x)$. Then $v$ has the following representation*

$$v(t, x) = \mathbb{E}_{\mathbb{P}}\Big(f(X_t^x) \exp\Big(-\int_0^t g(X_u^x)\, du\Big)\Big), \tag{A.75}$$

*where $X^x$ is a solution to the SDE*

$$dX_t^x = \mu(X_t^x)\, dt + \sigma(X_t^x)\, dW_t$$

*with the initial condition $X_0^x = x$.*

*Remarks.* The *infinitesimal generator* $\mathcal{A}$ of $X$ is given by the formula

$$\mathcal{A}f(x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2}(x) + \mu(x)\frac{\partial f}{\partial x}(x).$$

Then equation (A.74) can be rewritten as follows

$$\frac{\partial v}{\partial t} = \mathcal{A}v - gv.$$

In the proof of Proposition A.17.1 we will use the following lemma.

**Lemma A.17.1** *Assume that the $\sigma$-field $\mathcal{F}_0$ is trivial. Let $M = (M_t)_{t \in [0,T]}$ be a local martingale with respect to $\mathbb{F}$ such that the random variable $M_T^* = \sup_{t \le T} |M_t|$ is integrable. Then $M$ is an $\mathbb{F}$-martingale.*

*Proof.* Let $(\tau_n)_{n\in\mathbb{N}}$ be any reducing sequence of stopping times for $M$. For any $\in \mathbb{N}$ and $t \le T$ the martingale property of the stopped process $M^{\tau_n}$ yields

$$\mathbb{E}_{\mathbb{P}}(M_{T\wedge\tau_n} \,|\, \mathcal{F}_t) = M_{t\wedge\tau_n}. \tag{A.76}$$

Since the random variable $M_T^*$ dominates $M_{T\wedge\tau_n}$ and $X_{t\wedge\tau_n}$, using the dominated convergence theorem, we may pass to the limit in (A.76) and thus we obtain, for any $t \in [0, T]$, $\mathbb{E}_{\mathbb{P}}(M_T \,|\, \mathcal{F}_t) = M_t$, which is, of course, the desired martingale property of $M$. □

*Proof of Proposition A.17.1.* The proof relies on finding a martingale $M = (M_s)_{s\in[0,t]}$ such that $\mathbb{E}_{\mathbb{P}} M_0 = v(t, x)$ and

$$\mathbb{E}_{\mathbb{P}} M_t = \mathbb{E}_{\mathbb{P}}\Big( f(X_t^x) \exp\Big( -\int_0^t g(X_u^x)\, du\Big)\Big).$$

Since for any martingale $M$ we have that $\mathbb{E}_{\mathbb{P}} M_0 = \mathbb{E}_{\mathbb{P}} M_t$, this implies (A.75). Let us fix $t > 0$ and let us define the process $M = (M_s)_{s\in[0,t]}$ by the formula

$$M_s = v(t-s, X_s^x) \exp\Big( -\int_0^s g(X_u^x)\, du\Big). \tag{A.77}$$

Itô formula, when combined with the assumption that the function $u$ solves (A.74), yield

$$dM_s = \exp\Big( -\int_0^s g(X_u^x)\, du\Big) \sigma(X_t^x) \frac{\partial u}{\partial x}(t-s, X_s^x)\, dW_s$$

and thus $M$ is a local martingale. We assumed that $u, g$ are bounded functions, so that $u(x) \le L_1$ and $g(x) \le L_2$ for some constants $L_1$ and $L_2$. From (A.77), we thus get

$$\sup_{u\le t} |M_u| \le L_1 \exp(L_2 t) < \infty,$$

so that $\mathbb{E}_{\mathbb{P}} M_t^* < \infty$. It follows from Lemma A.17.1 that $M$ is a martingale. Hence

$$\begin{aligned} v(t, x) = \mathbb{E}_{\mathbb{P}} M_0 = \mathbb{E}_{\mathbb{P}} M_t &= \mathbb{E}_{\mathbb{P}}\Big( v(0, X_t^x) \exp\Big( \int_0^t g(X_u^x)\, du\Big)\Big) \\ &= \mathbb{E}_{\mathbb{P}}\Big( f(X_t^x) \exp\Big( \int_0^t g(X_u^x)\, du\Big)\Big), \end{aligned}$$

since obviously $v(0, X_t^x) = f(X_t^x)$. We conclude that (A.75) holds. □

**Multidimensional case.** Let $W$ be a $d$-dimensional standard Brownian motion on the underlying probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We consider the $k$-dimensional diffusion process $X$ given as

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0 = x, \tag{A.78}$$

where the coefficients $\mu : \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times d}$ satisfy the Lipschitz condition (A.51) and the linear growth condition (A.52). By virtue of the multidimensional version of Theorem A.12.1, we conclude that the SDE (A.78) admits a unique solution. Then $X$ is a continuous process and it enjoys the Markov property with respect to $\mathbb{F}$.

Let the $d$-dimensional matrix $a(x) = [a_{ij}(x)]$ be equal to $\sigma(x)\sigma^t(x)$, where $\sigma^t(x)$ is the transpose of the matrix $\sigma(x)$. We associate with the Markov process $X$ its *infinitesimal generator* $\mathcal{A}$, which is given by the formula

$$\mathcal{A}f(x) = \sum_{i=1}^{k} \mu_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{k} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x),$$

for any function $f : \mathbb{R}^k \to \mathbb{R}$ belonging to a suitably defined class of functions, referred to as the domain $D(\mathcal{A})$ of the differential operator $\mathcal{A}$.

Typically, it is rather difficult to determine explicitly the domain $D(\mathcal{A})$ of the infinitesimal generator $\mathcal{A}$ of a diffusion process $X$. However, $\mathcal{A}$ is always well defined on the class of all functions with compact support (i.e., vanishing outside a bounded interval) belonging to the class $C^2(\mathbb{R}^k)$. Let us denote by $C^2_c(\mathbb{R}^k)$ the class of all such functions. Then we have $D(\mathcal{A}) \subseteq C^2_c(\mathbb{R}^k)$.

We have the following result, referred as the *Feynman-Kac formula*.

**Proposition A.17.2** *Let $g \in C(\mathbb{R}^k)$ be a function bounded from below and let $f$ belong to the class $C^2_c(\mathbb{R}^k)$. Define the function $v : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}$ by the formula*

$$v(t,x) = \mathbb{E}_{\mathbb{P}}\Big( f(X^x_t) \exp\Big( -\int_0^t g(X^x_s)\, ds \Big)\Big).$$

*Then $v$ satisfies the PDE*

$$\frac{\partial v}{\partial t} = \mathcal{A}v - gv \tag{A.79}$$

*with the initial condition $v(0,x) = f(x)$ for $x \in \mathbb{R}^k$.*

Under assumptions of Proposition A.17.2 we also have the following converse result.

**Proposition A.17.3** *If $w \in C^{1,2}(\mathbb{R} \times \mathbb{R}^k)$ is bounded on $[-K,K] \times \mathbb{R}^d$ for any $K > 0$ (equivalently, the function $w$ is bounded on $A \times \mathbb{R}^d$ for any compact subset $A \subset \mathbb{R}$) and $w$ is a solution to the PDE (A.79) with the initial condition $w(0,x) = f(x)$ for $x \in \mathbb{R}^k$ then $w(t,x) = v(t,x)$.*

## A.18 First Passage Times

In this section, we present the basic results concerning first passage times and probability distributions of some relevant functionals of a Brownian motion with drift. In this section, we find it convenient to assume that $T = \infty$, that is, we deal with a standard Brownian motion $W = (W_t)_{t \in \mathbb{R}_+}$ with respect to a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$. Let us denote by $\mathcal{F}_\infty$ the smallest $\sigma$-field containing $\mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

Recall that a standard Brownian motion $W$ has the Markov property with respect to the reference filtration $\mathbb{F}$ (see Lemma A.11), specifically, for any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ and arbitrary $t, h \geq 0$ the following equality is valid

$$\mathbb{E}_{\mathbb{P}}(g(W_{t+h}) \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(g(W_{t+h}) \,|\, W_t). \tag{A.80}$$

It appears that this property can be extended to some random times.

A random variable $\tau : (\Omega, \mathbb{F}, \mathbb{P}) \to \mathbb{R}_+ \cup \{\infty\}$ is called an $\mathbb{F}$-*stopping time* if, for every $t \in \mathbb{R}_+$, the event $\{\tau \leq t\}$ belongs to the $\sigma$-field $\mathcal{F}_t$.

For any $\mathbb{F}$-stopping time time $\tau$ we denote by $\mathcal{F}_\tau$ the $\sigma$-field of all event in $\mathcal{F}_\infty$ occurring prior to or at $\tau$. Formally, the $\sigma$-field $\mathcal{F}_\tau$ is defined by the formula

$$\mathcal{F}_\tau = \big\{ A \in \mathcal{F}_\infty \,|\, A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \forall\, t \in \mathbb{R}_+ \big\}.$$

The $\sigma$-field $\mathcal{F}_\tau$ should not be confused with the $\sigma$-field $\mathcal{F}(\tau)$ generated by the random variable $\tau$. Recall that $\mathcal{F}(\tau) = \sigma\{\{\omega \in \Omega \,|\, \tau(\omega) \leq t\} \,|\, t \in \mathbb{R}_+\}$ and thus $\mathcal{F}(\tau) \subseteq \mathcal{F}_\tau$, but usually $\mathcal{F}_\tau$ and $\mathcal{F}(\tau)$ do not coincide.

For instance, if $\mathbb{F} = \mathbb{F}^X$ for some stochastic process $X$ then $\mathcal{F}_\tau$ is the $\sigma$-field generated by the stopped process $X^\tau$, that is, $\mathcal{F}_\tau = \sigma\{X_{\tau \wedge t} \,|\, t \in \mathbb{R}_+\}$.

A Brownian motion $W$ enjoys also the following *strong Markov property* with respect to $\mathbb{F}$: for any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$, any stopping time $\tau$ such that $\mathbb{P}\{\tau < \infty\} = 1$, and any real number $t \geq 0$

$$\mathbb{E}_{\mathbb{P}}(g(W_{\tau+t}) \,|\, \mathcal{F}_\tau) = \mathbb{E}_{\mathbb{P}}(g(W_{\tau+t}) \,|\, W_\tau). \tag{A.81}$$

It can be deduced from (A.81) that a Brownian motion $W$ starts afresh after any finite stopping time $\tau$, that is, the process $\tilde{W} = (\tilde{W}_t)_{t \in \mathbb{R}_+}$ given by the equality $\tilde{W}_t = W_{\tau+t} - W_\tau$ is a Brownian motion with respect to its natural filtration $\mathbb{F}^{\tilde{W}}$.

Given a one-dimensional standard Brownian motion $W$, let us denote by $M_t^W$ and $m_t^W$ the running maximum and minimum respectively. More explicitly, we set

$$M_t^W = \sup_{u \in [0,t]} W_u, \quad m_t^W = \inf_{u \in [0,t]} W_u.$$

It is well known that we have, for any $t > 0$,

$$\mathbb{P}\{M_t^W > 0\} = 1, \quad \mathbb{P}\{m_t^W < 0\} = 1. \tag{A.82}$$

The following result, commonly referred to as the *reflection principle,* is a rather straightforward consequence of the strong Markov property of a standard Brownian motion and its symmetry (cf. Harrison (1985), Karatzas and Shreve (1998a) or Revuz and Yor (1999)).

**Lemma A.18.1** *The formula*

$$\mathbb{P}\{W_t \leq x,\ M_t^W \geq y\} = \mathbb{P}\{W_t \geq 2y - x\} = \mathbb{P}\{W_t \leq x - 2y\} \tag{A.83}$$

*is valid for every* $t > 0,\ y \geq 0$ *and* $x \leq y$.

Let the process $Y$ follow a Brownian motion with the standard deviation $\sigma$ and the drift $\nu$, specifically,

$$dY_t = \nu\, dt + \sigma\, dW_t, \quad Y_0 = y_0,$$

where $W$ is a standard one-dimensional Brownian motion under $\mathbb{P}$ with respect to $\mathbb{F}$. Put another way

$$Y_t = y_0 + \nu t + \sigma W_t \tag{A.84}$$

for some constants $\nu \in \mathbb{R}$ and $\sigma > 0$. Let us note that $Y$ inherits from a Brownian motion $W$ the strong Markov property with respect to the reference filtration $\mathbb{F}$.

**Probability distribution of the first passage time.** Let $\tau$ stand for the *first passage time to zero* by the process $Y$, that is, the $\mathbb{F}$-stopping time $\tau$ given by

$$\tau = \inf\{ t \geq 0 \,|\, Y_t = 0\}.$$

It is well known that in an arbitrarily small time interval $[0, t]$ the sample path of the Brownian motion started at 0 passes through origin infinitely many times (see, for instance, Page 42 in Krylov (1995)). Using Girsanov's theorem and the strong Markov property of the Brownian motion, it is not difficult to show that first passage time by $Y$ to zero coincides with the first crossing time by $Y$ of the level 0, that is, with probability 1,

$$\tau = \inf\{ t \geq 0 \,|\, Y_t \leq 0\} = \inf\{ t \geq 0 \,|\, Y_t < 0\}.$$

Let us first prove the following well known result.

**Lemma A.18.2** *Let $W$ be a one-dimensional standard Brownian motion under $\mathbb{P}$ and let $X_t = \nu t + \sigma W_t$, $t \in \mathbb{R}_+$, where $\sigma > 0$ and $\nu \in \mathbb{R}$. Then for every $x > 0$ and any $s > 0$ we have*

$$\mathbb{P}\Big\{ \sup_{0 \leq u \leq s} X_u \leq x \Big\} = N\left(\frac{x - \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x} N\left(\frac{-x - \nu s}{\sigma\sqrt{s}}\right) \tag{A.85}$$

*and for every $x < 0$ and any $s > 0$*

$$\mathbb{P}\Big\{ \inf_{0 \leq u \leq s} X_u \geq x \Big\} = N\left(\frac{-x + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x} N\left(\frac{x + \nu s}{\sigma\sqrt{s}}\right). \tag{A.86}$$

*Proof.* To derive the first equality, we will use Girsanov's theorem and the reflection principle for a Brownian motion. Assume first that $\sigma = 1$. Let $\mathbb{P}^*$ be the probability measure on $(\Omega, \mathcal{F}_s)$ given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\nu W_s - \frac{\nu^2}{2}s}, \quad \mathbb{P}\text{-a.s.},$$

so that the process $W_t^* = X_t = W_t + \nu t$, $t \in [0, s]$, follows a standard Brownian motion under $\mathbb{P}^*$ and

$$\frac{d\mathbb{P}}{d\mathbb{P}^*} = e^{\nu W_s^* - \frac{\nu^2}{2}s}, \quad \mathbb{P}^*\text{-a.s.}$$

Moreover

$$\mathbb{P}\{\sup_{0\le u\le s} X_u > x,\ X_s \le x\} = \mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu W_s^* - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{\sup_{0\le u\le s} W_u^* > x,\ W_s^* \le x\}}\Big).$$

Let us define an $\mathbb{F}$-stopping time $\tau_x$ by setting

$$\tau_x = \inf\{\, t \ge 0 \,|\, W_t^* = x\},$$

and let us introduce an auxiliary process $\tilde{W}_t,\ t \in [0, s]$, defined by the formula

$$\tilde{W}_t = W_t^* \mathbb{1}_{\{\tau_x \ge t\}} + (2x - W_t^*)\mathbb{1}_{\{\tau_x < t\}}.$$

By virtue of the reflection principle, the process $\tilde{W}$ also follows a standard Brownian motion under $\mathbb{P}^*$. Moreover, we have

$$\{\sup_{0\le u\le s} \tilde{W}_u > x,\ \tilde{W}_s \le x\} = \{W_s^* \ge x\} \subset \{\tau_x \le s\}.$$

Let us denote

$$J = \mathbb{P}\{\sup_{0\le u\le s} (W_u + \nu u) \le x\}.$$

Then we obtain

$$\begin{aligned}
J &= \mathbb{P}\{X_s \le x\} - \mathbb{P}\Big\{\sup_{0\le u\le s} X_u > x,\ X_s \le x\Big\} \\
&= \mathbb{P}\{X_s \le x\} - \mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu W_s^* - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{\sup_{0\le u\le s} W_u^* > x,\ W_s^* \le x\}}\Big) \\
&= \mathbb{P}\{W_s + \nu s \le x\} - \mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu \tilde{W}_s - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{\sup_{0\le u\le s} \tilde{W}_u > x,\ \tilde{W}_s \le x\}}\Big) \\
&= \mathbb{P}\{X_s \le x\} - \mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu(2x - W_s^*) - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{W_s^* \ge x\}}\Big) \\
&= \mathbb{P}\{X_s \le x\} - e^{2\nu x}\,\mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu W_s^* - \frac{\nu^2}{2}s}\,\mathbb{1}_{\{W_s^* \le -x\}}\Big) \\
&= \mathbb{P}\{X_s \le x\} - e^{2\nu x}\,\mathbb{P}\{W_s + \nu s \le -x\} \\
&= N\left(\frac{x - \nu s}{\sqrt{s}}\right) - e^{2\nu x} N\left(\frac{-x - \nu s}{\sqrt{s}}\right).
\end{aligned}$$

This ends the proof of the first equality for $\sigma = 1$. We have, for any $\sigma > 0$,

$$\mathbb{P}\Big\{\sup_{0\le u\le s} (\sigma W_u + \nu u) \le x\Big\} = \mathbb{P}\Big\{\sup_{0\le u\le s} (W_u + \nu\sigma^{-1} u) \le x\sigma^{-1}\Big\},$$

and this implies (A.85). Since $-W$ follows a standard Brownian motion under $\mathbb{P}$ we have, for any $x < 0$,

$$\mathbb{P}\Big\{\inf_{0\le u\le s} (\sigma W_u + \nu u) \ge x\Big\} = \mathbb{P}\Big\{\sup_{0\le u\le s} (\sigma W_u - \nu u) \le -x\Big\},$$

and thus the second equality is a simple consequence of the first. □

**Proposition A.18.1** *Let $Y$ be given by (A.84), where $\nu \in \mathbb{R}$, $\sigma > 0$ and $W$ is a standard Brownian motion under $\mathbb{P}$. Then the random variable $\tau$ has an inverse Gaussian probability distribution under $\mathbb{P}$, that is, for $0 < s < \infty$,*

$$\mathbb{P}\{\tau \le s\} = \mathbb{P}\{\tau < s\} = N(h_1(s)) + e^{-2\nu\sigma^{-2}y_0} N(h_2(s)), \tag{A.87}$$

*where $N$ is the standard Gaussian cumulative distribution function and*

$$h_1(s) = \frac{-y_0 - \nu s}{\sigma\sqrt{s}}, \quad h_2(s) = \frac{-y_0 + \nu s}{\sigma\sqrt{s}}.$$

*Proof.* Let us first observe that

$$\mathbb{P}\{\tau \ge s\} = \mathbb{P}\{\inf_{0 \le u \le s} Y_u \ge 0\} = \mathbb{P}\{\inf_{0 \le u \le s} X_u \ge -y_0\}, \tag{A.88}$$

where $X_u = \sigma W_u + \nu u$. We know from Lemma A.18.2 that for every $x < 0$ we have

$$\mathbb{P}\{\inf_{0 \le u \le s} X_u \ge x\} = N\left(\frac{-x + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu\sigma^{-2}x} N\left(\frac{x + \nu s}{\sigma\sqrt{s}}\right).$$

When combined with (A.88), this yields (A.87). □

The following corollary is a consequence of Proposition A.18.1 and the strong Markov property of the process $Y$ with respect to the filtration $\mathbb{F}$.

**Corollary A.18.1** *Under the assumptions of Proposition A.18.1 we have, for any $t < s$ on the event $\{\tau > t\}$,*

$$\mathbb{P}\{\tau \le s \,|\, \mathcal{F}_t\} = N\left(\frac{-Y_t - \nu(s-t)}{\sigma\sqrt{s-t}}\right) + e^{-2\nu\sigma^{-2}Y_t} N\left(\frac{-Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right).$$

**Joint probability distribution of $Y$ and $\tau$.** In the next step, we will examine the joint distribution of the process $Y$ and its first passage time $\tau$. For the process $X_t = \nu t + \sigma W_t$, we write

$$M_s^X = \sup_{u \in [0,s]} X_u, \quad m_s^X = \inf_{u \in [0,s]} X_u.$$

By the Girsanov theorem, the process $\sigma^{-1}X$ is a Brownian motion under an equivalent probability measure and thus we can deduce that, for any $s > 0$ (cf. (A.83))

$$\mathbb{P}\{M_s^X > 0\} = 1, \quad \mathbb{P}\{m_s^X < 0\} = 1.$$

**Lemma A.18.3** *For any $s > 0$, the joint distribution of $X_s$ and $M_s^X$ is given by the formula*

$$\mathbb{P}\{X_s \le x,\ M_s^X \ge y\} = e^{2\nu y\sigma^{-2}}\,\mathbb{P}\{X_s \ge 2y - x + 2\nu s\}, \tag{A.89}$$

*for every $x, y \in \mathbb{R}$ such that $y \ge 0$ and $x \le y$.*

*Proof.* Since

$$I = \mathbb{P}\{X_s \le x,\ M_s^X \ge y\} = \mathbb{P}\{X_s^\sigma \le x\sigma^{-1},\ M_s^{X^\sigma} \ge y\sigma^{-1}\},$$

where $X_t^\sigma = W_t + \nu t\sigma^{-1}$, it is clear that we may and do assume, without loss of generality, that $\sigma = 1$.

We shall use similar arguments as in the proof of Lemma A.18.2. It follows from Girsanov's theorem that $X$ is a standard Brownian motion under the probability measure $\mathbb{P}^*$ given on $(\Omega, \mathcal{F}_s)$ by the formula

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\nu W_s - \frac{\nu^2}{2}s}, \quad \mathbb{P}\text{-a.s.}$$

Therefore

$$\frac{d\mathbb{P}}{d\mathbb{P}^*} = e^{\nu W_s^* - \frac{\nu^2}{2}s}, \quad \mathbb{P}\text{-a.s.},$$

where $W_t^* = X_t = W_t + \nu t,\ t \in [0, s]$, is a standard Brownian motion under $\mathbb{P}^*$. It is thus clear that

$$I = \mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu W_s^* - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{X_s \le x,\ M_s^X \ge y\}}\Big) = \mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu W_s^* - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W_s^* \le x,\ M_s^{W^*} \ge y\}}\Big).$$

Since $W^*$ is a standard Brownian motion under $\mathbb{P}^*$, an application of the reflection principle yields

$$\begin{aligned} I &= \mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu(2y - W_s^*) - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{2y - W_s^* \le x,\ M_s^{W^*} \ge y\}}\Big) \\ &= \mathbb{E}_{\mathbb{P}^*}\Big(e^{\nu(2y - W_s^*) - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W_s^* \ge 2y - x\}}\Big) \\ &= e^{2\nu y}\, \mathbb{E}_{\mathbb{P}^*}\Big(e^{-\nu W_s^* - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W_s^* \ge 2y - x\}}\Big), \end{aligned}$$

since under the present assumptions we have $2y - x \ge y$. Let us define an equivalent probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_s)$ by setting

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} = e^{-\nu W_s^* - \frac{\nu^2}{2}s}, \quad \mathbb{P}^*\text{-a.s.}$$

Then we have that

$$I = e^{2\nu y}\, \mathbb{E}_{\mathbb{P}^*}\Big(e^{-\nu W_s^* - \frac{\nu^2}{2}s}\, \mathbb{1}_{\{W_s^* \ge 2y - x\}}\Big) = e^{2\nu y}\, \tilde{\mathbb{P}}\{W_s^* \ge 2y - x\}.$$

The process $\tilde{W}_t = W_t^* + \nu t,\ t \in [0, s]$, is a standard Brownian motion under $\tilde{\mathbb{P}}$ and we have

$$I = e^{2\nu y}\, \tilde{\mathbb{P}}\{\tilde{W}_s + \nu s \ge 2y - x + 2\nu s\}.$$

The asserted formula (A.89) follows easily from the last equality. □

Let us observe that (a similar remark applies to any formula given below)

$$\mathbb{P}\{X_s \leq x,\ M_s^X \geq y\} = \mathbb{P}\{X_s < x,\ M_s^X > y\}.$$

The following result is a straightforward consequence of Lemma A.18.3.

**Proposition A.18.2** *We have, for any $x, y \in \mathbb{R}$ satisfying $y \geq 0$ and $x \leq y$,*

$$\mathbb{P}\{X_s \leq x,\ M_s^X \geq y\} = e^{2\nu y \sigma^{-2}}\, N\left(\frac{x - 2y - \nu s}{\sigma\sqrt{s}}\right). \tag{A.90}$$

*Hence for every $x, y \in \mathbb{R}$ such that $x \leq y$ and $y \geq 0$*

$$\mathbb{P}\{X_s \leq x,\ M_s^X \leq y\} = N\left(\frac{x - \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}}\, N\left(\frac{x - 2y - \nu s}{\sigma\sqrt{s}}\right). \tag{A.91}$$

*Proof.* For the first equality, note that

$$\mathbb{P}\{X_s \geq 2y - x + 2\nu s\} = \mathbb{P}\{-\sigma W_s \leq x - 2y - \nu s\} = N\left(\frac{x - 2y - \nu s}{\sigma\sqrt{s}}\right),$$

since $-\sigma W_t$ has Gaussian distribution with zero mean and variance $\sigma^2 t$. To establish (A.91), it is enough to observe that

$$\mathbb{P}\{X_s \leq x,\ M_s^X \leq y\} + \mathbb{P}\{X_s \leq x,\ M_s^X > y\} = \mathbb{P}\{X_s \leq x\}$$

and to apply (A.90). □

It is clear that, for every $y \geq 0$,

$$\mathbb{P}\{M_s^X \geq y\} = \mathbb{P}\{X_s \geq y\} + \mathbb{P}\{X_s \leq y,\ M_s^X \geq y\}$$

and thus

$$\mathbb{P}\{M_s^X \geq y\} = \mathbb{P}\{X_s \geq y\} + e^{2\nu y \sigma^{-2}}\, \mathbb{P}\{X_s \geq y + 2\nu s\}. \tag{A.92}$$

Hence

$$\mathbb{P}\{M_s^X \leq y\} = 1 - \mathbb{P}\{M_s^X \geq y\} = \mathbb{P}\{X_s \leq y\} - e^{2\nu y \sigma^{-2}}\, \mathbb{P}\{X_s \geq y + 2\nu s\}.$$

This leads to the following corollary.

**Corollary A.18.2** *The following formula holds for every $y \geq 0$*

$$\mathbb{P}\{M_s^X \leq y\} = N\left(\frac{y - \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}}\, N\left(\frac{-y - \nu s}{\sigma\sqrt{s}}\right). \tag{A.93}$$

Let us now focus on the distribution of the infimum of $X$. Observe that we have, for any $y \leq 0$,

$$\mathbb{P}\{\sup_{0 \leq u \leq s} (\sigma W_u - \nu u) \geq -y\} = \mathbb{P}\{\inf_{0 \leq u \leq s} (-\sigma W_u + \nu u) \leq y\} = \mathbb{P}\{\inf_{0 \leq u \leq s} X_u \leq y\},$$

where the last equality follows from the symmetry of the Brownian motion. Consequently we have, for every $y \le 0$,

$$\mathbb{P}\{m_s^X \le y\} = \mathbb{P}\{M_s^{\tilde{X}} \ge -y\},$$

where the process $\tilde{X}$ equals $\tilde{X}_t = \sigma W_t - \nu t$. The following result is thus not difficult to prove.

**Proposition A.18.3** *For every $s > 0$, the joint distribution of $(X_s, m_s^X)$ satisfies, for any $x, y \in \mathbb{R}$ such that $y \le 0$ and $y \le x$,*

$$\mathbb{P}\{\, X_s \ge x,\ m_s^X \ge y\} = N\left(\frac{-x + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}} N\left(\frac{2y - x + \nu s}{\sigma\sqrt{s}}\right).$$

**Corollary A.18.3** *The following formula is valid for every $y \le 0$*

$$\mathbb{P}\{m_s^X \ge y\} = N\left(\frac{-y + \nu s}{\sigma\sqrt{s}}\right) - e^{2\nu y \sigma^{-2}} N\left(\frac{y + \nu s}{\sigma\sqrt{s}}\right).$$

Recall that $Y_t = y_0 + X_t$, where $X_t = \nu t + \sigma W_t$. We write

$$m_s^X = \inf_{u \in [0,s]} X_u, \quad m_s^Y = \inf_{u \in [0,s]} Y_u.$$

**Corollary A.18.4** *For any $s > 0$ and $y \ge 0$ we have*

$$\mathbb{P}\{Y_s \ge y,\ \tau \ge s\} = N\left(\frac{-y + y_0 + \nu s}{\sigma\sqrt{s}}\right) - e^{-2\nu\sigma^{-2} y_0} N\left(\frac{-y - y_0 + \nu s}{\sigma\sqrt{s}}\right).$$

*Proof.* Since

$$\mathbb{P}\{Y_s \ge y,\ \tau \ge s\} = \mathbb{P}\{\, Y_s \ge y,\ m_s^Y \ge 0\} = \mathbb{P}\{\, X_s \ge y - y_0,\ m_s^X \ge -y_0\},$$

the formula is rather obvious. □

More generally, by applying the Markov property and time-homogeneity of $Y$ we obtain the following result.

**Lemma A.18.4** *Under the assumptions of Lemma A.18.1 we have, for any $t < s$ and $y \ge 0$ on the event $\{\tau > t\}$,*

$$\mathbb{P}\{Y_s \ge y,\ \tau \ge s \,|\, \mathcal{F}_t\} = N\left(\frac{-y + Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right) - e^{-2\nu\sigma^{-2} Y_t} N\left(\frac{-y - Y_t + \nu(s-t)}{\sigma\sqrt{s-t}}\right).$$

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