Appendix A
Some Auxiliary Results

A.1 Introduction

This chapter contains some auxiliary results which are also of independent interest, and some of them seem to be new.

In Sect. A.2, we find the asymptotic behavior of a series related to $Q(\varepsilon)$ from (11.1). In fact, the main result of Sect. A.2 describes the asymptotic behavior of the series corresponding to the stable limit law. Note that this result is valid even for a wider class of distributions.

In Sect. A.3, we provide some large deviation bounds for distributions attracted to a stable law. Known results of this type assert that $\mathsf{P}(|S_n| \geq x_n b_n)$ and $n\,\mathsf{P}(|X| \geq x_n b_n)$ have the same asymptotic behavior as $n \to \infty$, where $\{b_n\}_{n\geq 1}$ is the normalizing sequence in the weak convergence to a stable limit distribution and where $\{x_n\}_{n\geq 1}$ is an arbitrary sequence tending to infinity, that is, $x_n \to \infty$ as $n \to \infty$. Our bounds hold for all $x \geq 0$. Note also that our estimates are uniform in x, but we do not prove an asymptotic equivalence for the probabilities of large deviations.

In Sect. A.4, we discuss some key properties of slowly varying functions used in the proof of the main result.

Some necessary properties of stable distributions and of distributions attracted to stable laws are collected in Sect. A.5. We prove a known result about the normalizing and centering sequences in the weak convergence to the stable limit law, which is convenient in many particular cases. Another result of this section provides the existence of negative moments for stable distributions.

In Sect. A.6, we state a result of Aljančić et al. [9] on the asymptotic behavior of integrals with slowly varying functions.

Finally, in Sect. A.7, a new proof of a theorem of Parameswaran [290] is given, but for sums here instead of integrals.

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A.2 The Asymptotic Behavior of Series of Weighted Tails of Distributions

Given a random variable Z, a number $\varepsilon > 0$, and functions $w \in \mathcal{RV}_r$ and $\psi \in \mathcal{RV}_\gamma$, put

$$\mathcal{Q}(\varepsilon) = \sum_{k=1}^{\infty} w_k \, \mathsf{P}(|Z| \geq \varepsilon \psi_k), \tag{A.1}$$

where $w_k = w(k)$ and $\psi_k = \psi(k)$.

Lemma A.1 *Let $w \in \mathcal{RV}_r$, $r \geq -1$, and let the series $\sum w_k$ diverge. Let W be as defined in (11.12). Consider a positive, continuous, increasing function ψ such that $\psi \in \mathcal{RV}_\gamma$, for $\gamma > 0$. Assume that Z is a random variable such that*

$$\mathsf{E}\,|Z|^\eta < \infty, \quad \textit{for some} \quad \eta > \frac{r+1}{\gamma}. \tag{A.2}$$

Let f be a density of the distribution of $|Z|$. Then

$$\mathcal{Q}(\varepsilon) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{A.3}$$

In addition, if

$$\mathsf{E}\,|Z|^\eta < \infty, \quad \textit{for some} \quad \eta < \frac{r+1}{\gamma}, \tag{A.4}$$

then

$$\lim_{\varepsilon \downarrow 0} \frac{\mathcal{Q}(\varepsilon)}{U\,(1/\varepsilon)} = \mathsf{E}\,|Z|^{(r+1)/\gamma}, \tag{A.5}$$

where the function U is as defined in (11.18) and where ψ^{-1} is the inverse function of ψ.

Note that conditions (A.2) and (A.4) are equivalent if $r \neq -1$. On the other hand, they are different if $r = -1$, since each of them describes a different behavior of the distribution function of the random variable Z, namely, condition (A.2) means that a certain positive moment exists, while condition (A.4) says that a negative moment exists.

Remark A.2 Lemma A.1 is used in the proof of Theorem 11.1 for an α-stable random variable Z_α and for $\psi(x) = \varphi(x)/b(x)$, where $\varphi \in \mathcal{RV}_{1/p}$ appears in the large deviation probabilities in the series of (11.1) and where $b \in \mathcal{RV}_{1/\alpha}$ provides the normalizations in the attraction of the sums S_n to the random variable Z_α.

Proof of Lemma A.1 If η is such that condition (A.2) holds, then Chebyshev's inequality implies

$$\mathcal{Q}(\varepsilon) \le \frac{\mathsf{E}\,|Z|^{\eta}}{\varepsilon^{\eta}} \sum_{k=1}^{\infty} \frac{w_k}{\psi_k^{\eta}},$$

whence (A.3) follows, since $w\psi^{-\eta} \in \mathcal{RV}_{r-\eta\gamma}$ and $r - \eta\gamma < -1$ in view of condition (A.2) (see Lemma A.9).

To prove (A.5), we represent the sum as

$$\sum_{k=1}^{\infty} w_k \,\mathsf{P}(|Z| \ge \varepsilon\psi_k) = \sum_{k=1}^{\infty} w_k \sum_{j=k}^{\infty} \int_{I_j(\varepsilon)} f(t)\,dt,$$

where $I_j(\varepsilon) = [\varepsilon\psi_j, \varepsilon\psi_{j+1})$. Changing the order of summation we get

$$\begin{aligned}
\sum_{k=1}^{\infty} w_k \,\mathsf{P}(|Z| \ge \varepsilon\psi_k) &= \sum_{j=1}^{\infty} W(j) \int_{I_j(\varepsilon)} f(t)\,dt \\
&= \sum_{j=1}^{\infty} \int_{I_j(\varepsilon)} W\left(\psi^{-1}(t/\varepsilon)\right) f(t)\,dt \\
&\quad + \sum_{j=1}^{\infty} \int_{I_j(\varepsilon)} \left[W(j) - W\left(\psi^{-1}(t/\varepsilon)\right)\right] f(t)\,dt \\
&\equiv \mathcal{Q}_1(\varepsilon) + \mathcal{Q}_2(\varepsilon).
\end{aligned}$$

Now we show that

$$\mathcal{Q}_1(\varepsilon) \sim U(1/\varepsilon)\,\mathsf{E}\,|Z|^{(r+1)/\gamma} \qquad \text{as } \varepsilon \downarrow 0, \tag{A.6}$$

$$Q_2(\varepsilon) = o\left(U(1/\varepsilon)\right) \qquad \text{as } \varepsilon \downarrow 0. \tag{A.7}$$

In the proof below, we apply Theorem A.18 (i) and A.19 (i) together with conditions (A.2) and (A.4). To prove (A.6) note that

$$\mathcal{Q}_1(\varepsilon) = \int_{\varepsilon\psi_1}^{\infty} U(t/\varepsilon)\, f(t)\,dt.$$

The rest of the proof differs for the cases $\psi_1 = 0$ and $\psi_1 > 0$.

Case $\psi_1 = 0$ Fix $B > 0$. Then

$$\mathcal{Q}_1(\varepsilon) = \int_0^B U(t/\varepsilon)\, f(t)\,dt + \int_B^{\infty} U(t/\varepsilon)\, f(t)\,dt.$$

The asymptotic behavior of the terms on the right-hand side above has been obtained in Theorems A.18 (i) and A.19 (i), with $x = 1/\varepsilon$ and $\nu = (r+1)/\gamma$. Thus (A.6) follows from conditions (A.2) and (A.4), i.e.

$$\int_0^B U(t/\varepsilon)\, f(t)\, dt \sim U(1/\varepsilon) \int_0^B t^{(r+1)/\gamma} f(t)\, dt, \qquad \varepsilon \downarrow 0,$$
$$\int_B^\infty U(t/\varepsilon)\, f(t)\, dt \sim U(1/\varepsilon) \int_B^\infty t^{(r+1)/\gamma} f(t)\, dt, \qquad \varepsilon \downarrow 0.$$

Case $\psi_1 > 0$ Fix $A > 0$. Then

$$\int_A^\infty U(t/\varepsilon)\, f(t)\, dt \le \int_{\varepsilon\psi_1}^\infty U(t/\varepsilon)\, f(t)\, dt,$$

for $0 < \varepsilon < A/\psi_1$. Thus

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \int_{\varepsilon\psi_1}^\infty U(t/\varepsilon)\, f(t)\, dt \\
&\qquad \ge \lim_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \int_A^\infty U(t/\varepsilon)\, f(t)\, dt = \int_A^\infty t^{(r+1)/\gamma} f(t)\, dt,
\end{aligned}$$

in view of condition (A.2) and Theorem A.18 (i), with $x = 1/\varepsilon$. Since $A > 0$ is arbitrary, we conclude that

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \int_{\varepsilon\psi_1}^\infty U(t/\varepsilon)\, f(t)\, dt \ge \int_0^\infty t^{(r+1)/\gamma} f(t)\, dt = \mathsf{E}\,|Z|^{(r+1)/\gamma}. \tag{A.8}$$

For any given $B > 0$,

$$\int_{\varepsilon\psi_1}^\infty U(t/\varepsilon)\, f(t)\, dt \le \int_0^B U(t/\varepsilon)\, f(t)\, dt + \int_B^\infty U(t/\varepsilon)\, f(t)\, dt,$$

whence

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \int_{\varepsilon\psi_1}^\infty U(t/\varepsilon)\, f(t)\, dt \\
&\quad \le \lim_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \int_0^B U(t/\varepsilon)\, f(t)\, dt \\
&\qquad + \lim_{\varepsilon\downarrow 0} \frac{1}{U(1/\varepsilon)} \int_B^\infty U(t/\varepsilon)\, f(t)\, dt \\
&\quad = \int_0^B t^{(r+1)/\gamma} f(t)\, dt + \int_B^\infty t^{(r+1)/\gamma} f(t)\, dt = \mathsf{E}\,|Z|^{(r+1)/\gamma}.
\end{aligned} \tag{A.9}$$

Thus (A.6) follows from (A.8) and (A.9) in the case when $\psi_1 > 0$.

For the proof of (A.7), we put $u(x) = w\left(\psi^{-1}(x)\right)$. Note that $u \in \mathcal{RV}_{r/\gamma}$. Then

$$|\mathcal{Q}_2(\varepsilon)| \le \sum_{j=1}^{\infty} (W(j+1) - W(j)) \int_{I_j(\varepsilon)} f(t)\,dt \le \sum_{j=1}^{\infty} w(j+1) \int_{I_j(\varepsilon)} f(t)\,dt,$$

since the function $W \circ \psi^{-1}$ is nondecreasing. According to Theorem A.7, there exists a constant $C > 0$ such that

$$|\mathcal{Q}_2(\varepsilon)| \le C \sum_{j=1}^{\infty} \int_{I_j(\varepsilon)} u\,(t/\varepsilon)\, f(t)\,dt = C \int_{\varepsilon\psi_1}^{\infty} u\,(t/\varepsilon)\, f(t)\,dt.$$

By the same reasoning as above, with u instead of U, we obtain

$$\int_{\varepsilon\psi_1}^{\infty} u\,(t/\varepsilon)\, f(t)\,dt \sim u\,(1/\varepsilon)\,\mathsf{E}\,|Z|^{(r+1)/\gamma} \qquad \varepsilon \downarrow 0,$$

whence (A.7) results from Lemma A.20. Now equality (A.5) follows from relations (A.6) and (A.7). □

Remark A.3 The condition $\mathsf{E}\,U(|Z|) < \infty$ is, in fact, necessary and sufficient for the validity of $Q(\varepsilon) < \infty$, for all $\varepsilon > 0$. If ψ is not necessarily increasing, one may choose an increasing function ψ_1 such that $\psi_1 \sim \psi$, and then reformulate the necessary and sufficient convergence condition in terms of this function, i.e. $\mathsf{E}\,W\left(\psi_1^{-1}(|Z|)\right) < \infty$.

A.3 Large Deviations in the Case of Attraction to a Stable Law

In this section, we prove a result on the probabilities of large deviations for sums of independent, identically distributed random variables, which are attracted to a stable distribution of index $0 < \alpha < 2$.

Our uniform estimate given by (A.12) differs from the usual form of a large deviation result for heavy tailed distributions. One of the popular ways to express this property is

$$\lim_{n\to\infty} \sup_{x \ge t_n} \frac{\mathsf{P}(|S_n| \ge x)}{n\,\mathsf{P}(|X| \ge x)} = 1, \tag{A.10}$$

where $\{t_n\}_{n\ge 1}$ is a sequence such that

$$\frac{S_n}{t_n} \xrightarrow{\mathsf{P}} 0 \tag{A.11}$$

(see Remark A.6 below). Clearly (A.12) below provides a bound for the whole axis, while (A.10) describes precise asymptotics for large arguments (see also Remark A.6 below).

Throughout this section we denote by $\{b_n\}_{n\ge 1}$ the normalizing sequence in the weak convergence to the stable limit law.

Proposition A.4 *Assume that* $0 < \alpha < 2$. *Let* X, $\{X_n\}_{n\ge 1}$ *be independent, identically distributed random variables such that* $X \in \mathcal{DA}\,(\alpha, \{b_n\}, \{a_n\})$. *If* $\alpha = 1$, *then we additionally assume that* X *has a symmetric distribution. We also assume that* $\mathsf{E}\, X = 0$, *if* $\alpha > 1$. *Then*

$$\sup_{x\ge 0}\sup_{n\ge 1} \frac{\mathsf{P}(|S_n| \ge x b_n)}{n\,\mathsf{P}(|X| \ge x b_n)} < \infty. \tag{A.12}$$

One can drop the assumption of symmetry of the distribution in the case when $\alpha = 1$, if S_n in (A.12) is replaced by $S_n - a_n b_n$.

Corollary A.5 *Let* $\alpha = 1$ *and* $X \in \mathcal{DA}\,(1, \{b_n\}, \{a_n\})$. *Then*

$$\sup_{x\ge 0}\ \sup_{n\ge 1} \frac{\mathsf{P}(|S_n - a_n b_n| \ge x b_n)}{n\,\mathsf{P}(|X| \ge x b_n)} < \infty.$$

Remark A.6 The theory of large deviations is a huge area and many authors have studied this topic in detail. One of the popular ways to express a large deviation property is written in (A.10). To get an impression of why this kind of large deviation principle does not imply (A.12) we note that, for any $m \ge 1$,

$$\begin{aligned}\sup_{x\ge 0}\sup_{n\ge 1} \frac{\mathsf{P}(|S_n| \ge x b_n)}{n\,\mathsf{P}(|X| \ge x b_n)} &\ge \sup_{x\ge 0} \frac{\mathsf{P}(|S_m| \ge x b_m)}{m\,\mathsf{P}(|X| \ge x b_m)} = \sup_{x\ge 0} \frac{\mathsf{P}(|S_m| \ge x)}{m\,\mathsf{P}(|X| \ge x)} \\ &= \max\left\{ \sup_{x\le t_m} \frac{\mathsf{P}(|S_m| \ge x)}{m\,\mathsf{P}(|X| \ge x)},\ \sup_{x\ge t_m} \frac{\mathsf{P}(|S_m| \ge x)}{m\,\mathsf{P}(|X| \ge x)} \right\}.\end{aligned}$$

The second term on the right-hand side is bounded in m in view of (A.10). To prove that the first term on the right-hand side is bounded in m one can try to proceed as follows:

$$\sup_{x\le t_m} \frac{\mathsf{P}(|S_m| \ge x)}{m\,\mathsf{P}(|X| \ge x)} \le \frac{1}{m\,\mathsf{P}(|X| \ge t_m)}.$$

Unfortunately $m\,\mathsf{P}(|X| \ge t_m) \to 0$ as $m \to \infty$ in view of the necessary condition for the weak law of large numbers (A.11) and thus the right-hand side is unbounded. Therefore there is no simple way to extract (A.12) from (A.10).

Proof of Proposition A.4 We shall make use of some ideas of Heyde [178], where a similar result has been obtained in the case of attraction to a nondegenerate stable distribution of index α, if $\alpha \ne 1$ and $\alpha \ne 2$.

Let $b_0 = \inf_{n\geq 1} b_n$. Without loss of generality assume that $b_0 > 0$. Fix $x_0 > 0$. Then, according to (A.30),

$$\sup_{0\leq x\leq x_0}\sup_{n\geq 1}\frac{\mathsf{P}(|S_n|\geq xb_n)}{n\,\mathsf{P}(|X|\geq xb_n)}\leq \sup_{n\geq 1}\frac{1}{n\,\mathsf{P}(|X|\geq x_0 b_n)}<\infty.$$

This means that one can restrict the consideration to the case when $x \geq x_0$. The exact value of x_0 will be chosen below.

For the sake of simplicity, we may assume that $a_n = 0$, $n \geq 1$. In fact, by (11.13), a_n is in general involved in the attraction to Z_α, but Lemma A.16 allows us to switch to the case of $a_n = 0$, $n \geq 1$, if $\alpha \neq 1$. For $\alpha = 1$, we can do so as well, since the random variable X has a symmetric distribution.

Let $1/2 < \gamma < 1$. Fix $x \geq x_0$ and put $z = x^\gamma$. To choose x_0, let $\eta_1 > 0$, $\eta_2 > 0$, and $\eta_3 > 0$ be such that $(1-\gamma)\eta_1 + \gamma\eta_2 < (2\gamma - 1)\alpha$ and $\eta_3 < 2 - \alpha$. According to Lemma A.10, with $\ell = g$, where the function g is as defined in (A.26), there are numbers x_{1*}, x_{2*}, x_{3*} for which (A.20) is valid, with $\eta = \eta_1, \eta_2, \eta_3$, respectively, and with some arbitrary $D > 1$. Now, choose $x_0 = \frac{1}{b_0}\max\{x_{1*}^{1/\gamma}, x_{2*}^{1/\gamma}, x_{3*}^{1/\gamma}\}$. Then $zb_n \geq \max\{x_{1*}, x_{2*}, x_{3*}\}$, for all $n \geq 1$, and one may assume that $b_n \geq \max\{x_{1*}, x_{2*}, x_{3*}\}$, for all $n \geq 1$.

Fix $n \geq 1$ and put

$$X_{kn} = X_k I(|X_k| < zb_n), \qquad S_{nn} = \sum_{k\leq n} X_{kn},$$

for $1 \leq k \leq n$. Similar to Heyde [178], we have the following inclusion of random events, i.e.

$$\begin{aligned}\{|S_{nn}|\geq xb_n\} \subseteq &\left\{\exists k\leq n\colon |X_k|\geq \frac{xb_n}{2}\right\}\\ &\cup\left\{\exists k_1<k_2\leq n\colon |X_{k_1}|\geq zb_n, |X_{k_2}|\geq zb_n\right\}\\ &\cup\left\{|S_{nn}|\geq \frac{xb_n}{2}\right\},\end{aligned}$$

whence

$$\begin{aligned}\mathsf{P}(|S_n|\geq xb_n)\leq n\,\mathsf{P}\left(|X|\geq \frac{xb_n}{2}\right)+n^2\left[\mathsf{P}(|X|\geq zb_n)\right]^2\\ +\mathsf{P}\left(|S_{nn}|\geq\frac{xb_n}{2}\right).\end{aligned} \tag{A.13}$$

Each term on the right-hand side of (A.13) will be considered separately.

Step 1 Let $b = \min_{n \ge 1} b_n$. Then (A.26) implies that

$$\frac{\mathsf{P}(|X| \ge x b_n/2)}{\mathsf{P}(|X| \ge x b_n)} = 2^\alpha \frac{g(x b_n/2)}{g(x b_n)} \le 2^\alpha \sup_{t \ge b} \frac{g(t/2)}{g(t)} < \infty,$$

by Theorem A.7, since g is a slowly varying function. Thus

$$\sup_{x \ge 1} \sup_{n \ge 1} \frac{\mathsf{P}(|X| \ge x b_n/2)}{\mathsf{P}(|X| \ge x b_n)} < \infty.$$

Step 2 We use again (A.26) and conclude that

$$\frac{n\left[\mathsf{P}(|X| \ge z b_n)\right]^2}{\mathsf{P}(|X| \ge x b_n)} = n \cdot \frac{[g(z b_n)]^2}{(z b_n)^{2\alpha}} \cdot \frac{(x b_n)^\alpha}{g(x b_n)} = \frac{n g(b_n)}{b_n^\alpha} \cdot \frac{[g(z b_n)]^2}{g(b_n) g(x b_n)} \cdot \frac{x^\alpha}{z^{2\alpha}}. \tag{A.14}$$

By Lemma A.15 (iii), the term $n g(b_n)/b_n^\alpha$ is bounded. For the second factor on the right-hand side of (A.14), we apply Lemma A.10, with $\eta = \eta_1$ and then with $\eta = \eta_2$, and get

$$\frac{g(z b_n)}{g(x b_n)} \le D \left(\frac{x}{z}\right)^{\eta_1}, \qquad \frac{g(z b_n)}{g(b_n)} \le D z^{\eta_2}.$$

Therefore,

$$\frac{n\left[\mathsf{P}(|X| \ge z b_n)\right]^2}{\mathsf{P}(|X| \ge x b_n)} \le \text{const} \frac{z^{-\eta_1+\eta_2-2\alpha}}{x^{-\eta_1-\alpha}} = \text{const } x^{\gamma(-\eta_1+\eta_2-2\alpha)+\alpha+\eta_1}.$$

Since $\gamma(-\eta_1 + \eta_2 - 2\alpha) + \alpha + \eta_1 < 0$, we conclude that

$$\sup_{x \ge 1} \sup_{n \ge 1} \frac{n\left[\mathsf{P}(|X| \ge z b_n)\right]^2}{\mathsf{P}(|X| \ge x b_n)} < \infty. \tag{A.15}$$

Step 3 Consider the third term on the right-hand side of (A.13). By Chebyshev's inequality,

$$\mathsf{P}\left(|S_{nn}| \ge \frac{x b_n}{2}\right) \le \frac{4}{x^2 b_n^2} \left[n \,\mathsf{E}\, X_{1n}^2 + n^2 \left(\mathsf{E}\, X_{1n}\right)^2\right]. \tag{A.16}$$

On integrating by parts we have $\mathsf{E}\, X_{1n}^2 \le 2 \int_0^{z b_n} x \,\mathsf{P}(|X| \ge x)\, dx$. Thus

$$\frac{\mathsf{E}\, X_{1n}^2}{x^2 b_n^2 \,\mathsf{P}(|X| \ge x b_n)} \le 2 \frac{\int_0^{z b_n} x \,\mathsf{P}(|X| \ge x)\, dx}{z^2 b_n^2 \,\mathsf{P}(|X| \ge z b_n)} \frac{\mathsf{P}(|X| \ge z b_n)}{\mathsf{P}(|X| \ge x b_n)} \left(\frac{z}{x}\right)^2. \tag{A.17}$$

The first factor is bounded in view of Lemma A.11, with $\lambda = \alpha - 1$, since $\mathsf{P}(|X| \geq x)$ is a regularly varying function (see (A.26)) and $\alpha < 2$.

The other factors on the right-hand side of (A.17) can be estimated as follows:

$$\frac{\mathsf{P}(|X| \geq zb_n)}{\mathsf{P}(|X| \geq xb_n)} \cdot \left(\frac{z}{x}\right)^2 = \frac{g(zb_n)}{g(xb_n)} \cdot \left(\frac{z}{x}\right)^{2-\alpha} \leq D \left(\frac{z}{x}\right)^{2-\alpha-\eta_3}.$$

Hence

$$\sup_{x \geq 1} \sup_{n \geq 1} \frac{\mathsf{E}\, X_{1n}^2}{x^2 b_n^2\, \mathsf{P}(|X| \geq xb_n)} < \infty.$$

Next we consider the term $n^2\,(\mathsf{E}\, X_{1n})^2$ in (A.16).

Case $0 < \alpha < 1$ On integrating by parts, we obtain

$$|\,\mathsf{E}\, X_{1n}| \leq E|X_{1n}| \leq \int_0^{zb_n} \mathsf{P}(|X_1| \geq x)\, dx.$$

Thus (A.26) and Lemma A.11, with $\lambda = \alpha$, imply that

$$\begin{aligned} \frac{n\,(\mathsf{E}\, X_{1n})^2}{x^2 b_n^2\, \mathsf{P}(|X_1| \geq xb_n)} &\leq \frac{n}{x^2 b_n^2\, \mathsf{P}(|X_1| \geq xb_n)} \left[\int_0^{zb_n} \mathsf{P}(|X_1| \geq x)\, dx \right]^2 \\ &\leq \text{const}\, \frac{n}{x^2 b_n^2\, \mathsf{P}(|X_1| \geq xb_n)} \left[zb_n\, \mathsf{P}(|X_1| \geq zb_n \right]^2 \\ &= \text{const}\, \frac{n \left[\mathsf{P}(|X_1| \geq zb_n \right]^2}{\mathsf{P}(|X_1| \geq xb_n)} \cdot \left(\frac{z}{x}\right)^2. \end{aligned}$$

The first factor is bounded in view of (A.15). Since $z \leq x$, we get

$$\sup_{x \geq 1} \sup_{n \geq 1} \frac{n\,(\mathsf{E}\, X_{1n})^2}{x^2 b_n\, \mathsf{P}(|X| \geq xb_n)} < \infty. \tag{A.18}$$

Case $\alpha = 1$ Here, (A.18) is obvious, since the distribution of the random variable X is symmetric.

Case $1 < \alpha < 2$ Recall that $\mathsf{E}\, X = 0$. An integration by parts yields

$$|\,\mathsf{E}\, X_{1n}| = \int_{zb_n}^{\infty} \mathsf{P}(|X| \geq x)\, dx + zb_n (\mathsf{P}(|X| \geq zb_n).$$

Thus

$$\frac{n\,(\mathsf{E}\, X_{1n})^2}{x^2 b_n\, \mathsf{P}(|X| \geq xb_n)} = \left[\frac{\int_{zb_n}^{\infty} \mathsf{P}(|X| \geq x)\, dx}{zb_n\, \mathsf{P}(|X| \geq zb_n)} + 1 \right]^2 \cdot \frac{n \left[\mathsf{P}(|X| \geq zb_n) \right]^2}{\mathsf{P}(|X| \geq xb_n)} \cdot \left(\frac{z}{x}\right)^2.$$

The first factor is bounded, in view of Lemma A.12, and the boundedness of the second factor follows from (A.15). Since $z \le x$, this implies (A.18).

On combining the above bounds, the proof of (A.12) can be completed. □

Proof of Corollary A.5 Denote by ξ^s and h_ξ the symmetrized version and characteristic function, respectively, of the random variable ξ. Recall that $\xi^s = \xi_1 - \xi_2$, where ξ_1 and ξ_2 are independent copies of ξ. If $X \in \mathcal{DA}(1, \{b_n\}, \{a_n\})$, then there exists a 1-stable random variable Z_1 such that

$$h_{S_n}\left(\frac{t}{b_n}\right) e^{-ita_n} \to h_{Z_1}(t), \qquad t \in \mathbf{R}^1.$$

Thus

$$\left|h_{S_n}\left(\frac{t}{b_n}\right)\right|^2 \to \left|h_{Z_1}(t)\right|^2, \qquad t \in \mathbf{R}^1.$$

Note that $|h_{S_n}|^2$ and $|h_{Z_1}|^2$ are characteristic functions of the random variables S_n^s and Z_1^s. Therefore, $S_n^s/b_n \xrightarrow{d} Z_1^s$. Since $|h_{Z_1}|^2(t) = e^{-2c|t|}$, by (A.21), Z_1^s is a 1-stable random variable, that is, $X^s \in \mathcal{DA}(1, \{b_n\}, \{0_n\})$. Now we apply Proposition A.4 with $\alpha = 1$ and get

$$\sup_{x\ge 0} \sup_{n\ge 1} \frac{\mathsf{P}(|S_n^s| \ge xb_n)}{n\,\mathsf{P}(|X_1^s| \ge xb_n)} < \infty. \tag{A.19}$$

Then we use the symmetrization inequalities

$$\frac{\mathsf{P}(|S_n - \mu_n| \ge xb_n)}{4\,\mathsf{P}(|X_1| \ge xb_n/2)} \le \frac{\mathsf{P}(|S_n^s| \ge xb_n)}{n\,\mathsf{P}(|X_1^s| \ge xb_n)},$$

where μ_n is a median of S_n. We show that there exists a constant M for which $|\mu_n/b_n - a_n| \le M$ for all $n \ge 1$, whence

$$\mathsf{P}(|S_n - a_nb_n| \ge xb_n) \le \mathsf{P}(|S_n - \mu_n| \ge (x-M)b_n).$$

Thus

$$\sup_{x\ge M} \sup_{n\ge 1} \frac{\mathsf{P}(|S_n - a_nb_n| \ge xb_n)}{n\,\mathsf{P}(|X_1| \ge xb_n)}$$
$$\le \text{const} \sup_{x\ge M} \sup_{n\ge 1} \frac{\mathsf{P}(|S_n^s|) \ge (x-M)b_n)}{n\,\mathsf{P}(|X_1^s| \ge (x-M)b_n)} \frac{\mathsf{P}(|X_1^s| \ge (x-M)b_n)}{\mathsf{P}(|X_1^s| \ge 2xb_n)} < \infty,$$

by (A.19) and (A.26). Finally,

$$\sup_{0\le x\le M}\sup_{n\ge 1}\frac{\mathsf{P}(|S_n-a_nb_n|\ge xb_n)}{n\,\mathsf{P}(|X_1|\ge xb_n)}\le\sup_{n\ge 1}\frac{1}{n\,\mathsf{P}(|X_1|\ge Mb_n)}<\infty,$$

according to (A.26) and Lemma A.15 (iii).

It remains to prove that the sequence $x_n=\mu_n/b_n-a_n$ is bounded. This can be done via contradiction. If $\limsup x_n=\infty$, then there exists a subsequence $\{n_k\}$ such that $x_{n_k}\to\infty$ as $k\to\infty$. Let x_0 be such that $\mathsf{P}(Z_1\ge x_0)<\frac{1}{2}$ and choose k_0 such that $x_{n_k}\ge x_0$ for all $k\ge k_0$. Thus, for $k\ge k_0$,

$$\mathsf{P}(S_{n_k}\ge\mu_{n_k})=\mathsf{P}\left(\frac{S_{n_k}}{b_{n_k}}-a_{n_k}\ge x_{n_k}\right)\le\mathsf{P}\left(\frac{S_{n_k}}{b_{n_k}}-a_{n_k}\ge x_0\right)\to\mathsf{P}(Z_1\ge x_0),$$

since the distribution function of the random variable Z_1 is continuous (see (A.22)). This is a contradiction, since $\mathsf{P}(S_{n_k}\ge\mu_{n_k})\ge\frac{1}{2}$, which proves that $\limsup x_n<\infty$.

If $\liminf x_n=-\infty$, an analogous reasoning can be used. Choose x_0 such that $\mathsf{P}(Z_1\le x_0)<\frac{1}{2}$ and then find k_0 for which $x_{n_k}<x_0$ for all $k\ge k_0$. Thus, for $k\ge k_0$,

$$\mathsf{P}(S_{n_k}\le\mu_{n_k})=\mathsf{P}\left(\frac{S_{n_k}}{b_{n_k}}-a_{n_k}\le x_{n_k}\right)\le\mathsf{P}\left(\frac{S_{n_k}}{b_{n_k}}-a_{n_k}\le x_0\right)\to\mathsf{P}(Z_1\le x_0),$$

in view of (A.22). This is a contradiction again, since $\mathsf{P}(S_{n_k}\le\mu_{n_k})\ge\frac{1}{2}$, and thus the sequence $\{x_n\}$ is indeed bounded. □

A.4 Some Auxiliary Results on Slowly Varying Functions

Recall that a function ℓ is called *slowly varying* (in the Karamata sense) if it is positive for all arguments, finite on every bounded interval, and if

$$\lim_{t\to\infty}\frac{\ell(ct)}{\ell(t)}=1,\qquad\text{for all } c>1$$

(recall Definition 3.3). We further assume that the function ℓ is measurable.

A thorough treatise of slowly and regularly varying functions can be found in Seneta [324] and Bingham et al. [41]. Below we only provide a collection of simple properties to which we refer in our proofs.

One of the fundamental results concerning regularly varying functions is the uniform convergence theorem. Below is the statement of this result for slowly varying functions (see also Theorem 3.47, where the more general case of PRV-functions is studied).

Theorem A.7 *If ℓ is a measurable slowly varying function, then*

$$\lim_{t\to\infty} \sup_{c\in[a,b]} \frac{\ell(ct)}{\ell(t)} = 1$$

for any interval $[a, b]$ with $0 < a < b < \infty$.

Most of the properties of slowly varying functions, which are used here for our considerations, can easily be derived from the following result on the existence of a monotone version.

Proposition A.8 *Let ℓ be a measurable slowly varying function. Then, given an arbitrary $\eta > 0$, there exist slowly varying functions ℓ_1 and ℓ_2 such that*

(i) $x^{\eta}\ell_1(x)$ is nondecreasing in $[x_0, \infty)$ for some $x_0 \geq 0$,
(ii) $x^{-\eta}\ell_2(x)$ is nonincreasing in $[x_0, \infty)$ for some $x_0 \geq 0$,
(iii) $\ell \sim \ell_1$ and $\ell \sim \ell_2$,

where the symbol "$u \sim v$" stands for the asymptotic equivalence of functions u and v, that is, for $u(t)/v(t) \to 1$ as $t \to \infty$.

The following property of slowly varying functions follows immediately from Proposition A.8.

Lemma A.9 *Let ℓ be a measurable slowly varying function. Then, for all $\eta > 0$,*

$$\ell(x) = o\left(x^{\eta}\right), \qquad x \to \infty.$$

If $f \in \mathcal{RV}_a$, $a < -1$, then the series

$$\sum_{n=1}^{\infty} f(n)\ell(n)$$

converges.

Another useful corollary of Proposition A.8 provides the so-called Potter bounds. A similar result holds in the general case of PRV-functions (see Theorem 3.59).

Lemma A.10 *Let ℓ be a measurable slowly varying function. Given some arbitrary constants $D > 1$ and $\eta > 0$, there exists a number $x_* > 0$ such that*

$$\frac{1}{D}\left(\frac{x_2}{x_1}\right)^{-\eta} \leq \frac{\ell(x_1)}{\ell(x_2)} \leq D\left(\frac{x_2}{x_1}\right)^{\eta} \tag{A.20}$$

for all $x_* \leq x_1 \leq x_2$.

Proposition A.8 implies the following bounds for integrals of slowly varying functions.

Lemma A.11 *Let ℓ be a measurable slowly varying function and let $\lambda < 1$. If ℓ is uniformly bounded away from zero in $[0, x_0]$ for some $x_0 > 0$, then*

$$\int_0^T \frac{\ell(x)}{x^\lambda}\,dx \asymp \frac{\ell(T)}{T^{\lambda-1}}$$

for all sufficiently large $T > 0$.

Recall that the symbol $f \asymp g$ means that there are two constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 f(x) \le g(x) \le c_2 f(x) \quad \text{for all} \quad x.$$

(The constants c_1 and c_2 depend on the functions f and g and are universal in x.)

Lemma A.12 *Let ℓ be a measurable slowly varying function and let $\lambda > 1$. Then*

$$\int_T^\infty \frac{\ell(x)}{x^\lambda}\,dx \asymp \frac{\ell(T)}{T^{\lambda-1}}.$$

A more advanced technique allows us to obtain more precise asymptotics of the integrals in both lemmas under the same assumptions (see Theorems A.18 and A.19).

On the other hand, neither Lemma A.11 nor Lemma A.12 holds for $\lambda = 1$ (for a corresponding result, see Parameswaran [290]). An analogue of Lemma A.11 for $\lambda = 1$, with sums instead of integrals, is obtained in Sect. A.7.

The proof of Lemma A.11 can be modified to obtain similar bounds for sums (instead of integrals).

Lemma A.13 *Let ℓ be a slowly varying function and let $\lambda < 1$. Then*

$$\sum_{k=1}^m \frac{\ell(k)}{k^\lambda} \asymp \frac{\ell(m)}{m^{\lambda-1}}$$

for all $m \ge 1$.

Lemma A.14 *Let ℓ be a slowly varying function and let $\lambda > 1$. Then*

$$\sum_{k=m}^\infty \frac{\ell(k)}{k^\lambda} \asymp \frac{\ell(m)}{m^{\lambda-1}}$$

for all $m \ge 1$.

The main idea for the proof of the latter four lemmas is demonstrated in the following proof of Lemma A.11.

Proof of Lemma A.11 First we obtain upper bounds. Let $\eta > 0$ be such that $\lambda + \eta < 1$. Choose a function ℓ_1 according to Proposition A.8, that is, $\ell_1(x)x^\eta$ is a nondecreasing function. Now, find an x_1 such that $\ell(x) \le 2\ell_1(x)$ and $\ell_1(x) \le 2\ell(x)$ for all $x \ge x_1$. Put $x_* = \max\{x_0, x_1\}$. The rest of the proof is easy for $T \ge x_*$. Indeed,

$$\int_0^T \frac{\ell(x)}{x^\lambda}\,dx = \int_0^{x_*} \frac{\ell(x)}{x^\lambda}\,dx + \int_{x_*}^T \frac{\ell(x)}{x^\lambda}\,dx = o\left(\ell(T)T^{1-\lambda}\right) + \int_{x_*}^T \frac{\ell(x)}{x^\lambda}\,dx,$$

since $\ell(T)T^{1-\lambda} \to \infty$, by Proposition A.8. The second term can be estimated as

$$\int_{x_*}^T \frac{\ell(x)}{x^\lambda}\,dx = \int_{x_*}^T \frac{\ell(x)}{\ell_1(x)} \cdot \ell_1(x)x^\eta \frac{1}{x^{\lambda+\eta}}\,dx \le 2\ell_1(T)T^\eta \frac{T^{1-\lambda-\eta}}{1-\lambda-\eta}$$
$$= O\left(\ell(T)T^{1-\lambda}\right).$$

The proof of the lower bound is even simpler. Let $\eta > 0$. Find a function ℓ_2 such that $\ell_2(x)x^{-\eta}$ is nonincreasing. Then choose a number x_1 such that $\ell(x) \le 2\ell_2(x)$ and $\ell_2(x) \le 2\ell(x)$ for all $x \ge x_1$. Put $x_* = \max\{x_0, x_1\}$. Then, for $T \ge x_*$,

$$\int_0^T \frac{\ell(x)}{x^\lambda}\,dx \ge \int_{x_*}^T \frac{\ell(x)}{x^\lambda}\,dx = \int_{x_*}^T \frac{\ell(x)}{\ell_2(x)}\ell_2(x)x^{-\eta} \cdot \frac{1}{x^{\lambda-\eta}}\,dx$$
$$\ge \frac{1}{2}\ell_2(T)T^{-\eta}\frac{T^{1+\eta-\lambda} - x_*^{1+\eta-\lambda}}{1+\eta-\lambda} = O\left(\ell(T)T^{1-\lambda}\right),$$

which completes the proof. □

A.5 Distributions Attracted to Stable Laws

Denote by Z_α a stable random variable with index α, $0 < \alpha \le 2$, and let G_α and h_α be their distribution function and characteristic function, respectively. The following canonical representation of the characteristic function h_α is well known, i.e.

$$h_\alpha(t) = \exp\left\{iat - c|t|^\alpha\,(1 - i\beta\,\mathrm{sign}(t)\omega_\alpha(t))\right\}, \tag{A.21}$$

where a is a real number, c and β are real numbers such that $c \ge 0$ and $|\beta| \le 1$, and

$$\mathrm{sign}(t) = \begin{cases} -1, & t < 0, \\ 0, & t = 0, \\ 1, & t > 0, \end{cases} \qquad \omega_\alpha(t) = \begin{cases} \tan\frac{\pi\alpha}{2}, & \alpha \ne 1, \\ -\frac{2}{\pi}\ln|t|, & \alpha = 1. \end{cases}$$

The case $c = 0$ corresponds to a degenerate distribution function G_α and will not be considered below. Therefore, throughout this section we assume that $c > 0$.

Then

$$G_\alpha \text{ is a continuous function} \tag{A.22}$$

(see Gnedenko and Kolmogorov [151], p. 183). Moreover, the distribution function G_α is infinitely often differentiable in $\mathbf{R}$. Further, there exists a finite constant $C_\alpha \geq 0$ (with $C_\alpha > 0$ if $\alpha \neq 2$) such that

$$\lim_{x\to\infty} x^\alpha \, \mathsf{P}(|Z_\alpha| \geq x) = C_\alpha \tag{A.23}$$

(see Gnedenko and Kolmogorov [151], p. 182). This, for example, implies that

$$\mathsf{E}\,|Z_\alpha|^\eta < \infty \tag{A.24}$$

for all $0 < \eta < \alpha$.

Recall that $X \in \mathcal{DA}\,(\alpha, \{c_n\}, \{a_n\})$ means that the weak convergence in (11.13) holds. It is well-known that the normalizing sequence $\{c_n\}_{n\geq 1}$ is of the form

$$c_n = n^{1/\alpha} f(n), \qquad \text{where } f \text{ is a slowly varying function} \tag{A.25}$$

(see Theorem 2.1.1 in Ibragimov and Linnik [187]), while the tail of the distribution function of the random variable X satisfies the following condition

$$\mathsf{P}(|X| \geq x) = \frac{g(x)}{x^\alpha}, \qquad \text{with a slowly varying function } g. \tag{A.26}$$

This implies, for example, that for an arbitrary $0 < \eta < \alpha$,

$$\mathsf{E}\,|X|^\eta < \infty \tag{A.27}$$

(see Gnedenko and Kolmogorov [151], p. 179). The functions f in (A.25) and g in (A.26) are related to each other, since

$$\lim_{n\to\infty} n \, \mathsf{P}(|X| \geq c_n) = \lim_{n\to\infty} \frac{g\left(n^{1/\alpha} f(n)\right)}{f^\alpha(n)} = K_\alpha, \tag{A.28}$$

where K_α is a finite constant that may differ from the constant C_α in (A.23) (see (9) and (10) on p. 176 in Gnedenko and Kolmogorov [151]).

Below we provide further properties of distributions attracted to nondegenerate stable laws. We also list some properties of the centering and normalizing sequences involved in the weak convergence of (11.13).

Lemma A.15 *Let $\{S_n, n \geq 1\}$ be partial sums of independent, identically distributed random variables $\{X_n, n \geq 1\}$ and let X be a copy of the random variable X_1. Assume that $X \in \mathcal{DA}(\alpha, \{c_n\}, \{a_n\})$. Then, given a function $\varphi \in \mathcal{RV}_{1/p}$, with $0 < p < \alpha$, there exist a nondegenerate α-stable random variable Z_α and a real sequence $\{b_n, n \geq 1\}$ such that*

$$\frac{S_n}{b_n} - a_n \xrightarrow{d} Z_\alpha, \qquad n \to \infty, \tag{A.29}$$

where $\xrightarrow{d}$ denotes weak convergence and

(i) $b_n = b(n)$, $b(x) = x^{1/\alpha} h(x)$, *with a continuous, slowly varying function h,*
(ii) the function $\varphi(x)/b(x)$ is increasing,
(iii) $\lim_{n\to\infty} n \,\mathsf{P}(|X| \geq b_n) = K_\alpha$, *where the constant K_α is as defined in (A.28) (with $K_\alpha > 0$ if $\alpha < 2$).*

Property (iii) implies that

$$\lim_{x\to\infty} x \,\mathsf{P}(|X| \geq \theta b(x)) = \frac{K_\alpha}{\theta^\alpha}, \tag{A.30}$$

for all $\theta > 0$.

Lemma A.16 *Let X, $\{X_n, n \geq 1\}$ be independent, identically distributed random variables with distribution function F. Assume that $X \in \mathcal{DA}(\alpha, \{b_n\}, \{a_n\})$. Then the weak convergence in (A.29) holds with*

$$a_n = \frac{n}{b_n} \int_{|x|<b_n} x \, dF(x). \tag{A.31}$$

Moreover,

(A) if $\alpha < 1$, then

$$X \in \mathcal{DA}(\alpha, \{b_n\}, \{0_n\}); \tag{A.32}$$

(B) relation (A.32) holds for $1 < \alpha \leq 2$ as well, if $\mathsf{E}\, X = 0$; in the latter case, the limiting random variable Z_α in (A.32) is such that $\mathsf{E}\, Z_\alpha = 0$.

Lemma A.16 is known (see, for example, the translator's remark in Gnedenko and Kolmogorov [151], p. 175, or Theorem 3 in §5 of Chapter XVII in Feller [129]). For the sake of completeness, we give a new proof here based on some properties of slowly varying functions.

Proof of Lemma A.15 Let f be the slowly varying function as given in (A.25). Let $\widehat{f}$ be an asymptotically equivalent continuous version of f satisfying $\widehat{f}(n) = f(n)$, $n \geq 1$. Such a version exists in view of the statement at the end of Section 1.4 in

Seneta [324]. By Slutsky's theorem (see, e.g., Theorem 1 in Chow and Teicher [89], p. 249) the weak convergence in (11.13) together with (A.25) yield

$$\frac{S_n}{n^{1/\alpha}\widehat{f}(n)} - a_n \xrightarrow{d} Z_\alpha, \qquad n \to \infty.$$

Now we apply statement 4° of Section 1.5 in Seneta [324], with $\gamma = (\alpha - p)/\alpha p$, to conclude that there exists a slowly varying function h such that $h \sim \widehat{f}$ and $n^{(\alpha-p)/\alpha p}/h(n)$ is increasing. Put $b_n = n^{1/\alpha}h(n)$. Then $b_n \sim c_n$, so that, by Slutsky's theorem, the weak convergence in (A.29) retains. Hence property (ii) is obvious. Moreover, from (A.26),

$$n\,\mathsf{P}(|X| \geq b_n) = n\,\mathsf{P}(|X| \geq c_n)\frac{g(b_n)}{g(c_n)}\frac{c_n^\alpha}{b_n^\alpha},$$

hence property (iii) is proved by recalling (A.28), since g is slowly varying and $b_n \sim c_n$. □

Proof of Lemma A.16 The constants a_n can be obtained from a general result in the theory of weak convergence for sums of independent random variables in series schemes (see, e.g., Theorem 4 in Chapter 25 of Gnedenko and Kolmogorov [151]), that is,

$$a_n = \sum_{k=1}^{k_n} \int_{|x|<\tau} x\,dF_{nk}(x) - \gamma_n(\tau),$$

where $\tau > 0$ is an arbitrary fixed number for which $\pm\tau$ are continuity points of the spectral function H in the Lévy representation of the limit distribution function and where $\gamma_n(\tau)$ is a convergent sequence. In the case of partial sums of independent, identically distributed random variables, we have $k_n = n$ and $F_{nk}(x) = \mathsf{P}(X < xb_n)$, $k \leq n$, whence

$$a_n = \frac{n}{b_n} \int_{|x|<\tau b_n} x\,dF(x) - \gamma_n(\tau).$$

The limiting behavior of $\gamma_n(\tau)$ is also known, namely,

$$\lim_{n\to\infty} \gamma_n(\tau) = a + \int_{|x|<\tau} x\,dH(x) + \int_{|x|\geq\tau} \frac{dH(x)}{x},$$

where a is the constant in the canonical representation (A.21) and where H is the spectral function of G_α (see, e.g., relation (9) in Chapter 3 of Gnedenko and Kolmogorov [151]). Note that the spectral function H of G_α is continuous, that is, one can take, for example, $\tau = 1$. To prove (A.31), we choose $\tau = 1$ and apply Slutsky's theorem once more.

Turning to the proof of (A) we first show that the limit

$$\lim_{n\to\infty} \frac{n}{b_n} \int_{|x|<b_n} x\, dF(x) \tag{A.33}$$

exists if $\alpha < 1$. Put $\overline{F} = 1 - F$. On integrating by parts, we get

$$\begin{aligned}\int_{0\le x<b_n} x\, dF(x) &= -\int_{0<x<b_n} x\, d\overline{F}(x) = b_n\overline{F}(b_n) + \int_{0<x<b_n} \overline{F}(x)\, dx \\ &= b_n\overline{F}(b_n) + b_n \int_{0<y<1} \overline{F}(yb_n)\, dy.\end{aligned}$$

By Theorem 5, Chapter 35, in Gnedenko and Kolmogorov [151], we have $\overline{F} \in \mathcal{RV}_{-\alpha}$, whence

$$\frac{n}{b_n} \int_{0\le x<b_n} x\, dF(x) = n\overline{F}(b_n) + \frac{n}{b_n^\alpha} \int_{0<y<1} \frac{g_1(yb_n)}{y^\alpha}\, dy, \tag{A.34}$$

with some slowly varying function g_1. Now we use Theorem A.19, with $U = g_1$, $\nu = 0$, and $f(t) = t^{-\alpha}$, to prove that

$$\int_{0<y<1} \frac{g_1(yb_n)}{y^\alpha}\, dy \sim g_1(b_n) \int_0^1 \frac{dy}{y^\alpha}.$$

The limit of $n\overline{F}(b_n)$ in (A.34), as $n \to \infty$, exists in view of relation (10) in Chapter 35 of Gnedenko and Kolmogorov [151]. Thus Lemma A.15 together with property (iii) implies that the total limit in (A.34) also exists. An analogous argument proves that the limit

$$\lim_{n\to\infty} \frac{n}{b_n} \int_{-b_n<x<0} x\, dF(x)$$

exists, too. Note, however, that the proof of this property is given for F instead of $\overline{F}$ and uses relation (9) instead of (10), Chapter 35, in Gnedenko and Kolmogorov [151]. In doing so, another slowly varying function comes in instead of g_1. Combining these results, we prove (A.33), whence (A.32) follows, for $\alpha < 1$.

To prove (B), we show that (A.33) holds, for $\alpha > 1$, as well. Note that

$$\frac{n}{b_n} \int_{|x|<b_n} x\, dF(x) = -\frac{n}{b_n} \int_{|x|\ge b_n} x\, dF(x).$$

We follow the same lines of proof as above, but now for $\alpha > 1$. The areas of the integration are $x \ge b_n$ and $x \le -b_n$ instead of $0 \le x < b_n$ and $-b_n < x < 0$

above. Another difference is that we use Theorem A.18 instead of Theorem A.19. As a consequence, the integral $\int_1^\infty t^{-\alpha}\,dt$ appears in the limit instead of $\int_0^1 t^{-\alpha}\,dt$.

Finally, Lemma 5.2.2 in Ibragimov and Linnik [187] implies that, for $0 < \eta < \alpha$,

$$\sup_{n\ge 1} \mathsf{E}\left[\frac{|S_n|}{b_n}\right]^\eta < \infty, \tag{A.35}$$

if (A.32) holds. Using this result for $\eta = 1$ together with Corollary 7 of Chow and Teicher [89], p. 254, we conclude that

$$\mathsf{E}\left(\frac{S_n}{b_n}\right) \to \ \mathsf{E}\, Z_\alpha, \qquad n \to \infty,$$

whence $\mathsf{E}\, Z_\alpha = 0$ follows. This completes the proof. □

The final result of this section states an assertion on the existence of moments (of negative order) for stable distributions. This result has been used to apply Lemma A.1 in the proof of Theorem 11.1, for the case $r = -1$.

Lemma A.17 *Let $0 < \alpha \le 2$ and g_α be the density of some α-stable distribution function. Then*

$$\int_{-1}^{1} |x|^{-\delta} g_\alpha(x)\,dx < \infty, \tag{A.36}$$

for all $0 \le \delta < 1$.

Proof of Lemma A.17 Note that (A.36) is obvious if $c = 0$ in the representation (A.21). Thus we consider the case when $c > 0$. Now, (A.21) implies that $|h_\alpha(t)| = \exp\{-c|t|^\alpha\}$ and, in addition, h_α is absolutely integrable on $\mathbf{R}$. Moreover, the function

$$h_\alpha(t) \int_{-1}^{1} |x|^{-\delta} e^{-itx}\,dx$$

is also absolutely integrable on $\mathbf{R}$, since $0 \le \delta < 1$. It is well-known that

$$g_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} h_\alpha(t)\,dt.$$

Thus

$$\begin{aligned} 2\pi \int_{-1}^{1} |x|^{-\delta} g_\alpha(x)\,dx &= \int_{-1}^{1} |x|^{-\delta} \int_{-\infty}^{\infty} e^{-itx} h_\alpha(t)\,dt\,dx \\ &= \int_{-\infty}^{\infty} \left[h_\alpha(t) \int_{-1}^{1} |x|^{-\delta} e^{-itx}\,dx \right] dt < \infty, \end{aligned}$$

completing the proof. □

A.6 The Asymptotic Behavior of Integrals and Sums with Slowly Varying Functions

The proof of Theorem 11.1 is based on the following obvious generalization of two results of Aljančić et al. [9], where the case of slowly varying functions U is considered, that is, the special case $\nu = 0$.

Theorem A.18 *Let f be a real-valued function and $U \in \mathcal{RV}_\nu$. Assume that the Lebesgue integral*

$$\int_A^\infty t^\eta f(t)\, dt \tag{A.37}$$

is well-defined, for some $\eta \geq \nu$ and $A > 0$. Then the integral

$$\int_A^\infty U(xt) f(t)\, dt$$

is also well-defined if either

(i) $\eta > \nu$, *or*
(ii) $\eta = \nu$ *and the function* $U(t)/t^\nu$ *is nonincreasing on* (t_0, ∞), *for some* $t_0 > 0$.

In each of these cases,

$$\int_A^\infty U(xt) f(t)\, dt \sim U(x) \int_A^\infty t^\nu f(t)\, dt \qquad \text{as } x \to \infty.$$

Theorem A.19 *Let f be a real-valued function and $U \in \mathcal{RV}_\nu$. Assume that the Lebesgue integral*

$$\int_0^B t^\eta f(t)\, dt \tag{A.38}$$

is well defined, for some $\eta \leq \nu$ and $B > 0$. Then

$$\int_0^B U(xt) f(t)\, dt \sim U(x) \int_0^B t^\nu f(t)\, dt \qquad \text{as } x \to \infty,$$

if either

(i) $\eta < \nu$, *or*
(ii) $\eta = \nu$ *and the function* $U(t)/t^\nu$ *is nonincreasing on* (t_0, ∞), *for some* $t_0 > 0$.

A.7 The Parameswaran Lemma for Sums

The following result is classical for the case when $r > -1$ and for integrals instead of sums (the case $r = -1$ has been treated in Parameswaran [290], Lemma 1).

Lemma A.20 *Let $w \in \mathcal{RV}_r$, $r \geq -1$, and let the function W be as defined in (11.12). Then $W \in \mathcal{RV}_{r+1}$ and, moreover,*

$$\lim_{t\to\infty} \frac{tw(t)}{W(t)} = r + 1. \tag{A.39}$$

Note that Lemma A.20 asserts that W is a slowly varying function if $r = -1$.

The proof in Parameswaran [290] has been given for integrals, but we need such a result for sums. Even more important is that the proof in [290] uses the l'Hospital rule for differentiable W, that is, for continuous w. In our case, however, W is a step function defined by (11.12). The Parameswaran method can be modified by using a continuous version w^* of w such that $w^*(t) \sim w(t)$, as $t \to \infty$, and by applying the uniform convergence theorem for slowly varying functions. For convenience, we provide below a new proof of Parameswaran's lemma based on an elementary argument.

As usual, the symbol $[x]$ denotes the integer part of a real number x.

Lemma A.21 *Let $w \in \mathcal{RV}_{-1}$ and let the function W be defined by equality (11.12). Then*

(i) $W([cn]) \sim W(n)$ as $n \to \infty$, for all $c > 1$, that is, W is a slowly varying function;
(ii) $nw_n = o(W(n))$ as $n \to \infty$, that is, relation (A.39) holds for $r = -1$ as well.

Proof of Lemma A.21 First we prove (ii). Fix $D > 0$ and choose $0 < \delta < 1$ with $\ln\left(\delta^{-1}\right) \geq 3D$. Let n_1 be such that

$$\sum_{k=[\delta n]}^{n} \frac{1}{k} \geq 2D \quad \text{for all} \quad n \geq n_1.$$

Such an n_1 exists, since

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(n) + E + o(1) \qquad \text{as } n \to \infty, \tag{A.40}$$

where $E = 0.577\ldots$ is a universal constant, known as the *Euler constant*. Now choose n_2 such that

$$\inf_{[\delta n]\leq k\leq n} kw_k \geq \frac{1}{2} nw_n \quad \text{for all} \quad n \geq n_2.$$

Such an n_2 exists in view of Theorem A.7. Therefore,

$$\sum_{k=1}^{n} w_k \geq \sum_{k=[\delta n]}^{n} w_k \geq \frac{1}{2} n w_n \sum_{k=[\delta n]}^{n} \frac{1}{k} \geq D n w_n$$

for $n \geq \max\{n_1, n_2\}$, whence

$$\liminf_{n\to\infty} \frac{1}{n w_n} \sum_{k=1}^{n} w_k \geq D,$$

so that property (ii) is proved, since $D > 0$ can be chosen arbitrarily large.

To prove property (i), note that

$$\frac{W([cn])}{W(n)} = 1 + \frac{1}{W(n)} \sum_{k=n+1}^{[cn]} w_k,$$

for $c > 1$. Thus, by (A.40) and Theorem A.7,

$$\sum_{k=n+1}^{[cn]} w_k \leq \sup_{n\leq k\leq [cn]} k w_k \sum_{k=n+1}^{[cn]} \frac{1}{k} \sim n w_n \ln(c),$$

as $n \to \infty$. Therefore, for all fixed $c > 1$,

$$1 \leq \frac{W([cn])}{W(n)} \leq 1 + O\left(\frac{n w_n}{W(n)}\right),$$

whence (i) follows in view of (ii). □

A.8 Comments

Section A.3 Relation (A.10) is proved by Cline and Hsing [91] for the more general case of subexponential distributions. The Cline and Hsing [91] result generalizes a theorem of Heyde [178] proved for an attraction to stable laws, stating that

$$\lim_{n\to\infty} \frac{\mathsf{P}(|S_n| \geq x_n b_n)}{n\,\mathsf{P}(|X| \geq x_n b_n)} = 1 \tag{A.41}$$

for any sequence $\{x_n\}_{n\geq 1}$ such that $x_n \to \infty$ as $n \to \infty$. Related results are obtained by A.V. Nagaev [280, 281], S.V. Nagaev [282], Vonogradov [361], and Mikosch and Nagaev [273]. A similar result for dominated variation distributions (distribution functions being monotone ORV functions in our language) is proved by Baltrunas [22]. Several extensions are known for dependent random variables (see, for example, Liu [252]). Some applications of large deviations in insurance and finance are discussed by Klüppelberg and Mikosch [232], Mikosch and Nagaev [274], and Tang et al. [356].

An overview of earlier results on this topic is given in Chapter 8 of Embrechts et al. [119] (see also Christoph and Wolf [84], Saulis and Statulevičius [319], and Samorodnitsky and Taqqu [317]). Further references can be found in Borovkov and Borovkov [48] and Denisov et al. [101] (see, in addition, Foss et al. [132]).

Section A.4 The proof of Proposition A.8 can be found in Seneta [324] (Section 1.4, statement 4°). It also follows from the more general Theorem 3.79, which is valid for POV-functions.

Section A.5 See the discussion in Hall [170] and the last remark in the Introduction of the monograph of Zolotarev [375] concerning the choice of signs and constants in (A.21) as well as in the definition of $\omega_\alpha(t)$. A short and elegant proof of (A.26) and an explicit determination of the constants C_α and K_α in (A.23) and (A.28) can be found in Samorodnitsky and Taqqu [317].

Section A.6 Theorems A.18 and A.19 are obvious generalizations of two results of Aljančić et al. [9], where the case of slowly varying functions U is considered, that is, the special case $\nu = 0$ (see also Seneta [324]).

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