Appendix A
The Fibred Isomorphism Conjecture

The Setup

Let $\mathcal{S}\colon$ TOP $\to$ Ω-SPECTRA be a covariant homotopy functor. Let **F** be the category of continuous surjective maps: objects in **F** are continuous surjective maps $p\colon E \to B$, where E, B are objects in TOP, and morphisms between pairs of maps $p_1\colon E_1 \to B_1$ and $p_2\colon E_2 \to B_2$ consist of continuous maps $f\colon E_1 \to E_2$ and $g\colon B_1 \to B_2$ that make the following diagram commute:

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\big\downarrow & & {\scriptstyle p_2}\big\downarrow \\ B_1 & \xrightarrow{g} & B_2. \end{array} \tag{A.1}$$

Within this framework, Quinn constructed a functor between the categories **F** and Ω-SPECTRA [1]. The value of this Ω-spectrum at the object $(p\colon E \to B)$ is denoted by $\mathbb{H}(B; \mathcal{S}(p))$, and the value at the object $(E \to *)$ is $\mathcal{S}(E)$. The map of spectra $\mathbb{A}\colon \mathbb{H}(B_1; \mathcal{S}(p_1)) \to \mathbb{H}(B_2; \mathcal{S}(p_2))$ associated to the commutative diagram (A.1) is known as the *Quinn assembly map*. Other ingredients for the fibred isomorphism conjecture may be found in [2].

The Conjecture

Given a discrete group Γ, let $E_{\mathcal{VC}}\Gamma$ be a universal Γ-space for the family of virtually cyclic subgroups of Γ, let $\mathcal{B}_{\mathcal{VC}}\Gamma$ denote the orbit space $E_{\mathcal{VC}}\Gamma/\Gamma$, and let X be a space on which Γ acts freely and properly discontinuously. If (f, g) is the following morphism in **F**:

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$$\begin{array}{ccc} E_{\mathcal{VC}}\Gamma \times_\Gamma X & \xrightarrow{f} & X/\Gamma \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow \\ \mathcal{B}_{\mathcal{VC}}\Gamma & \xrightarrow{g} & * \end{array}$$

then the *Fibred Isomorphism Conjecture* for the functor $\mathcal{S}$ and the group Γ is the assertion that

$$\mathbb{A}\colon\ \mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{S}(p_1)) \to \mathcal{S}(X/\Gamma)$$

is a homotopy equivalence, and hence the induced map

$$\mathbb{A}_*\colon\ \pi_n(\mathbb{H}(\mathcal{B}_{\mathcal{VC}}\Gamma; \mathcal{S}(p_1))) \to \pi_n(\mathcal{S}(X/\Gamma))$$

is an isomorphism for all $n \in \mathbb{Z}$. This conjecture was stated in [2] for the functors $\mathcal{S} = \mathcal{P}_*(\cdot)$, $\mathcal{K}(\cdot)$ and $\mathcal{L}^{-\infty}$, the pseudoisotopy, algebraic K-theory and $\mathcal{L}^{-\infty}$-theory functors respectively. In this manuscript, we use the functor $\mathcal{S} = \mathcal{K}_*(\cdot)$. The validity of this conjecture for K-theory and braid groups of $\mathbb{S}^2$ is proved in [3]. Other cases in which the conjecture holds may be found in [4].

Appendix B
Braid Groups

In this appendix, we recall briefly some basic facts and results about braid groups for the convenience of the reader. More information about braid groups may be found in [5–8]. We refer the reader to [9] for a recent survey on surface braid groups.

If $n \geq 1$, the n-string Artin braid group, denoted by B_n, may be defined by the following presentation [10]:

generators: $\sigma_1, \ldots, \sigma_{n-1}$ (known as the *Artin generators*).
relations: (known as the *Artin relations*)

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{if } |i - j| \geq 2 \text{ and } 1 \leq i, j \leq n-1 \tag{B.1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } 1 \leq i \leq n-2. \tag{B.2}$$

The generator σ_i may be regarded geometrically as the braid with a single positive crossing of the ith string with the $(i+1)^{\text{st}}$ string, while all other strings remain vertical (see Fig. B.1). It is convenient to view a geometric braid as being a collection of pairwise-disjoint arcs (or strings) in the Cartesian product $\mathbb{D}^2 \times [0, 1]$, where $\mathbb{D}^2$ is the 2-disc, and each string joins two points of the form $(x, 0)$ to $(y, 1)$, where x and y belong to a set X of n distinguished basepoints lying in the interior of $\mathbb{D}^2$. The group operation in B_n corresponds to concatenation of these geometric braids. The group B_1 is trivial, B_2 is infinite cyclic generated by σ_1, and for all $n \geq 2$, B_n is infinite. For all $n \in \mathbb{N}$, B_n is torsion free [11]. The map $\sigma\colon B_n \to S_n$ defined on the generators by $\sigma(\sigma_i) = (i, i+1)$ for all $1 \leq i \leq n-1$ may be seen to be a surjective homomorphism. Its kernel, denoted by P_n, is known as the n-string pure Artin braid group. Thus a braid $\beta \in B_n$ is *pure* if for all $x \in X$, there is a string of β that joins $(x, 0)$ to $(x, 1)$. The 'half twist' braid Δ_n is defined by:

$$\Delta_n = \prod_{i=1}^{n-1} \sigma_1 \cdots \sigma_{n-i}.$$

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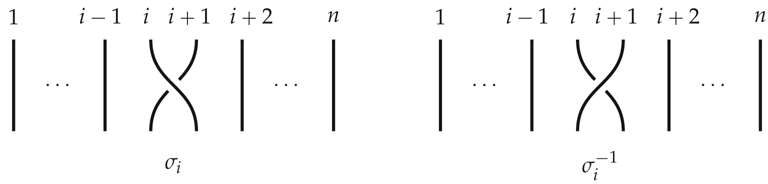
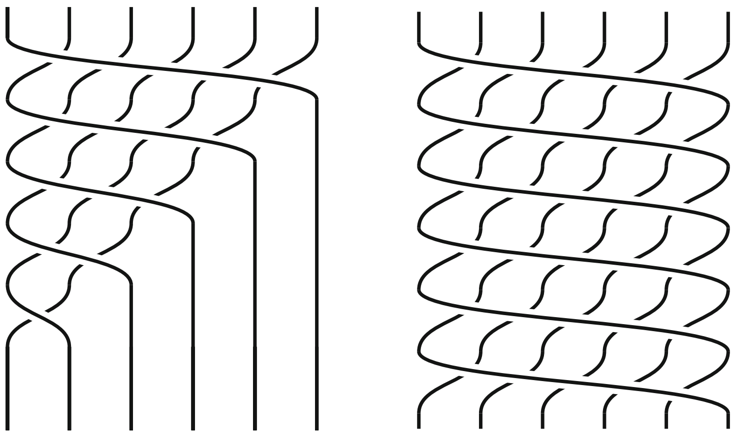


Fig. B.1 The braid σ_i and its inverse



(a) The half twist braid Δ_6 of B_6. (b) The full twist braid Δ_6^2 of B_6.

Fig. B.2 The braids Δ_6 and Δ_6^2 of B_6

Using the braid relations, one may check that the square Δ_n^2 of Δ_n, known as the 'full twist' braid is given by:

$$\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n \in B_n. \tag{B.3}$$

The braids Δ_n and Δ_n^2 are illustrated in Fig. B.2a and b in the case $n = 6$. One may check that Δ_n^2 is a pure braid. If $n \geq 3$, $Z(B_n) = Z(P_n) = \langle \Delta_n^2 \rangle$, where $Z(G)$ denotes the centre of the group G [12]. The Artin pure braid group is generated by the set $\{A_{i,j}\}_{1 \leq i < j \leq n}$ [8, Lemma I.4.2], where:

$$A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}. \tag{B.4}$$

Geometrically, $A_{i,j}$ may be represented by a braid all of whose strings are vertical, with the exception of the jth string that wraps around the ith string as in Fig. B.3. In particular, for all $i = 1, \dots, n-1$, $A_{i,i+1} = \sigma_i^2$.

The Artin braid groups admit many different generalisations, one being that of *surface braid groups*. If M is a surface, orientable or not, with or without boundary, and with a finite number (possibly zero) of punctures, the n-string braid group $B_n(M)$

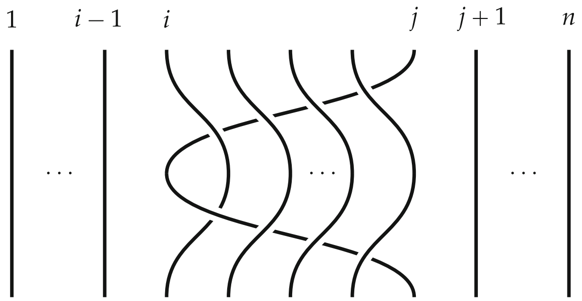


Fig. B.3 The element $A_{i,j}$ of B_n

may be defined geometrically simply by replacing $\mathbb{D}^2$ by M. The subgroup $P_n(M)$ of n-string pure braids is defined in a manner similar to that for P_n. A number of presentations of $B_n(M)$ and $P_n(M)$ may be found in the literature, see [13–15] for example.

Braid groups may also defined topologically in terms of configuration spaces as follows. Let $F_n(M)$ denote the nth *configuration space* of M defined by:

$$F_n(M) = \left\{(p_1, \ldots, p_n) \in M^n \middle| p_i \neq p_j \text{ for all } i, j \in \{1, \ldots, n\}, i \neq j\right\}.$$

We equip $F_n(M)$ with the topology induced by the product topology on M^n. A transversality argument shows that $F_n(M)$ is a connected $2n$-dimensional open manifold. There is a natural free action of the symmetric group S_n on $F_n(M)$ given by permutation of coordinates. The resulting orbit space $F_n(M)/S_n$ shall be denoted by $D_n(M)$, the nth *permuted configuration space* of M, and may be thought of as the configuration space of n *unordered* points. The associated canonical projection $p\colon F_n(M) \to D_n(M)$ is thus a regular $n!$-fold covering map [8, p. 14]. Fox and Neuwirth showed that $P_n(M) \cong \pi_1(F_n(M))$ and $B_n(M) \cong \pi_1(D_n(M))$ [16]. If $n = 1$ then $F_1(M) = M$, and thus $B_1(M) = P_1(M) = \pi_1(M)$, so braid groups generalise the notion of fundamental group. The map p gives rise to the following short exact sequence:

$$1 \to P_n(M) \to B_n(M) \xrightarrow{p_*} S_n \to 1. \tag{B.5}$$

In the case where M is the disc, p_* is the surjective homomorphism σ described on page 75.

This topological definition is very useful in practice, and may be used as follows to obtain fibrations involving the configuration spaces, and (short) exact sequences bringing into play the homotopy groups of these spaces. Suppose that M is a surface with empty boundary, and let $m > n \geq 1$. Then the map $p_{m,n}\colon F_m(M) \to F_n(M)$ given by $p_{m,n}(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$ that forgets the last $m - n$ coordinates is a locally-trivial fibration, known as the *Fadell-Neuwirth fibration*, whose fibre is $F_{m-n}(M \setminus \{z_1, \ldots, z_n\})$, where $(z_1, \ldots, z_n)$ is a basepoint of $F_n(M)$ [17]. The fibre is known to be an Eilenberg–Mac Lane space of type $K(\pi, 1)$. Taking the long exact sequence in homotopy of the fibration, and using Fox and Neuwirth's isomorphisms mentioned above, we obtain the *Fadell-Neuwirth short exact sequence of surface pure braid groups*:

$$1 \to \pi_1(F_{m-n}(M \setminus \{z_1, \ldots, z_n\})) \to P_m(M) \xrightarrow{(p_{m,n})_*} P_n(M) \to 1. \tag{B.6}$$

The homomorphism $(p_{m,n})_*$ induced by the map $p_{m,n}$ may be visualised geometrically as the map that 'forgets' the last $m-n$ strings of a braid in $P_m(M)$. Due to the fact that the higher homotopy groups of the braid groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$ are non trivial, in order to obtain the short exact sequence (B.6) for these two surfaces, we need to suppose additionally that $n \geq 3$ (resp. $n \geq 2$). In particular, if $m = n+1$, then (B.6) becomes:

$$1 \to \pi_1(M \setminus \{z_1, \ldots, z_n\}) \to P_{n+1}(M) \xrightarrow{(p_{n+1,n})_*} P_n(M) \to 1. \tag{B.7}$$

The braid groups of $\mathbb{S}^2$ and $\mathbb{R}P^2$ are of particular interest, partly because they are the only surface braid groups to possess torsion, and as we explained in the introduction, the methods of [18, 19] cannot be applied to study their lower algebraic K-theory. The isomorphism classes of the maximal finite subgroups of $B_n(\mathbb{S}^2)$ are given in Theorem 2. An analogous result for the braid groups of $\mathbb{R}P^2$ may be found in [20]. The braid groups of the sphere were initially studied by Fadell, Van Buskirk and Gillette [21–23]. A presentation of $B_n(\mathbb{S}^2)$ due to the first two of these authors is given in Theorem 34. From a geometric point of view, the space $\mathbb{S}^2 \times [0, 1]$ in which geometric braids of the sphere are defined may be visualised as that between two concentric spheres (see [8, pp. 41, 42 and 45] or [24, Fig. 2.1(c), p. 193] for example), and the geometric representation of the generators of that presentation is as in Fig. B.1. Using such figures, the reader may convince himself or herself of the validity of the relations given in Theorem 34, in particular the 'surface relation' (3.1). Other properties of $B_4(\mathbb{S}^2)$ that we use in this manuscript are given in Sect. 3.1. The full twist braid Δ_n^2 also plays an important rôle in $B_n(\mathbb{S}^2)$. If $n \geq 3$, it is the unique element of $B_n(\mathbb{S}^2)$ of order 2, it is the unique non-trivial torsion element of $P_n(\mathbb{S}^2)$, and it generates the centre of $B_n(\mathbb{S}^2)$ [22, 25]. The pure braid group $P_4(\mathbb{S}^2)$ is generated by the set $\{A_{i,j}\}_{1 \leq i < j \leq 4}$, where in terms of the generators σ_1, σ_2 and σ_3 of $B_4(\mathbb{S}^2)$, $A_{i,j}$ is given by (B.4), and its geometric representation within $\mathbb{S}^2 \times [0, 1]$ is as in Fig. B.3. If $m \geq 1$, a presentation of $P_m(\mathbb{S}^2)$ may be obtained using techniques similar to those of [26, Proposition 7]. Note that if one takes $n = 0$ in that proposition, one does indeed obtain a presentation of $P_m(\mathbb{S}^2)$ whose generating set is $\{A_{i,j}\}_{1 \leq i < j \leq m}$, and whose relations are given by those of [8, Lemma I.4.2] for P_m, and by the 'surface relations' that are of the form:

$$\left(\prod_{i=1}^{j-1} A_{i,j}\right)\left(\prod_{k=j+1}^{m} A_{j,k}\right) = 1 \tag{B.8}$$

for all $1 \le j \le m$.

Taking $M = \mathbb{S}^2$ and $n = 3$ in (B.7) yields:

$$1 \to \pi_1(\mathbb{S}^2 \setminus \{z_1, z_2, z_3\}) \to P_4(\mathbb{S}^2) \xrightarrow{(p_{4,3})_*} P_3(\mathbb{S}^2) \to 1. \tag{B.9}$$

The kernel is a free group of rank 2 that may be identified with the subgroup of $P_4(\mathbb{S}^2)$ generated by $(A_{1,4}, A_{2,4})$, and the quotient $P_3(\mathbb{S}^2)$ is equal to $\langle \Delta_3^2 \rangle$, and is isomorphic to $\mathbb{Z}_2$. The map $s\colon P_3(\mathbb{S}^2) \to P_4(\mathbb{S}^2)$ defined by $s(\Delta_3^2) = \Delta_4^2$ is a homomorphism, and is a section for $(p_{4,3})_*$ since removal of the last string of Δ_4^2 in $P_4(\mathbb{S}^2)$ yields the braid Δ_3^2 in $P_3(\mathbb{S}^2)$, *i.e.* $(p_{4,3})_*(\Delta_4^2) = \Delta_3^2$. So the short exact sequence (B.9) splits, and since $\Delta_4^2 \in Z(P_4(\mathbb{S}^2))$, it follows that:

$$P_4(\mathbb{S}^2) \cong \mathbb{F}_2 \times \mathbb{Z}_2. \tag{B.10}$$

From this, it follows also that $Z(P_4(\mathbb{S}^2)) = \langle \Delta_4^2 \rangle$.

References

1. F. Quinn, Ends of maps II. Invent. Math. **68**, 353–424 (1982)
2. F.T. Farrell, L.E. Jones, Isomorphism conjectures in algebraic K-theory. J. Am. Math. Soc. **6**, 249–297 (1993)
3. D. Juan-Pineda, J. Sánchez, The K and L theoretic Farrell-Jones isomorphism conjecture for braid groups, in Topology and Geometric Group Theory, ed. by M.W. Davis et al, Springer Proceedings in Mathematics & Statistics **184**, 33–43 (2016)
4. C. Wegner, The K-theoretic Farrell-Jones conjecture for $CAT(0)$-groups. Trans. Am. Math. Soc. **140**, 779–793 (2012)
5. A.J. Berrick, F.R. Cohen, E. Hanbury, Y.-L. Wong, J. Wu, Braids: introductory lectures on braids, configurations and their applications. Lecture Notes Series, vol. 19 (Institute for Mathematical Sciences, National University of Singapore, World Scientific, 2010)
6. J.S. Birman, Braids, links and mapping class groups. Ann. Math. Stud. **82** (Princeton University Press, 1974)
7. J. González-Meneses, Basic results on braid groups. Ann. Math. Blaise Pascal **18**, 15–59 (2011)
8. V.L. Hansen, *Braids and coverings: selected topics*. London Mathematics Society Student Texts, vol. 18 (Cambridge University Press, 1989)
9. J. Guaschi, D. Juan-Pineda, A survey of surface braid groups and the lower algebraic K-theory of their group rings, in *Handbook of Group Actions*, ed. by L. Ji, A. Papadopoulos et S.-T. Yau. Advanced Lectures in Mathematics, vol II, 32 (International Press of Boston Inc., 2015), pp. 23–76
10. E. Artin, Theorie der Zöpfe. Abh. Math. Sem. Univ. Hamburg **4**, 47–72 (1925)
11. J.L. Dyer, The algebraic braid groups are torsion-free: an algebraic proof. Math. Z. **172**, 157–160 (1980)
12. W.-L. Chow, On the algebraical braid group. Ann. Math. **49**, 654–658 (1948)
13. P. Bellingeri, On presentations of surface braid groups. J. Algeb. **274**, 543–563 (2004)
14. D.L. Gonçalves, J. Guaschi, On the structure of surface pure braid groups. J. Pure Appl. Algeb. **182**, 33–64 (2003) (due to a printer's error, this article was republished in its entirety with the reference **186** (2004) 187–218)
15. J. González-Meneses, New presentations of surface braid groups. J. Knot Th. Ramif. **10**, 431–451 (2001)

16. R.H. Fox, L. Neuwirth, The braid groups. Math. Scand. **10**, 119–126 (1962)
17. E. Fadell, L. Neuwirth, Configuration spaces. Math. Scand. **10**, 111–118 (1962)
18. C.S. Aravinda, F.T. Farrell, S.K. Roushon, Algebraic K-theory of pure braid groups. Asian J. Math. **4**, 337–343 (2000)
19. F.T. Farrell, S.K. Roushon, The Whitehead groups of braid groups vanish. Int. Math. Res. Notices **10**, 515–526 (2000)
20. D.L. Gonçalves, J. Guaschi, Surface braid groups and coverings. J. London Math. Soc. **85**, 855–868 (2012)
21. E. Fadell, J. Van Buskirk, The braid groups of $\mathbb{E}^2$ and $\mathbb{S}^2$. Duke Math. J. **29**, 243–257 (1962)
22. R. Gillette, J. Van Buskirk, The word problem and consequences for the braid groups and mapping class groups of the 2-sphere. Trans. Am. Math. Soc. **131**, 277–296 (1968)
23. J. Van Buskirk, Braid groups of compact 2-manifolds with elements of finite order. Trans. Am. Math. Soc. **122**, 81–97 (1966)
24. K. Murasugi, B.I. Kurpita, *A Study of Braids*. Mathematics and its Applications, vol. 484 (Kluwer Academic Publishers, Dordrecht, 1999)
25. D.L. Gonçalves, J. Guaschi, The roots of the full twist for surface braid groups. Math. Proc. Camb. Phil. Soc. **137**, 307–320 (2004)
26. D.L. Gonçalves, J. Guaschi, The braid group $B_{m,n}(\mathbb{S}^2)$ and a generalisation of the Fadell-Neuwirth short exact sequence. J. Knot Th. Ramif. **14**, 375–403 (2005)