Appendix A

Differential Operators and Equations in Various Curvilinear Coordinate Systems

Here we present the differential operators and Helmholtz equations in orthogonal coordinate systems used through the book (Figs. A.1, A.2, A.3 and A.4). In an arbitrary orthogonal curvilinear coordinate system (Fig. A.1).

$$\operatorname{grad} F = \frac{\vec{e}_1}{h_1}\frac{\partial F}{\partial q_1} + \frac{\vec{e}_2}{h_2}\frac{\partial F}{\partial q_2} + \frac{\vec{e}_3}{h_3}\frac{\partial F}{\partial q_3}, \tag{A.1}$$

where h_1, h_2, h_3 are Lame coefficients, $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are unit vectors, and $F = F(q_1, q_2, q_3)$ is a scalar function.

$$\operatorname{div} \vec{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 F_1) + \frac{\partial}{\partial q_2}(h_1 h_3 F_2) + \frac{\partial}{\partial q_3}(h_1 h_2 F_3)\right], \tag{A.2}$$

where $\vec{F} = F_1(q_1, q_2, q_3)\,\vec{e}_1 + F_2(q_1, q_2, q_3)\,\vec{e}_2 + F_3(q_1, q_2, q_3)\,\vec{e}_3$;

$$\begin{aligned}\operatorname{rot} \vec{F} = {}& \frac{\vec{e}_1}{h_2 h_3}\left[\frac{\partial}{\partial q_2}(h_3 F_3) - \frac{\partial}{\partial q_3}(h_2 F_2)\right] + \frac{\vec{e}_2}{h_3 h_1}\left[\frac{\partial}{\partial q_3}(h_1 F_1) - \frac{\partial}{\partial q_1}(h_3 F_3)\right] \\ & + \frac{\vec{e}_3}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(h_2 F_2) - \frac{\partial}{\partial q_2}(h_1 F_1)\right].\end{aligned} \tag{A.3}$$

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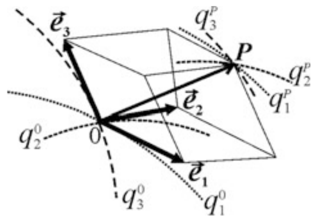

Fig. A.1 The curvilinear coordinate system

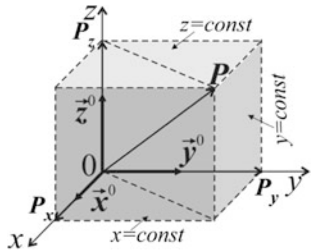

Fig. A.2 The rectangular coordinate system

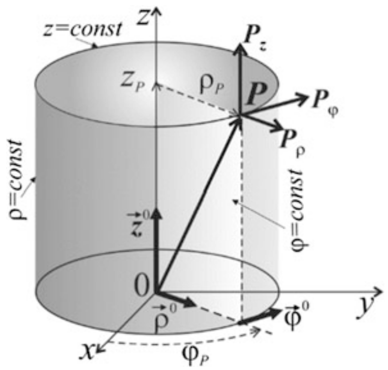

Fig. A.3 The cylindrical coordinate system

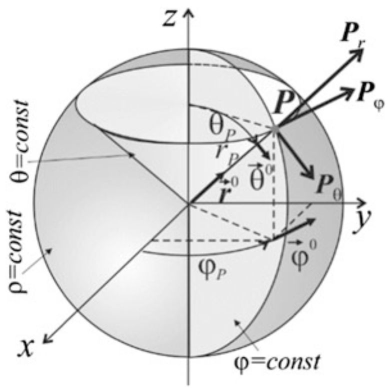

Fig. A.4 The spherical coordinate system

The Laplacian of a scalar function is

$$\Delta F \equiv \operatorname{div} \operatorname{grad} F = \frac{1}{h_1 h_2 h_3}\left[\begin{array}{c}\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial F}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial F}{\partial q_2}\right) \\ + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial F}{\partial q_3}\right)\end{array}\right]. \tag{A.4}$$

The Laplacian of a vector function is defined by the formula $\Delta\vec{F} \equiv \operatorname{grad}\operatorname{div}\vec{F} - \operatorname{rot}\operatorname{rot}\vec{F}$. The inhomogeneous Helmholtz equation for the tensor Green's function $\hat{G}(q_1, q_2, q_3; q_1', q_2', q_3')$ has the form.

$$\Delta\hat{G} + k^2\hat{G} = -4\pi\hat{I}\frac{\delta(q_1 - q_1')\delta(q_2 - q_2')\delta(q_3 - q_3')}{h_1 h_2 h_3}, \tag{A.5}$$

where $\hat{I}$ is the unit tensor, $\delta(q - q')$ is the Dirac delta function, (q_1', q_2', q_3') are coordinates of the source point, the operator $\Delta \equiv \operatorname{div}\operatorname{grad}$ is apply to all nine tensor components.

In the Cartesian coordinate system (Fig. A.2) $q_1 = x, q_2 = y, q_3 = z, h_1 = h_2 = h_3 = 1$, $\vec{e}_1 = \vec{x}^0, \vec{e}_1 = \vec{x}^0, \vec{e}_2 = \vec{y}^0, \vec{e}_3 = \vec{z}^0$, and,

$$\operatorname{grad} F = \vec{x}^0\frac{\partial F}{\partial x} + \vec{y}^0\frac{\partial F}{\partial y} + \vec{z}^0\frac{\partial F}{\partial z}, \tag{A.6}$$

$$\operatorname{div}\vec{F} = \frac{\partial F_x}{\partial_x} + \frac{\partial F_y}{\partial_y} + \frac{\partial F_z}{\partial_z}, \tag{A.7}$$

$$\operatorname{rot}\vec{F} = \vec{x}^0\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \vec{y}^0\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \vec{x}^0\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right), \tag{A.8}$$

$$\Delta F \equiv \operatorname{div}\operatorname{grad} F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}. \tag{A.9}$$

An inhomogeneous Helmholtz equation for the tensor Green's function $\hat{G}(x, y, z; x', y', z')$ in this coordinate system has the form.

$$\Delta\hat{G} + k^2\hat{G} = -4\pi\hat{\dot{I}}\delta(x - x')\delta(y - y')\delta(z - z'). \tag{A.10}$$

In a cylindrical coordinate system (Fig. A.3), $q_1 = \rho, q_1 = \varphi, q_3 = z$, $h_1 = h_3 = 1$, $h_2 = \rho$ and $\vec{e}_1 = \vec{\rho}^0, \vec{e}_2 = \vec{\varphi}^0, \vec{e}_3 = \vec{z}^0$

$$\operatorname{grad} F = \vec{\rho}^0\frac{\partial F}{\partial\rho} + \frac{\vec{\varphi}^0}{\rho}\frac{\partial F}{\partial\varphi} + \vec{z}^0\frac{\partial F}{\partial z}, \tag{A.11}$$

$$\operatorname{div}\vec{F} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z}, \tag{A.12}$$

$$\operatorname{rot}\vec{F} = \vec{\rho}^0\left[\frac{1}{\rho}\frac{\partial F_z}{\partial\varphi} - \frac{\partial F_\varphi}{\partial z}\right] + \vec{\varphi}^0\left[\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right] + \frac{\vec{z}^0}{\rho}\left[\frac{\partial}{\rho}\left(\rho F_\varphi\right) - \frac{\partial F_\rho}{\partial\varphi}\right], \tag{A.13}$$

$$\Delta F \equiv \operatorname{div}\operatorname{grad} F = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial F}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial\varphi^2} + \frac{\partial^2 F}{\partial z^2}. \tag{A.14}$$

An inhomogeneous Helmholtz equation for the tensor Green's function $\hat{G}(\rho, \varphi, z; \rho', \varphi', z')$ in this coordinate system has the form.

$$\Delta\hat{G} + k^2\hat{G} = -4\pi\hat{I}\,\frac{\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')}{\rho}. \tag{A.15}$$

$$\operatorname{grad} F = \vec{r}^0\frac{\partial F}{\partial r} + \frac{\vec{\theta}^0}{r}\frac{\partial F}{\partial\theta} + \frac{\vec{\varphi}^0}{r\sin\theta}\frac{\partial F}{\partial\varphi} \tag{A.16}$$

$$\operatorname{div}\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}, \tag{A.17}$$

$$\begin{aligned}\operatorname{rot}\vec{F} = {} & \frac{\vec{r}^0}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,F_\varphi\right) - \frac{\partial F_\theta}{\partial\varphi}\right] + \frac{\vec{\theta}^0}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rF_\varphi\right)\right] \\ & + \frac{\vec{\varphi}^0}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right],\end{aligned} \tag{A.18}$$

$$\begin{aligned}\Delta F \equiv \operatorname{div}\operatorname{grad} F = {} & \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) \\ & + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial\varphi^2}.\end{aligned} \tag{A.19}$$

The inhomogeneous Helmholtz equation for the tensor Green's function $\hat{G}(r, \theta, \varphi; r', \theta', \varphi')$ in this coordinate system has the form.

$$\Delta\hat{G} + k^2\hat{G} = -4\pi\hat{I}\,\frac{\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')}{r^2\sin\theta}. \tag{A.20}$$

Appendix B

Components of the Green's Tensor in the Spherical Coordinate System

According to (1.43), the tensor Green's function can be written as,

$$\begin{aligned}\hat{G} = (r, \theta, \varphi; r', \theta', \varphi') = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \{ & \vec{F}_{nm}(r; r', \theta', \varphi') \otimes [\nabla_* \chi_{nm}(\theta, \varphi), \vec{r}^0] \\ & + \vec{G}_{nm}(r; r', \theta', \varphi') \otimes \nabla_* \chi_{nm}(\theta, \varphi) \\ & + \vec{H}_{nm}(r; r', \theta', \varphi') \otimes \vec{r}^0 \chi_{nm}(\theta, \varphi)\}\end{aligned} \tag{B.1}$$

where symbol $\otimes$ denotes the tensor multiplication,

$$\nabla_* \chi_{nm} = \frac{\partial \chi_{nm}}{\partial \theta} \vec{\theta}^0 + \frac{1}{\sin\theta}\frac{\partial \chi_{nm}}{\partial \varphi} \vec{\varphi}^0,$$
$$\chi_{nm}(\theta, \varphi) = (a_{nm}\cos m\varphi + b_{nm}\sin m\varphi) p_n^m(\cos\theta),$$

a_{nm} and b_{nm} are unknown constant coefficients of tesseral harmonics of degree n and order m.

To determine the vector functional coefficients $\vec{F}_{nm}(r; r', \theta', \varphi')$, $\vec{G}_{nm}(r; r', \theta', \varphi')$, and $\vec{H}_{nm}(r; r', \theta', \varphi')$, we substitute Green's function (B.1) into Eq. (A.20). The operator divgrad can be represented as the sum of radial Δ_r and angular Δ_φ operators.

$$\operatorname{divgrad} F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\Delta_\varphi F = \Delta_r F + \frac{1}{r^2}\Delta_\varphi F. \tag{B.2}$$

By changing the order of summation and differentiation, and interchanging the sequence of operators ∇_* and Δ_φ, we arrive at the following expression.

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$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)\vec{F}_{nm}(r;r',\theta',\varphi')\otimes\left[\nabla_*\chi_{nm}(\theta,\varphi),\vec{r}^0\right]\right.$$
$$+\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)\vec{G}_{nm}(r;r',\theta',\varphi')\otimes\nabla_*\chi_{nm}(\theta,\varphi)+\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)$$
$$\left.\times\vec{H}_{nm}(r;r',\theta',\varphi')\otimes\vec{r}^0\chi_{nm}(\theta,\varphi)\right\}=-4\pi\hat{I}\frac{\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')}{r^2\sin\theta}. \tag{B.3}$$

Here we take into account the Eq. (1.39).

Let us combine constant and unknown vector functional coefficients, and introduce the notation.

$$\begin{aligned}
\overrightarrow{F^a_{nm}}(r;r',\theta',\varphi')&=a_{nm}\vec{F}_{nm}(r;r',\theta',\varphi'),\\
\overrightarrow{F^b_{nm}}(r;r',\theta',\varphi')&=b_{nm}\vec{F}_{nm}(r;r',\theta',\varphi'),\\
\overrightarrow{G^a_{nm}}(r;r',\theta',\varphi')&=a_{nm}\vec{G}_{nm}(r;r',\theta',\varphi'),\\
\overrightarrow{G^b_{nm}}(r;r',\theta',\varphi')&=b_{nm}\vec{G}_{nm}(r;r',\theta',\varphi'),\\
\overrightarrow{H^a_{nm}}(r;r',\theta',\varphi')&=a_{nm}\vec{H}_{nm}(r;r',\theta',\varphi'),\\
\overrightarrow{H^b_{nm}}(r;r',\theta',\varphi')&=b_{nm}\vec{H}_{nm}(r;r',\theta',\varphi'),
\end{aligned} \tag{B.4}$$

$$P_{nm}(\theta)=(n-m+1)p^m_{n+1}(\cos\theta)-(n+1)\cos\,\theta\,P^m_n(\cos\theta)=\sin\theta\frac{dP^n_m(\cos\theta)}{d\theta}.$$

Then the equality (B.3) can be represented as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)\overrightarrow{F^a_{nm}}(r;r',\theta',\varphi')\otimes\left\{\begin{matrix}-\frac{m\sin m\phi}{\sin\theta}P^m_n(\cos\theta)\vec{\theta}^0\\-\frac{\cos m\varphi}{\sin\theta}P_{nm}(\theta)\vec{\varphi}^0\end{matrix}\right\}$$
$$+\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)\overrightarrow{F^b_{nm}}(r;r',\theta',\varphi')\otimes\left\{\frac{m\cos m\varphi}{\sin\theta}P^m_n(\cos\theta)\vec{\theta}^0-\frac{\sin m\varphi}{\sin\theta}P_{nm}(\theta)\vec{\varphi}^0\right\}$$
$$+\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)\overrightarrow{G^a_{nm}}(r;r',\theta',\varphi')\otimes\left\{\frac{\cos m\varphi}{\sin\theta}P_{nm}(\theta)\vec{\theta}^0-\frac{m\sin m\varphi}{\sin\theta}P^m_n\cos(\theta)\vec{\varphi}^0\right\}$$
$$+\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)\overrightarrow{G^b_{nm}}(r;r',\theta',\varphi')\otimes\left\{\frac{\sin m\varphi}{\sin\theta}P_{nm}(\theta)\vec{\theta}^0-\frac{m\cos m\varphi}{\sin\theta}P^m_n\cos(\theta)\vec{\varphi}^0\right\}$$
$$+\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)\overrightarrow{H^a_{nm}}(r;r',\theta',\varphi')\otimes\left\{\vec{r}^0\cos m\varphi P^m_n(\cos\theta)\right\}$$
$$+\left(\Delta_r+k^2-\frac{n(n+1)}{r^2}\right)\overrightarrow{H^b_{nm}}(r;r',\theta',\varphi')\otimes\left\{\vec{r}^0\sin m\varphi P^m_n(\cos\theta)\right\}$$
$$=-4\pi\hat{I}\frac{\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')}{r^2\sin\theta}. \tag{B.5}$$

Let us introduce new functions,

$$\chi^{(1)}_{\mu\nu}(\theta,\varphi) = \cos\mu\varphi P^{\mu}_{\nu}(\cos\theta), \quad \chi^{(2)}_{\mu\nu}(\theta,\varphi) = \sin\mu\varphi P^{\mu}_{\nu}(\cos\theta),$$

which indexes μ and ν coincide with m and n, and define a set of vector functions similar to (1.38). Then, we apply the tensor multiplication to the left and right hand side of the Eq. (B.5) by the testing vector functions and integrate each equality over the intervals $0 \le \varphi \le 2\pi$ and $0 \le \theta \le \pi$. Using the orthogonality conditions of the vector eigenfunctions (1.44), we obtain the equations for determining the unknown vector coefficients:

$$\begin{aligned}
&\frac{\partial}{\partial r}\left(r^2\frac{\partial\overrightarrow{F^a_{nm}}(r;r',\theta',\varphi')}{\partial r}\right) + \left[k^2 - \frac{n(n+1)}{r^2}\right]\overrightarrow{F^b_{nm}}(r;r',\theta',\varphi') \\
&= \frac{2\pi(2-\delta_{om})\delta(r-r')}{n(n+1)C_{nm}\sin\theta'}\left[m\ \sin m\,\varphi' P^m_n(\cos\ \theta')\vec{\varphi}^0 + \cos m\,\phi' P_{nm}(\theta')\vec{\varphi}^0\right], \\
&\frac{\partial}{\partial r}\left(r^2\frac{\partial\overrightarrow{F^b_{nm}}(r;r',\theta',\varphi')}{\partial r}\right) + \left[k^2 - \frac{n(n+1)}{r^2}\right]\overrightarrow{F^b_{nm}}(r;r',\theta',\varphi') \\
&= \frac{4\pi\delta(r-r')}{n(n-1)C_{nm}\sin\theta'}\left[-m\ \cos m\,\varphi' P^m_n\left(\cos\vec{\theta}^0\right) + \sin m\,\varphi' P_{nm}(\theta')\vec{\varphi}^0\right], \\
&\frac{\partial}{\partial r}\left(r^2\frac{\partial\overrightarrow{G^a_{nm}}(r;r',\theta',\varphi')}{\partial r}\right) + \left[k^2 - \frac{n(n+1)}{r^2}\right]\overrightarrow{G^a_{nm}}(r;r',\theta',\varphi') \\
&= \frac{2\pi(2-\delta_{om})\delta(r-r')}{n(n+1)C_{nm}\sin\theta'}\left[-\cos m\varphi' P_{nm}(\theta')\vec{\theta}^0 + m\ \sin m\varphi' P^m_n(\cos\theta')\vec{\varphi}^0\right], \\
&\frac{\partial}{\partial r}\left(r^2\frac{\partial\overrightarrow{G^b_{nm}}(r;r',\theta',\varphi')}{\partial r}\right) + \left[k^2\frac{n(n+1)}{r^2}\right]\overrightarrow{G^b_{nm}}(r;r',\theta',\varphi') \\
&= -\frac{4\pi\delta(r-r')}{n(n-1)C_{nm}\sin\theta'}\left[\sin m\varphi' P^m_n(\theta')\vec{\theta}^0 + m\ \cos m\,\varphi' P^m_n\right] \\
&\frac{\partial}{\partial r}\left(r^2\frac{\partial\overrightarrow{H^a_{nm}}(r;r',\theta',\varphi')}{\partial r}\right) + \left[k^2\frac{n(n+1)}{r^2}\right]\overrightarrow{H^a_{nm}}(r;r',\theta',\varphi') \\
&= \frac{-2\pi(2-\delta_{\mathrm{om}})\delta(\mathrm{r}-\mathrm{r}')}{C_{nm}}\cos m\varphi' P^m_n(\cos\theta')\vec{\mathrm{r}}^0, \\
&\frac{\partial}{\partial r}\left(r^2\frac{\partial\overrightarrow{H^b_{nm}}(r;r',\theta',\varphi')}{\partial r}\right) + \left[k^2\frac{n(n+1)}{r^2}\right]\overrightarrow{H^b_{nm}}(r;r',\theta',\varphi') \\
&= \frac{-4\pi\delta(r-r')}{C_{nm}}\sin m\varphi' P^m_n(\cos\theta')\vec{r}^0, \quad C_{nm} = \frac{2\pi}{2n+1}\frac{(n+m)!}{n-m!}.
\end{aligned} \tag{B.6}$$

Then we represent the unknown vector coefficients as the products,

$$\begin{aligned}
\overrightarrow{F_{nm}^{a}}(r;r',\theta',\varphi') &= \overrightarrow{F_{nm}^{a}}(\theta',\varphi')f_{nm}^{a}(r,r'),\\
\overrightarrow{F_{nm}^{b}}(r;r',\theta',\varphi') &= \overrightarrow{F_{nm}^{b}}(\theta',\varphi')f_{nm}^{b}(r,r'),\\
\overrightarrow{G_{nm}^{a}}(r;r',\theta',\varphi') &= \overrightarrow{G_{nm}^{a}}(\theta',\varphi')g_{nm}^{a}(r,r'),\\
\overrightarrow{G_{nm}^{b}}(r;r',\theta',\varphi') &= \overrightarrow{G_{nm}^{b}}(\theta',\varphi')g_{nm}^{b}(r,r'),\\
\overrightarrow{H_{nm}^{a}}(r;r',\theta',\varphi') &= \overrightarrow{H_{nm}^{a}}(\theta',\varphi')h_{nm}^{a}(r,r'),\\
\overrightarrow{H_{nm}^{b}}(r;r',\theta',\varphi') &= \overrightarrow{H_{nm}^{b}}(\theta',\varphi')h_{nm}^{b}(r,r'),
\end{aligned} \tag{B.7}$$

Substituting (B.7) in (B.6) we can immediately determine that,

$$\begin{aligned}
\overrightarrow{F_{nm}^{a}}(\theta',\varphi') &= \tfrac{2-\delta_{om}}{2n(n+1)C_{nm}\sin\theta'}\left[m\ \sin m\varphi' P_n^m(\cos\theta')\vec{\theta}^0 + \cos m\varphi' P_{nm}(\theta')\vec{\varphi}^0\right],\\
\overrightarrow{F_{nm}^{b}}(\theta',\varphi') &= \tfrac{1}{n(n+1)C_{nm}\sin\theta'}\left[-m\ \cos m\varphi' P_{nm}(\cos\theta')\vec{\theta}^0 + \sin m\varphi' P_{nm}(\theta')\vec{\varphi}^0\right],\\
\overrightarrow{G_{nm}^{a}}(\theta',\varphi') &= \tfrac{2-\delta_{om}}{2n(n+1)C_{nm}\sin\theta'}\left[-\cos m\varphi' P_n^m(\theta')\vec{\theta}^0 + m\ \sin m\varphi' P_n^m(\cos\theta')\vec{\varphi}^0\right],\\
\overrightarrow{G_{nm}^{b}}(\theta',\varphi') &= \tfrac{1}{n(n+1)C_{nm}\sin\theta'}\left[\sin m\varphi' P_{nm}(\theta')\vec{\theta}^0 + m\ \cos m\varphi' P_n^m(\cos\theta')\vec{\varphi}^0\right],\\
\overrightarrow{H_{nm}^{a}}(\theta',\varphi') &= -\tfrac{(2-\delta_{om})\cos m\varphi'}{2C_{nm}}P_n^m(\cos\theta')\vec{r}^0,\\
\overrightarrow{H_{nm}^{b}}(\theta',\varphi') &= -\tfrac{\sin m\varphi'}{2C_{nm}}P_n^m(\cos\theta')\vec{r}^0,
\end{aligned} \tag{B.8}$$

and all functions $f_{nm}^{a}(r,r'), f_{nm}^{b}(r,r'), g_{nm}^{a}(r,r'), g_{nm}^{b}(r,r'), h_{nm}^{a}(r,r'), h_{nm}^{b}(r,r')$, satisfy the equation,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial U_{nm}(r,r')}{\partial r}\right) + \left[k^2 - \frac{n(n+1)}{r^2}\right]U_{nm}(r,r')r^2 = 4\pi\delta(r-r'). \tag{B.9}$$

Using these results, we can write expression (B.1) as,

$$\hat{G}(r,\theta,\varphi;r',\theta',\varphi') = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{\overrightarrow{F^a_{nm}}(\theta',\varphi')f^a_{nm}(r,r')\otimes\left[-\frac{m\,\sin\,m\varphi}{\sin\theta}P^m_n(\cos\theta)\vec{\theta}^0\right.\right.$$
$$\left.-\frac{\cos m\varphi}{\sin\theta}P_{nm}(\theta)\vec{\varphi}^0\right] + \overrightarrow{F^b_{nm}}(\theta',\varphi')f^b_{nm}(r,r')\otimes\left[\frac{m\,\cos m\varphi}{\sin\theta}P^m_n(\cos\theta)\vec{\theta}^0\right.$$
$$\left.-\frac{\sin m\varphi}{\sin\theta}P_{nm}(\theta)\vec{\varphi}^0\right] + \overrightarrow{G^a_{nm}}(\theta',\varphi')g^a_{nm}(r,r')\otimes\left[\frac{\cos m\varphi}{\sin\theta}P_{nm}(\theta)\vec{\theta}^0\right.$$
$$\left.-\frac{m\,\sin m\varphi}{\sin\theta}P^m_n(\cos\theta)\vec{\varphi}^0\right] + \overrightarrow{G^b_{nm}}(\theta',\varphi')g^b_{nm}(r,r')\otimes\left[\frac{\sin m\varphi}{\sin\theta}P_{nm}(\theta)\vec{\theta}^0\right.$$
$$\left.+\frac{m\,\cos m\varphi}{\sin\varphi}P^m_n(\cos\theta)\vec{\varphi}^0\right] + \overrightarrow{H^a_{nm}}(\theta',\varphi')h^a_{nm}(r,r')\otimes\left[\cos m\varphi P^m_n(\cos\theta)\vec{r}^0\right]$$
$$\left.+\overrightarrow{H^b_{nm}}(\theta',\varphi')h^b_{nm}(r,r')\otimes\left[\sin m\,\phi\,P^m_n(\cos\theta)\vec{r}^0\right]\right\}. \tag{B.10}$$

Using the tensor multiplication of vectors (see [7, 8] in Chap. 1), we can write the expression for the components of the tensor Green's function,

$$\hat{G}(r,\theta,\varphi;r',\theta',\varphi') = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix} \tag{B.11}$$

Thus,

$$\begin{aligned}
\hat{G}_{11}(r,\theta,\varphi;r',\theta',\varphi') &= -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2-\delta_{\mathrm{om}}}{2C_{nm}}P^m_n(\cos\theta)P^m_n(\cos\theta') \\
&\quad\times\left[h^a_{nm}(r,r')\cos m\,\varphi\,\cos m\,\varphi' + h^b_{nm}(r,r')\sin m\,\varphi\sin m\,\varphi'\right], \\
\hat{G}_{12}(r,\theta,\varphi;r',\theta',\varphi') &= \hat{G}_{21}(r,\theta,\varphi;r',\theta',\varphi') = 0; \\
\hat{G}_{13}(r,\theta,\varphi;r',\theta',\varphi') &= \hat{G}_{31}(r,\theta,\varphi;r',\theta',\varphi') = 0; \\
\hat{G}_{22}(r,\theta,\varphi;r',\theta',\varphi') &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2-\delta_{\mathrm{om}}}{2n(n+1)C_{nm}\sin\theta\sin\theta'}\left\{m^2P^m_n(\cos\theta)P^m_n(\cos\theta)P^m_n(\cos\theta')\right. \\
&\quad\times\left[f^a_{nm}(r,r')\sin m\,\varphi\sin m\,\varphi' + f^b_{nm}(r,r')\cos m\varphi\cos m\varphi'\right] \\
&\quad\left.+P_{nm}(\theta)P_{nm}(\theta')\left[\cos m\varphi\cos m\varphi' + g^b_{nm}(r,r')\sin m\varphi\sin m\varphi'\right]\right\}, \\
\hat{G}_{23}(r,\theta,\varphi;r',\theta',\varphi') &= -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{m}{n(n+1)C_{nm}\sin\theta\sin\theta'}\left\{P_{nm}(\theta)P^m_n(\cos\theta')\right. \\
&\quad\times\left[f^a_{nm}(r,r')\cos m\varphi\sin m\varphi' - f^b_{nm}(r,r')\sin m\varphi\cos m\varphi'\right] \\
&\quad\left.+P^m_n(\cos\theta)P_{nm}(\theta')\left[g^b_{nm}(r,r')\cos m\varphi\cos m\varphi' - g^a_{nm}(r,r')\sin m\varphi\cos m\varphi'\right]\right\} \\
\hat{G}_{32}(r,\theta,\varphi;r',\theta',\varphi') &= -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{m}{n(n+1)C_{nm}\sin\theta\sin\theta'}\left\{P^m_n(\cos\theta)P_{nm}(\theta')\right. \\
&\quad\times\left[f^a_{nm}(r,r')\sin m\varphi\cos m\varphi' - f^b_{nm}(r,r')\cos m\varphi\sin m\varphi'\right] \\
&\quad\left.+P_{nm}(\theta)P^n_m(\cos\theta')\left[g^b_{nm}(r,r')\sin m\varphi\cos m\varphi' - g^a_{nm}(r,r')\cos m\varphi\sin m\varphi'\right]\right\}, \\
\hat{G}_{33}(r,\theta,\varphi;r',\theta',\varphi') &= -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2-\delta_{om}}{2n(n+1)C_{nm}\sin\theta\sin\theta'}\left\{P_{nm}(\theta)P_{nm}(\cos\theta')\right. \\
&\quad\times\left[f^a_{nm}(r,r')\cos m\varphi\cos m\varphi' + f^b_{nm}(r,r')\sin m\varphi\sin m\varphi'\right] \\
&\quad\left.+m^2P^m_n(\cos\theta)P^m_n(\cos\theta')\left[g^a_{nm}(r,r')\sin m\varphi\sin m\varphi' + g^b_{nm}(r,r')\cos m\varphi\cos m\varphi'\right]\right\}.
\end{aligned} \tag{B.12}$$

Analysing the expressions (B.12), we can conclude that the functions $h^a_{nm}(r,r')$ and $h^b_{nm}(r,r')$, $f^a_{nm}(r,r')$ and $f^b_{nm}(r,r')$, $g^a_{nm}(r,r')$ and $g^b_{nm}(r,r')$, can be found from the Eq. (B.9) subjected to the same boundary conditions for each pair of functions, hence,

$$\begin{aligned} h_{nm}(r,r') &= h^a_{nm}(r,r') = h^b_{nm}(r,r'),\\ f_{nm}(r,r') &= f^a_{nm}(r,r') = f^b_{nm}(r,r'),\\ g_{nm}(r,r') &= g^a_{nm}(r,r') = g^b_{nm}(r,r'). \end{aligned} \tag{B.13}$$

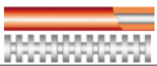
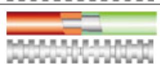
The final expressions for the components of tensor electric or magnetic Green's functions can be presented as follows.

$$\begin{aligned}
G^{e(m)}_{11}(r,\theta,\varphi;r',\theta',\varphi') &= -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(2-\delta_{om})\left(h^{e(m)}_{nm}\right)(r,r')}{2C_{nm}}P^m_n(\cos\theta)P^m_n(\cos\theta')\cos m(\varphi-\varphi'),\\
G^{e(m)}_{22} = (r,\theta,\varphi;r',\theta',\varphi') &= -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2-\delta_{om}}{2n(n+1)C_{nm}\sin\theta\sin\theta'}\Big[m^2 f^{e(m)}_{nm}(r,r')\\
&\quad\times P^m_n(\cos\theta)P^m_n(\cos\theta') + g^{e(m)}_{nm}(r,r')\sin\theta\sin\theta'\frac{dP^m_n(\cos\theta)}{d\theta}\frac{dP^m_n(\cos\theta')}{d\theta'}\Big]\\
&\quad\times\cos m(\varphi-\varphi'),\\
G^{e(m)}_{23}(r,\theta,\varphi;r',\theta',\varphi') &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{m}{n(n+1)C_{nm}}\Big[f^{e(m)}_{nm}(r,r')\\
&\quad\times\frac{dP^m_n(\cos\theta)}{d\theta}\frac{P^m_n(\cos\theta')}{\sin\theta'} + g^{e(m)}_{nm}(r,r')\frac{P^m_n(\cos\theta)}{\sin\theta}\frac{dP^m_n(\cos\theta')}{d\theta'}\Big]\sin m(\varphi-\varphi'),\\
G^{e(m)}_{32}(r,\theta,\varphi;r',\theta',\varphi') &= -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{m}{n(n+1)C_{nm}}\Big[g^{e(m)}_{nm}(r,r')\\
&\quad\times\frac{dP^m_n(\cos\theta)}{d\theta}\frac{P^m_n(\cos\theta')}{\sin\theta'} + f^{e(m)}_{nm}(r,r')\frac{P^m_n(\cos\theta)}{\sin\theta}\frac{dP^m_n(\cos\theta')}{d\theta'}\Big]\sin m(\varphi-\varphi'),\\
G^{e(m)}_{33}(r,\theta,\varphi;r',\theta',\varphi') &= -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2-\delta_{om}}{2n(n+1)C_{nm}\sin\theta\sin\theta'}\Big[m^2 g^{e(m)}_{nm}(r,r')\\
&\quad\times P^m_n(\cos\theta)P^m_n(\cos\theta') + f^{e(m)}_{nm}(r,r')\sin\theta\sin\theta'\frac{dP^m_n(\cos\theta)}{d\theta}\frac{dP^m_n(\cos\theta')}{d\theta'}\Big]\\
&\quad\times\cos m(\varphi-\varphi').
\end{aligned} \tag{B.14}$$

Appendix C

Distributed Surface Impedances of Electrically Thin Vibrators

Formulas determining the distributed surface impedance of electrically thin vibrators (material parameters are ε, μ, σ) have the following form.

No.	The vibrator design	Vibrator model	Impedance
1	Solid metal cylinder. The radius satisfy inequality $r \gg \Delta^\circ$, Δ° is skin layer thickness		$\overline{Z}_S = \frac{1+i}{120\pi\sigma\Delta^\circ}$
2	Metallized dielectric cylinder. Metal layer thickness is $h_R \ll \Delta^\circ$		$\overline{Z}_S = \frac{1}{120\pi\sigma h_R + ikr(\varepsilon-1)/2}$
3	Metal-dielectric cylinder. L_1 is the thickness of a metal disc, L_2 is the thickness of a dielectric disk		$\overline{Z}_S = -i\frac{L_1}{L_2+L_2}\frac{2}{kr\varepsilon}$
4	Magnetodielectric metalized cylinder. r_i is the radius of internal conducting cylinder		$\overline{Z}_S = \frac{1}{120\pi\sigma h_R - i/kr\mu \ln\left(\frac{r}{r_i}\right)}$
5	Metal cylinder coated with magnetodielectric layer, which thickness is $r - r_i$, or corrugated cylinder $(L_1 + L_2) \ll \lambda$, where L_1 is crests thickness where $\overline{Z}_S = 0$, L_1 is the notch width where $\overline{Z}_S \neq 0$		$\overline{Z}_S = ikr\mu \ln(r/r_i)$
			$\overline{Z}_S(s) = \overline{Z}_S \phi(s)$

(continued)

Y. M. Penkin et al., *Electromagnetic Fields Excited in Volumes with Spherical Boundaries*, Lecture Notes in Electrical Engineering 523,
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(continued)

No.	The vibrator design	Vibrator model	Impedance
6	Metal monofilar helix. r is helix radius $kr \ll 1$, Ψ is winding angle		$\overline{Z}_S = (i/2)kr\ \mathrm{ctg}^2\psi$

Formulas for surface impedances of vibrators where derive in the frame of the impedance concept ([22] in Chap. 2). They are valid for thin cylinders $\left|\left(k\sqrt{\varepsilon\mu}r\right)^2 \ln\left(k/\sqrt{\varepsilon\mu}r_i\right)\right| \ll 1$ both for finite and infinite cylinders, located in the hollow electrodynamic volume. If vibrators are in a material medium with parameters ε_1 and μ_1, all above formulas must contain the factor $\sqrt{\varepsilon_1\mu_1}$.

Appendix D

Green's Functions of Various Electrodynamic Volumes

I. **Electrical Dyadic Green's Functions**

1. The hollow half-infinite rectangular waveguide of the section $\{a \times b\}$ with the perfectly conducting walls.

$$\hat{G}^e(\vec{r},\vec{r}') = \frac{2\pi}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{k_z}\left\{(\vec{e}_x\otimes\vec{e}_{x'})\Phi_x^e(x,y;x',y')\begin{bmatrix} e^{-k_z|z-z'|} \\ -e^{-k_z(z+z')}\end{bmatrix}\right.$$
$$+(\vec{e}_y\otimes\vec{e}_{y'})\Phi_y^e(x,y;x',y')\begin{bmatrix} e^{-k_z|z-z'|} \\ -e^{-k_z(z+z')}\end{bmatrix} \qquad \text{(D.1)}$$
$$\left.+(\vec{e}_z\otimes\vec{e}_{z'})\Phi_z^e(x,y;x',y')\begin{bmatrix} e^{-k_z|z-z'|} \\ -e^{-k_z(z+z')}\end{bmatrix}\right\}.$$

where

$$\Phi_x(x,y;x',y') = \cos k_x x \cos k_x x' \sin k_y y \sin k_y y',$$
$$\Phi_y(x,y;x',y') = \sin k_x x \sin k_x x' \cos k_y y \cos k_y y',$$
$$\Phi_z(x,y;x',y') = \sin k_x x \sin k_x x' \sin k_y y \sin k_y y',$$

$\varepsilon_{m,n}\begin{cases} 1, m,n=0 \\ 2, m,n,\neq 0\end{cases}$, $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$, $k_z = \sqrt{k_x^2 + k_y^2 - k^2}$, m and n are the integer numbers, $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are the unit vectors of the rectangular coordinate system, $\hat{\mathrm{I}} = (\vec{e}_x\otimes\vec{e}_{x'}) + (\vec{e}_y\otimes\vec{e}_{y'}) + (\vec{e}_z\otimes\vec{e}_{z'})$ is the unit dyadic, and "⊗" stands for dyadic product.

Y. M. Penkin et al., *Electromagnetic Fields Excited in Volumes with Spherical Boundaries*, Lecture Notes in Electrical Engineering 523,
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2. Space outside the perfectly conducting sphere of the radius $\tilde{R}$ with the medium permittivity ε_1 and permeability μ_1 (Fig. D.1).

$$\hat{G}^e(\rho,\theta,\varphi;\rho',\theta',\varphi') = \begin{vmatrix} G^e_{\rho\rho'} & 0 & 0 \\ 0 & G^e_{\theta\theta'} & G^e_{\theta\varphi'} \\ 0 & G^e_{\varphi\theta'} & G^e_{\varphi\varphi'} \end{vmatrix}, \tag{D.2}$$

$$G^e_{\rho\rho'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\varepsilon_m h_n(\rho,\rho')}{2C_{nm}}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi-\varphi'),$$

$$G^e_{\theta\theta'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = -\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\varepsilon_m u_n(\rho,\rho')\cos m(\varphi-\varphi')}{2n(n+1)C_{nm}\sin\theta\sin\theta'}$$
$$\times\left[m^2P_n^m(\cos\theta)P_n^m(\cos\theta')+\sin\theta\sin\theta'\frac{\mathrm{d}P_n^m(\cos\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}P_n^m(\cos\theta')}{\mathrm{d}\theta'}\right]$$

$$G^e_{\theta\varphi'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{m\,u_n(\rho,\rho')\sin m(\varphi-\varphi')}{n(n+1)C_{nm}}$$
$$\times\left[\frac{dP_n^m(\cos\theta)}{\mathrm{d}\theta}\frac{P_n^m(\cos\theta')}{\sin\theta'}+\frac{P_n^m(\cos\theta)}{\sin\theta}\frac{\mathrm{d}P_n^m(\cos\theta')}{\mathrm{d}\theta'}\right],$$

$$G^e_{\varphi\theta'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = -G^e_{\theta\varphi'}(\rho,\theta,\varphi;\rho',\theta',\varphi'),$$
$$G^e_{\theta\theta'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = -G^e_{\theta\theta'}(\rho,\theta,\varphi;\rho',\theta',\varphi').$$

Here $P_n^m(\cos\theta)$ is the associated Legendre functions of the first sort,

$$C_{nm} = \frac{2\pi(n+m)!}{(2n+1)(n-m)!},$$

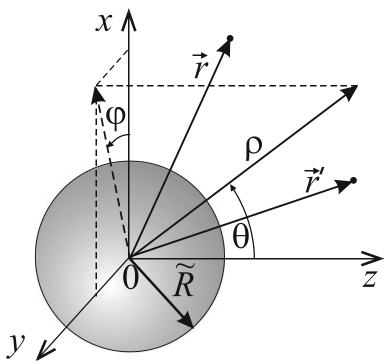

Fig. D.1 Geometry of the structure in a spherical coordinate system

$$h_p(\rho,\ \rho') = \begin{cases} 4\pi\, k_1 h_n^{(2)}(k_1\rho')\begin{bmatrix} j_n(k_1\rho)Q_n(y_n(k_1\tilde{R})) \\ -y_n(k_1\rho)Q_n(j_n(k_1\tilde{R})) \end{bmatrix}, \tilde{R} \le \rho < \rho', \\ 4\pi\, k_1 h_n^{(2)}(k_1\rho)\begin{bmatrix} j_n(k_1\rho')Q_n(y_n(k_1\tilde{R})) \\ -y_n(k_1\rho')Q_n(j_n(k_1\tilde{R})) \end{bmatrix}, \rho > \rho', \end{cases}$$

$$Q_n(\mathrm{f}_n(k_1R)) = \frac{n\mathrm{f}_n(k_1\tilde{R}) - k_1R\mathrm{f}_{n+1}(k_1\tilde{R})}{nh_n^{(2)}(k_1\tilde{R}) - k_1Rh_{n+1}^{(2)}(k_1\tilde{R})},$$

$$u_n(\rho, \rho') = \begin{cases} 4\pi\, k_1 \frac{h_n^{(2)}(k_1\rho')}{h_n^{(2)}(k_1\tilde{R})}\left[j_n(k_1\rho)y_n(k_1\tilde{R}) - y_n(k_1\rho)j_n(k_1\tilde{R})\right], \tilde{R} \le \rho < \rho', \\ 4\pi\, k_1 \frac{h_n^{(2)}(k_1\rho)}{h_n^{(2)}(k_1\tilde{R})}\left[j_n(k_1\rho')y_n(k_1\tilde{R}) - y_n(k_1\rho')j_n(k_1\tilde{R})\right], \rho > \rho', \end{cases}$$

$h_n^{(2)}(k_1\rho) = j_n(k_1\rho) - iy_n(k_1\rho) = \sqrt{\frac{\pi}{2k_1\rho}}H_{n+1/2}^{(2)}(k_1\rho)$ is the Hankel spherical function of the second kinde, $j_n(k_1\rho) = \sqrt{\frac{\pi}{2k_1\rho}}J_{n+1/2}(k_1\rho)$ and $y_n(k_1\rho) = \sqrt{\frac{\pi}{2k_1\rho}}N_{n+1/2}(k_1\rho)$ are the Bessel and Neumann spherical function, $J_{n+1/2}(k_1\rho)$ is the Bessel function, $N_{n+1/2}(k_1\rho)$ is the Neumann function and $H_{n+1/2}^{(2)}(k_1\rho)$ is the Hankel function of the second kind with the half-integer indices.

The expression for the component of the Green's function $G^e_{\rho\,\rho'}(\rho, \theta, \varphi; \rho', \theta', \varphi')$ in a more suitable form for numerical realization, can be obtained by the transition from the double to single series using the summation theorem for the Legendre polynomials.

$$G^e_{\rho\,\rho'}(\rho, \theta, \varphi; \rho', \theta', \varphi') = -\sum_{n=0}^{\infty}\frac{n+1/2}{2\pi}h_n(\rho, \rho')P_n(\cos\,\theta\,\cos\,\theta' + \sin\,\theta\,\sin\,\theta'\,\cos(\varphi - \varphi')).$$

II. **Magnetic Dyadic Green's Functions**

1. The hollow half-infinite rectangular waveguide with the cross-section $\{a \times b\}$ with the perfectly conducting walls.

$$\hat{G}^m(\vec{r}, \vec{r}') = \frac{2\pi}{ab}\sum_{m,n}\frac{\varepsilon_m\varepsilon_n}{k_z}\left\{\begin{array}{l} (\vec{e}_x \otimes \vec{e}_{x'})\Phi_x^m(x,y,x',y')\left[e^{-k_z|z-z'|} + e^{-k_z|z+z'|}\right] \\ + (\vec{e}_y \otimes \vec{e}_{y'})\Phi_y^m(x,y,x',y')\left[e^{-k_z|z-z'|} + e^{-k_z|z+z'|}\right] \\ + (\vec{e}_z \otimes \vec{e}_{z'})\Phi_x^m(x,y,x',y')\left[e^{-k_z|z-z'|} + e^{-k_z|z+z'|}\right] \end{array}\right\} \quad \text{(D.3)}$$

2. Semi-infinite rectangular waveguide with impedance $(\overline{Z}_S)$ end wall if in the case where extrinsic current sources are located on the end-wall surface.

$$\hat{G}^m(\vec{r},\vec{r}') = \frac{2\pi}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{k_z}\left\{(\vec{e}_y\otimes\vec{e}_{x'})\Phi_x^m(x,y;x',y')f_{\mathrm{II}}(k_z,\overline{Z}_S)2e^{-k_z z}\right.$$
$$\left.+(\vec{e}_y\otimes\vec{e}_{y'})\,\Phi_y^m(x,y;x',y')f_{\mathrm{II}}(k_z,\overline{Z}_S)2e^{-k_z z}\right\}, \tag{D.4}$$
$$f_{\mathrm{II}}(k_z,\overline{Z}_S) = \frac{kk_z(1+\overline{Z}_S^2)}{(ik+k_z\overline{Z}_S)(k\overline{Z}_S-ik)}.$$

3. Semi-infinite rectangular waveguide with impedance end $(\overline{Z}_S)$ excited by longitudinal extrinsic current.

$$\hat{G}^m(\vec{r},\vec{r}') = \frac{2\pi}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{k_z}\left\{(\vec{e}_z\otimes\vec{e}_{z'})\Phi_z^m(x,y;x',y')\begin{bmatrix}e^{-k_z|z-z'|}\\-f_\perp(k_z,\overline{Z}_S e^{-k_z(z-z')})\end{bmatrix}\right\},$$
$$f_\perp(k_z,\overline{Z}_S) = \frac{ik-k_z\overline{Z}_S}{ik+k_z\overline{Z}_S}. \tag{D.5}$$

4. The hollow rectangular resonator $\{a_R\times b_R\times H\}$ with perfectly conducting walls:

$$\hat{G}^m(\vec{r},\vec{r}') = \frac{2\pi}{a_R b_R}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{k_z}$$
$$\times\left\{(\vec{e}_x\otimes\vec{e}_{x'})\Phi_x^m(x,y;x',y')\left[\frac{\mathrm{ch}k_z(H-|z-z'|)+\mathrm{ch}k_z(H-|z+z'|)}{\mathrm{sh}k_z H}\right]\right.$$
$$+(\vec{e}_y\otimes\vec{e}_{y'})\Phi_y^m(x,y;x',y')\left[\frac{\mathrm{ch}k_z(H-|z-z'|)+\mathrm{ch}k_z(H-|z+z'|)}{\mathrm{sh}k_z H}\right] \tag{D.6}$$
$$\left.+(\vec{e}_z\otimes\vec{e}_{z'})\Phi_x^m(x,y;x',y')\left[\frac{\mathrm{ch}k_z(H-|z-z'|)+\mathrm{ch}k_z(H-|z+z'|)}{\mathrm{sh}k_z H}\right]\right\}$$

Where,

$$\Phi_x^m(x,y;x',y') = \sin\,k_x x\,\sin\,k_x x'\,\cos\,k_y y\,\cos\,k_y y',$$
$$\Phi_y^m(x,y;x',y') = \sin\,k_x x\,\cos\,k_x x'\,\sin\,k_y y\,\cos\,k_y y',$$
$$\Phi_z^m(x,y;x',y') = \cos\,k_x x\,\cos\,k_x x'\,\cos\,k_y y\,\cos\,k_y y'.$$

in (D.3–D.6), and remaining notations are the same as in (D.1).

5. Space outside the perfectly conducting sphere of the radius $\tilde{R}$ with the permittivity ε_1 and the permeability μ_1 of the medium (Fig. D.1):

$$\hat{G}^m(\rho,\theta,\varphi;\rho',\theta',\varphi') = \begin{vmatrix} G^m_{\rho\,\rho'} & 0 & 0 \\ 0 & G^m_{\theta\theta'} & G^m_{\theta\varphi'} \\ 0 & G^m_{\varphi\,\theta} & G^m_{\varphi\,\varphi'} \end{vmatrix}, \tag{D.7}$$

$$G^m_{\rho\,\rho^{prime}}(\rho,\theta,\varphi;\rho',\theta',\varphi') = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m h^m_n(\rho,\rho')}{2C_{nm}}P^m_n(\cos\,\theta)P^m_n(\cos\,\theta')\cos\,m(\varphi-\varphi'),$$

$$G^m_{\theta\,\theta'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m u^m_n(\rho,\rho')\cos\,m(\varphi-\varphi')}{2n(n+1)C_{nm}\sin\,\theta\,\sin\,\theta'}$$
$$\times\left[m^2P^m_n(\cos\,\theta)P^m_n(\cos\,\theta')+\sin\,\theta\,\sin\,\theta'\frac{\mathrm{d}P^m_n(\cos\,\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}P^m_n(\cos\,\theta')}{\mathrm{d}\theta'}\right],$$

$$G^m_{\theta\,\varphi'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{m\,u^m_n(\rho,\rho')\sin\,m(\varphi-\varphi')}{n(n+1)C_{nm}}$$
$$\times\left[\frac{\mathrm{d}P^m_n(\cos\,\theta)}{\mathrm{d}\theta}\frac{P^m_n(\cos\,\theta')}{\sin\,\theta'}+\frac{P^m_n(\cos\,\theta)}{\sin\,\theta}\frac{\mathrm{d}P^m_n(\cos\,\theta')}{\mathrm{d}\theta'}\right]$$

$$G^m_{\varphi\theta'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = -G^m_{\theta\varphi'}(\rho,\theta,\varphi;\rho',\theta',\varphi'),$$
$$G^m_{\varphi\varphi'}(\rho,\theta,\varphi;\rho',\theta',\varphi') = -G^m_{\theta\theta'}(\rho,\theta,\varphi;\rho',\theta',\varphi'),$$

where $h^m_n(\rho,\rho') = u^e_n(\rho,\rho')$, $u^m_n(\rho,\rho') = h^e_n(\rho,\rho')$, and remaining notations are the same as in (D.2).

Appendix E

Hertz Pseudovectors in Spherical Coordinate System

Since tensor Green's functions of Hertz vectors $\vec{\Pi}^{e(m)}(r, \theta, \phi)$ in spherical coordinates (r, θ, ϕ) can be constructed by eigenfunctions only for the classical inhomogeneous wave equation.

$$\Delta \hat{G} + k_1^2 \hat{G} = -4\pi \hat{I} \frac{\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')}{r^2 \sin \theta} \tag{E.1}$$

Hertz vectors defined by the integral operators include combinations of eigenfunctions similar to $u(r, \theta, \phi)$ and $v(r, \theta, \phi)$, which were introduced in Sect. 1.7. On the other hand, a physically correct determination of electromagnetic fields through rotor operators is not possible by direct usage of the Hertz vectors $\vec{\Pi}^{e(m)}(r, \theta, \phi)$ obtained in this way. This assertion is also confirmed by the Debye solution for the single-component Hertz vectors. Using the Debye approach, we can assume that the true Hertz vectors $\vec{\Pi}^{e(m)}(r, \theta, \phi)$ can be applied in two options, as products of the diffraction radius $(k_1 r)$ and either the true Hertz vectors $\vec{\Pi}^{e(m)}(r, \theta, \phi)$ or only their longitudinal components, i.e., as the Hertz pseudovectors:

$$\vec{\tilde{\Pi}}^{e(m)}(r, \theta, \phi) = k_1 r \Pi_r^{e(m)} \vec{r}^0 + \vec{\theta}^0 \Pi_\theta^{e(m)} + \vec{\phi}^0 \Pi_\phi^{e(m)}, \tag{E.2}$$

where $\vec{r}^0, \vec{\theta}^0, \vec{\phi}^0$ are unit vectors of the spherical coordinate system.

Let us now define operational actions, which can guarantee that the formulas for components of the electromagnetic field are coordinated among themselves, and the field components satisfy the boundary conditions. We will first analyze the second option. We can write,

Y. M. Penkin et al., *Electromagnetic Fields Excited in Volumes with Spherical Boundaries*, Lecture Notes in Electrical Engineering 523,
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$$\vec{H} = i\omega\varepsilon_1 \text{rot}(\vec{\vec{\Pi}}^e), \tag{E.3}$$

$$\frac{\vec{H}}{i\omega\varepsilon_1} = \frac{\vec{r}^0}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,\Pi^e_\phi) - \frac{\partial\Pi^e_\theta}{\partial\phi}\right] + \frac{\vec{\theta}^0}{r}\left[\frac{1}{\sin\theta}\cdot\frac{\partial(k_1 r\Pi^e_r)}{\partial\phi} - \frac{\partial}{\partial r}(r\Pi^e_\phi)\right] + \frac{\vec{\phi}^0}{r}\left[\frac{\partial}{\partial r}(r\Pi^e_\theta) - \frac{\partial(k_1 r\Pi^e_r)}{\partial\theta}\right].$$

If only one field component, $\vec{\Pi}^e_r(r,\theta,\phi)$ is specified, and $\vec{\Pi}^e_\theta(r,\theta,\phi) = \vec{\Pi}^e_\phi(r,\theta,\phi)$, we get,

$$H_r = 0, \quad H_\theta = \frac{i\omega\varepsilon_1}{r\sin\theta}\frac{\partial(k_1 r\Pi^e_r)}{\partial\phi}, \quad H_\phi = -\frac{i\omega\varepsilon_1}{r}\frac{\partial(k_1 r\Pi^e_r)}{\partial\theta}. \tag{E.4}$$

Since rot $\vec{H} = i\omega\varepsilon_1\vec{E}$, $\vec{E} = \frac{\text{rot }\vec{H}}{i\omega\varepsilon_1}$, we can write,

$$\vec{E} = \frac{\vec{r}^0}{r\sin\theta}\begin{bmatrix}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\left[\frac{\partial}{\partial r}(r\Pi^e_\theta) - \frac{\partial(k_1 r\Pi^e_r)}{\partial\theta}\right]\right) \\ -\frac{1}{r}\frac{\partial}{\partial\phi}\left[\frac{1}{\sin\theta}\cdot\frac{\partial(k_1 r\Pi^e_r)}{\partial\phi} - \frac{\partial}{\partial r}(r\Pi^e_\phi)\right]\end{bmatrix} + \frac{\vec{\theta}^0}{r}\begin{bmatrix}\frac{1}{r\sin^2\theta}\cdot\frac{\partial}{\partial\phi}\left[\frac{\partial}{\partial\theta}(\sin\theta\,\Pi^e_\phi) - \frac{\partial\Pi^e_\theta}{\partial\phi}\right] \\ -\frac{\partial}{\partial r}\left(\left[\frac{\partial}{\partial r}(r\Pi^e_\theta) - \frac{\partial(k_1 r\Pi^e_r)}{\partial\theta}\right]\right)\end{bmatrix} + \frac{\vec{\phi}^0}{r}\begin{bmatrix}\frac{\partial}{\partial r}\left(\left[\frac{1}{\sin\theta}\cdot\frac{\partial(k_1 r\Pi^e_r)}{\partial\phi} - \frac{\partial}{\partial r}(r\Pi^e_\phi)\right]\right) \\ -\frac{\partial}{\partial\theta}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,\Pi^e_\phi) - \frac{\partial\Pi^e_\theta}{\partial\phi}\right]\end{bmatrix} \tag{E.5}$$

Under these conditions we can also obtain $E_\theta = \frac{1}{r}\frac{\partial^2(k_1 r\Pi^e_r)}{\partial r\partial\theta}$, $E_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(k_1 r\Pi^e_r)}{\partial r\partial\phi}$, $E_r = \frac{\partial^2(kr\Pi^e_r)}{\partial r^2} + \omega^2\varepsilon\mu\cdot(kr\Pi^e_r)$, since,

$$\frac{k_1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Pi^e_r}{\partial\theta}\right) + \frac{k_1}{r\sin^2\theta}\frac{\partial^2\Pi^e_r}{\partial\phi^2} = -\frac{k_1}{r}\cdot\frac{\partial}{\partial r}\left(r_2\frac{\partial\Pi^e_r}{\partial r}\right) - \omega^2\varepsilon_1\mu_1(k_1 r\Pi^e_r)$$
$$= -\frac{\partial^2(k_1 r\Pi^e_r)}{\partial r^2} - \omega^2\varepsilon_1\mu_1(k_1 r\Pi^e_r).$$

Let us analyze the consistency of the boundary conditions for the electromagnetic field. For example, at the surface of the perfectly conducting sphere $E_\theta = 0$ and $E_\phi = 0$, therefore we can write,

$$\frac{\overrightarrow{\theta}^{0}}{r}\left[\frac{1}{r\sin^2\theta}\cdot\frac{\partial}{\partial\phi}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\Pi^e_\phi\right)-\frac{\partial\Pi^e_\theta}{\partial\phi}\right]-\frac{\partial}{\partial r}\left(\left[\frac{\partial}{\partial r}(r\Pi^e_\theta)-\frac{\partial(k_1 r\Pi^e_r)}{\partial\theta}\right]\right)\right]=0$$
$$\frac{1}{r^2\theta^2\sin^2\theta}\frac{\partial^2}{\partial\phi\partial\theta}\left(\sin\theta\,\Pi^e_\phi\right)-\frac{1}{r^2\sin^2\theta}\frac{\partial^2(\Pi^e_\theta)}{\partial\phi^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\Pi^e_\theta\right)+\frac{1}{r}\frac{\partial^2(k_1 r\Pi^e_r)}{\partial r\partial\theta}=0$$

Since,

$$-\frac{1}{r^2\sin^2\theta}\frac{\partial^2(\Pi^e_\theta)}{\partial\phi^2}-\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Pi^e_\theta)=\frac{1}{r^2\sin\,\theta}\frac{\partial}{\partial\theta}\left(\sin\,\theta\frac{\partial\Pi^e_\theta}{\partial\theta}\right)+\omega^2\varepsilon_1\mu_1\Pi^e_\theta$$

we arrive at,

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi\partial\theta}(\sin\,\theta\,\Pi^e_\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\,\theta\frac{\partial\Pi^e_\theta}{\partial\theta}\right)+\omega^2\varepsilon_1\mu_1\Pi^e_\theta+\frac{1}{r}\frac{\partial^2(k_1 r\Pi^e_r)}{\partial r\partial\theta}=0. \tag{E.6}$$

That is, the second derivative with respect to the radius for the transverse θ-component can be excluded using the wave equation,

$$\frac{\vec{\phi}^{0}}{r}\left[\frac{\partial}{\partial r}\left(\left[\frac{1}{\sin\,\theta}\cdot\frac{\partial(k_1 r\Pi^e_\phi)}{\partial\phi}-\frac{\partial}{\partial r}(r\Pi^e_\phi)\right]\right)-\frac{1}{r\,\sin\,\theta}\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial\theta}(\sin\,\theta\Pi^e_\phi)-\frac{\partial\Pi^e_\theta}{\partial\phi}\right]\right]=0,$$
$$\frac{1}{r\,\sin\theta}\frac{\partial^2(k_1 r\Pi^e_r)}{\partial r\partial\phi}-\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Pi^e_\phi)-\frac{\partial}{\partial\theta}\left[\frac{1}{r^2\sin\,\theta}\frac{\partial}{\partial\theta}(\sin\,\theta\,\Pi^e_\phi)\right]+\frac{1}{r^2\sin\,\theta}\frac{\partial^2\Pi^e_\theta}{\partial\theta\partial\phi}=0.$$

The term $\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Pi^e_\phi)$ can also be excluded using the wave equation. The condition can be fulfilled for any arbitrarily radii only if $k_1\frac{\partial(r\Pi^e_r)}{\partial r}=0$ and $\Pi^e_\theta=\Pi^e_\phi=0$. These conditions are consistent between the two equations. It is not difficult to see that we could not be able to exclude the second derivatives with respect to the radius for the transverse functions by using the wave equation if the first option, i.e., the product $k_1 r\vec{\Pi}^{e(m)}(r,\theta,\phi)$ is used. This means that the first option is unacceptable. Thus the boundary conditions for electric functions in the structure of the Green's function (Sect. 1.6) on the perfectly conducting sphere can be presented as,

$$\frac{\partial(rh^e_n(r,r'))}{\partial r}=0\quad\text{and}\quad u^e_n(r,r')=0. \tag{E.7}$$

Next, let us consider the fields excited by the pseudovector of the magnetic type.

$$\vec{E} = -i\omega\mu_1 \,\mathrm{rot}(\vec{\tilde{\Pi}}^m),$$

$$\frac{\vec{E}}{-i\omega\mu_1} = \frac{\vec{r}^0}{r\,\sin\,\theta}\left[\frac{\partial}{\partial\theta}(\sin\,\theta\,\Pi_\phi^m) - \frac{\partial\Pi_\theta^m}{\partial\phi}\right] + \frac{\vec{\theta}^0}{r}\left[\frac{1}{\sin}\cdot\frac{\partial(k_1 r\Pi_\phi^m)}{\partial\phi} - \frac{\partial}{\partial r}(r\Pi_\phi^m)\right]$$
$$+ \frac{\vec{\theta}^0}{r}\left[\frac{\partial}{\partial r}(r\Pi_\theta^m) - \frac{\partial(k_1 r\Pi_r^m)}{\partial\theta}\right].$$

From the conditions $E_\theta = 0$ and $E_\phi = 0$ we can determine,

$$h_n^m(r, r') = 0 \quad \text{and} \quad \frac{\partial(ru_n^m(r, r'))}{\partial r} = 0. \tag{E.8}$$

Let us also consider how the formulas $\vec{E} = (\text{grad div} + k_1^2)\vec{\Pi}^e$ and $\vec{H} = (\text{grad div} + k_1^2)\vec{\Pi}^m$ can be applied to find electromagnetic fields in the spherical coordinates. Of course, these vector formulas, as indicated in Sect. 1.7, cannot be used directly. The three-component fields can be determined only through the rotor operators of Hertz pseudovectors as,

$$\vec{E} = \mathrm{rot}\,\mathrm{rot}(\vec{\tilde{\Pi}}^e),$$
$$\vec{H} = i\omega\varepsilon_1 \,\mathrm{rot}(\vec{\tilde{\Pi}}^e),$$
$$\vec{H} = \mathrm{rot}\,\mathrm{rot}(\vec{\tilde{\Pi}}^m),$$
$$\vec{E} = i\omega\varepsilon_1 \,\mathrm{rot}(\vec{\tilde{\Pi}}^m).$$

However, it is of interest to compare the expressions $\vec{E} = \mathrm{rot}\;\mathrm{rot}(\vec{\tilde{\Pi}}^e)$ and $\vec{E} = (\text{grad div} + k_1^2)\vec{\Pi}^e$ in explicit form. The first formula correspond to the expressions (E.4). To obtain the relations for the second formula, we take into account that $\vec{E} = (\text{grad div} + \omega^2\varepsilon\mu)\vec{\Pi}^e = \mathrm{rot}\;\mathrm{rot}\,\vec{\Pi}^e$ and write,

$$\vec{E} = (\text{grad div} + k_1^2)\vec{\Pi}^e = \mathrm{rot}\;\mathrm{rot}\,\vec{\Pi}^e = \mathrm{rot}\Bigg(\frac{\vec{r}^0}{r\,\sin\,\theta}\left[\frac{\partial}{\partial\theta}(\sin\,\theta\,\Pi_\phi^e) - \frac{\partial\Pi_\theta^e}{\partial\phi}\right]$$
$$+ \frac{\vec{\theta}^0}{r}\left[\frac{1}{\sin\,\theta}\cdot\frac{\partial\Pi_r^e}{\partial\phi} - \frac{\partial}{\partial r}(r\Pi_\phi^e)\right] + \frac{\vec{\phi}^0}{r}\left[\frac{\partial}{\partial r}(r\Pi_\theta^e) - \frac{\partial\Pi_r^e}{\partial\theta}\right]\Bigg)$$
$$= \frac{\vec{\theta}^0}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin\,\theta}{r}\left[\frac{\partial}{\partial r}(r\vec{\Pi}_\theta^e) - \frac{\partial\vec{\Pi}_r^e}{\partial\theta}\right]\right) - \frac{1}{r}\frac{\partial}{\partial\phi}\left[\frac{1}{\sin\,\theta}\cdot\frac{\partial\Pi_r^e}{\partial\phi} - \frac{\partial}{\partial r}(r\Pi_\phi^e)\right]\right]$$
$$+ \frac{\vec{\theta}^0}{r}\left[\frac{1}{r\,\sin^2\theta}\cdot\frac{\partial}{\partial\phi}\left[\frac{\partial}{\partial\theta}(\sin\,\theta\,\Pi_\phi^e) - \frac{\partial\Pi_\theta^e}{\partial\phi}\right] - \left(\left[\frac{\partial}{\partial r}(r\Pi_\theta^e) - \frac{\partial\Pi_r^e}{\partial\theta}\right]\right)\right]$$
$$+ \frac{\vec{\phi}^0}{r}\left[\frac{\partial}{\partial r}\left(\left[\frac{1}{\sin\,\theta}\cdot\frac{\partial\Pi_r^e}{\partial\phi} - \frac{\partial}{\partial r}(r\Pi_\phi^e)\right]\right)\right] - \frac{\partial}{\partial\theta}\frac{1}{r\,\sin\,\theta}\left[\begin{matrix}\frac{\partial}{\partial\theta}(\sin\,\theta\,\Pi_\phi^e)\\ -\frac{\partial\Pi_\theta^e}{\partial\phi}\end{matrix}\right]\Bigg] \tag{E.9}$$

The transverse components of the electromagnetic field under condition that the radial component is equal to zero have the following form.

$$\begin{aligned}&\frac{\vec{\theta}^0}{r}\left[\frac{1}{r\sin^2\theta}\cdot\left[\frac{\partial}{\partial\theta}\left(\sin\frac{\partial}{\partial\phi}\Pi^e_\phi\right)-\frac{\partial^2\Pi^e_\theta}{\partial\phi^2}\right]-\frac{\partial^2(r\Pi^e_\theta)}{\partial r^2}\right]\\&-\frac{\vec{\phi}^0}{r}\left[\frac{\partial^2(r\Pi^e_\phi)}{\partial r^2}+\frac{\partial}{\partial\theta}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,\Pi^e_\phi)-\frac{\partial\Pi^e_\theta}{\partial\phi}\right]\right].\end{aligned} \tag{E.10}$$

The same expressions can be also obtained from (E.5).

$$\begin{aligned}\vec{E}\rightarrow&\frac{\vec{\theta}^0}{r}\left[\frac{1}{r\sin^2\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\phi}\Pi^e_\phi\right)-\frac{\partial^2\Pi^e_\theta}{\partial\phi^2}\right]-\frac{\partial^2(r\Pi^e_\theta)}{\partial r^2}\right]\\&-\frac{\vec{\phi}^0}{r}\left[\frac{\partial^2(r\Pi^e_\phi)}{\partial r^2}+\frac{\partial}{\partial\theta}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,\Pi^e_\phi)-\frac{\partial\Pi^e_\theta}{\partial\phi}\right]\right].\end{aligned} \tag{E.11}$$

As one might suppose, the expressions (E.10) and (E.11) turn out to be identical. Thus, if only the transverse components of the Hertz vector are considered, i.e., the Hertz vector has no radial component, then the formula $\vec{E}=(\text{grad div}+k_1^2)\vec{\Pi}^e$ is conditionally correct and can be used to determine the electromagnetic fields. Of course, this conclusion also applies to the formula $\vec{H}=(\text{grad div }+k_1^2)\vec{\Pi}^m$.