Appendix A
Conventions and Notation Concerning Differential Forms and $U(1)$ Connections

This chapter is meant for definition purposes and as a short review. A thorough and more pedagoigcal explination is out of the scope of this text, but there are several good references one can consult. One of the most popular, among physicists, is Ref. [1].

A.1 Differential Forms

One-forms are fields on a manifold that specify a flow, and they can be considered as maps from curves to real numbers. (Remember that when we make reference to a curve (or other manifold or submanifold) an orientation is always implicit, so also here.) In index notation, a one-form is written as a set of real numbers, here labeled by μ, $a_\mu(X)$, where a specific basis $\{x^\mu\}_{\mu=1,\dots,\mathscr{d}}$ is in mind. The map, from curves to the real numbers, is

$$\Gamma \to \sum_i \int a_i \frac{dx^i(s)}{ds} ds, \tag{A.1}$$

where

$$s \to X(s) \doteq (x^1(s), x^2(s), \dots) \tag{A.2}$$

is a parametrization of Γ. We will simply write

$$a = a_1 dx^1 + a_2 dx^2 + \cdots \tag{A.3}$$

without reference to any coordinate system, and the above equation is simply written as $\Gamma \to \int_\Gamma a$. Completely analogous, two-forms are maps from surfaces to reals and are denoted by

$$F = F_{12} dx^1 \wedge dx^2 + F_{13} dx^1 \wedge dx^3 + \cdots . \tag{A.4}$$

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Here, $dx^1 \wedge dx^2 = -dx^2 \wedge dx^1$ and, just as the notation implies, the wedge can be interpreted as a product of two one-forms. The map from surfaces $\{\mathcal{S}\}$ to reals is then given by

$$\mathcal{S} \to \int_{\mathcal{S}} F = \sum_{ij} \int \int_a^b F_{ij} \left(\frac{\partial x^i(s,t)}{\partial s} \frac{\partial x^j(s,t)}{\partial t} - \frac{\partial x^j(s,t)}{\partial s} \frac{\partial x^i(s,t)}{\partial t} \right) dsdt, \tag{A.5}$$

where

$$(s,t) \to X(s,t) \doteq (x^1(s,t), x^2(s,t), \ldots) \tag{A.6}$$

is a parametrization of $\mathcal{S}$. Analogous definitions hold for higher-order forms.

A.2 Normalization

We will reserve the letters a and b for $U(1)$ connections. A $U(1)$ connection specifies an element in $U(1)$ for each curve. These elements are given by integrals, e.g.,

$$e^{i \int_\Gamma a} \in U(1). \tag{A.7}$$

The $U(1)$ connections could thus be thought of as one-forms up to gauge transformations

$$a \to a + i\xi^* d\xi, \tag{A.8}$$

where ξ is a function from the manifold to unit complex numbers. The usual electromagnetic vector potential is also a $U(1)$ connection and the same conventions apply. In other words we will absorb a factor of $\frac{2\pi}{\phi_0}$ in the definition of A (ϕ_0 is the flux quantum, $\phi_0 = h/e$).

Appendix B
Definitions

As a reminder: we always implicitly assume an orientation of curves, surfaces and manifolds in general. In other words, "S is a surface" is short for "S is a surface with a given orientation".

When discussing space-time we will, like in the rest of the thesis (unless otherwise specified), assume that there at least implicity is a notion of absolute rest and absolute time.

Definition 1 (*algebraic definition*) *The exterior derivative*
A lower case d denotes the exterior derivative. Acting on functions it gives a one-form,

$$df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 \dots \tag{B.1}$$

and on forms it is defined through the recursive relation

$$d(df) = 0; \qquad\qquad d(\mathfrak{a} \wedge \mathfrak{b}) = d\mathfrak{a} \wedge \mathfrak{b} + (-1)^p \mathfrak{a} \wedge d\mathfrak{b}.$$

Example The exterior derivative When d acts on the one-form $a = B(r)d\theta$ (where r and θ are coordinates) we get

$$da = \frac{\partial B}{\partial r}dr \wedge d\theta - B(r)d(d\theta) = \frac{\partial B}{\partial r}dr \wedge d\theta. \tag{B.2}$$

Definition 1, is usually not the best way to think of the exterior derivative. Instead, the following, equivalent definition, is better:

Definition 2 (*geometric definition*) *The exterior derivative*
The exterior derivative of a k-form $\mathfrak{a}$ is the unique $k+1$ form $d\mathfrak{a}$ which for all $k+1$ dimensional submanifolds $\mathfrak{m}$ uphold

$$\int_{\mathfrak{m}} d\mathfrak{a} = \int_{\partial \mathfrak{m}} \mathfrak{a} \ . \tag{B.3}$$

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Definition 3 *The functional derivative*
Say that we have a local functional $S[a]$ of a k-form field a in a $\mathcal{D}$ dimensional manifold. If there exits a unique $\mathcal{D}-k$ form field $\frac{\delta S[a]}{\delta a}$ which makes the equality

$$\lim_{\epsilon \to 0} \frac{S[a+\epsilon\eta]-S[a]}{\epsilon} = \int \eta \wedge \frac{\delta S[a]}{\delta a} \tag{B.4}$$

hold for all k-form fields η, we say that the functional derivative of $S[a]$ with respect to a is that field.

Definition 4 *The Poincaré dual*
A Poincaré dual of a k-dimensional submanifold m_k is denoted by $\mathcal{P}(m_k)$ and is defined such that for all l-dimensional ($l \geq k$) submanifolds m_l and k-forms a, the following equality holds:

$$\int_{m_l} \mathcal{P}(m_k) \wedge a = \int_{m_l \cap m_k} a. \tag{B.5}$$

Lemma 5 *An explicit form of Poincaré duals*
If we consider m_k as a subset of a $\mathcal{D}$-dimensional manifold $\mathcal{M}$, it is straight forward to write an explicit form of the Poincaré duals. Say that m_k is defined implicitly by the equations $f_1(X) = f_2(X) = \cdots = f_{D-k}(X) = 0$ and $f_{\mathcal{D}-k+1}(X) > 0$. Then the Poincaré dual is given by

$$\mathcal{P}(m_k) = \Theta(f_{D-k+1})\,\delta(f_1)\delta(f_2)\cdots df_1 \wedge df_2 \wedge df_3 \cdots, \tag{B.6}$$

where Θ denotes the Heavyside step function. (The implicit equations do not determine an orientation for m_k but an orientation is implied by the order of the functions, $f_1, f_2, \ldots$.)

Corollary 6 *The exterior derivative of a Poincaré dual*
Using the above form of the Poincaré duals, we get

$$\begin{aligned} d\mathcal{P}(m_k) &= d\Theta(f_{D-k+1})\,\delta(f_1)\delta(f_2)\cdots df_1 \wedge df_2 \cdots \\ &= (-1)^p \delta(f_1)\delta(f_2)\cdots df_1 \wedge df_2 \wedge df_3 \cdots = (-1)^p \mathcal{P}(\partial m_k). \end{aligned} \tag{B.7}$$

Example 7 The linking number
In a three-dimensional manifold $\mathcal{M}$,

$$\int_{\mathcal{M}} \mathcal{P}(\Gamma) \wedge \mathcal{P}(S) \tag{B.8}$$

equals the number of times the curve Γ passes through S in the positive direction. So, the linking number between ∂S and Γ is given by the above integral.

Definition 8 (*algebraic definition*) *The Hodge isomorphism*
In a $\mathscr{d}$-dimensional oriented manifold equipped with a metric h, there is a natural map from k-forms to $\mathscr{d} - k$-forms, called the *Hodge isomorphism*. The image of a k-form $\mathscr{a}$ is denoted $\star_h \mathscr{a}$ and is the unique $\mathscr{d} - k$-form, such that the equality

$$\mathscr{b} \wedge \star_h \mathscr{a} = \langle \mathscr{b}, \mathscr{a} \rangle_h V$$

holds for all k-forms $\mathscr{b}$. Above, V denotes the volume form (which is defined by the metric and the orientation), *i.e.,* written in terms of a coordinate system $\{x^\mu\}_{\mu=1,\ldots,\mathscr{d}}$,

$$V = \sqrt{\det\left(h_{\mu\nu}\right)} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \tag{B.9}$$

and $\langle \cdot, \cdot \rangle_h$ denotes the inner product between k-forms defined by h, *i.e.,*

$$\langle \mathscr{b}, \mathscr{a} \rangle_h \equiv h^{\mu_1 \nu_1} h^{\mu_1 \nu_1} \cdots h^{\mu_k \nu_k} \mathscr{b}_{\mu_1, \mu_2, \ldots, \mu_k} \mathscr{a}_{\nu_1, \nu_2, \ldots, \nu_k}. \tag{B.10}$$

Definition 8 is good for calculations, but it is not the intuitive definition one should have in mind. Let us consider the Hodge isomorphism of a two-form $\mathcal{B}$ in a $3\mathscr{d}$ manifold $\mathcal{M}$. (If it helps we can then think of $\mathcal{B}$ as the magnetic field two-form and of $\star_h \mathcal{B}$ as the magnetic field co-vector.)

The one-form $\star_h \mathcal{B}$ then points in the direction of the positive normals of the surfaces with the largest $\mathcal{B}$-flux through them, and the modulus of $\star_h \mathcal{B}$ is given by the flux density of $\mathcal{B}$.

Let us make this a bit more precise: In $\mathscr{d} = 3$, a surface S has a normal direction which can be written in terms of a one-form field n_S, which has the property that

$$\int_{\Gamma_\parallel} n_S = 0 \tag{B.11}$$

for any curve $\Gamma_\parallel$ in the surface S. This only specifies n_S up to multiplication of a scalar field, but we also define n_S such that it points in the positive direction of S. Then, n_S is defined up to a multiplication of a positive scalar field. (Since S and $\mathcal{M}$ have an orientation, we have a prescription of when a curve $\Gamma_\perp$, which intersects S at one point, passes through S in a positive direction (see Definition 9). We define n_S such that the integral

$$\int_{\Gamma_\perp} n_S \tag{B.12}$$

is positive for short enough curves $\Gamma_\perp$, which pass through S in a positive direction.)

Since we have a metric, all surfaces have a well defined area and boundary length. If we now consider surfaces $\{s\}$ with sufficiently small area and boundary length, the integral $\int_s \mathcal{B}$ will only depend on the normal $n_s(X)$ of s at X and area(s), *i.e.,* the area of s. The direction of $\star_h \mathcal{B}$ is the positive normal to the surface with the largest value of $\int_s \mathcal{B}/\text{area}(s)$, and the modulus of $\star_h \mathcal{B}$ is given by $\int_s \mathcal{B}/\text{area}(s)$ for the same surface.

In Definition 10, we make it apply to forms of all orders. But, to do so, we first need to introduce the notion of a normal form of a submanifold.

Definition 9 *The normal form of a submanifold*
In a manifold $\mathcal{M}$ with dimension $\mathscr{d}$, a k-dimensional submanifold m has a normal form n_m of order $\mathscr{d} - k$. The normal form is defined such that

$$\int_{m_\parallel} n_m = 0 \tag{B.13}$$

for all $\mathscr{d} - k$ submanifolds $m_\parallel$, where either $m_\parallel \subset m$ or there exists an open subset $\mathcal{N}_m \subset m$ where $\mathcal{N}_m \subset m_\parallel$. The direction n_m is such that $\int_{m_\perp} n_m > 0$ for a $\mathscr{d} - k$ submanifold $m_\perp$ with an orientation such that the wedge product $V_m \wedge V_{m_\perp}$ between the volume form V_m of m and the volume form $V_{m_\perp}$ of $m_\perp$ is, up to a positive scalar field, the volume form of $\mathcal{M}$.

Definition 10 (*geometric definition*) *The Hodge isomorphism*
In a $\mathscr{d}$-dimensional oriented manifold equipped with a metric h, there is a natural map from k-forms to $\mathscr{d} - k$-forms, called the *Hodge isomorphism*. The image of a k-form a is called the *Hodge dual* and is denoted $\star_h a$.

Consider a set $S_{\epsilon,X}$ of k-dimensional, open balls parametrized by the ball radius ϵ and the center point X. By $B_{\epsilon,X}$ we denote a ball in $S_{\epsilon,X}$ with the property

$$\int_{B_{\epsilon,X}} a \geq \int_B a; \forall B \in S_{\epsilon,X}, \tag{B.14}$$

and by n_ϵ we denote the normal $\mathscr{d} - k$-form of $B_{\epsilon,X}$ that at X has the modulus $\int_{B_{\epsilon,X}} a/\text{area}(B_{\epsilon,X})$.

The Hodge dual $\star_h a$ is

$$\star_h a = \lim_{\epsilon \to 0} n_\epsilon. \tag{B.15}$$

Fact *Hodge dual and wedge products*

$$\int a \wedge \star_h b = \int \star_h a \wedge b \tag{B.16}$$

Fact *Inverse Hodge isomorphism*
If a is a k-form on a $\mathscr{d}$-dimensional manifold we have

$$\star_h \star_h a = (-1)^{k(\mathscr{d}-k)} a. \tag{B.17}$$

Fact *Hodge isomorphism and the Laplace operator*
The Laplace operator in a $\mathscr{d}$-dimensional manifold with metric h takes the form

$$(-1)^{\mathscr{d}+1} \star_h d \star_h d. \tag{B.18}$$

(To be clear, with Laplace operator we mean the operator which in Euclidian coordinates, i.e., where the metric has components $h_{\mu\nu} = \delta_{\mu\nu}$, takes the form $-\sum_{\mu} \partial_{\mu}\partial_{\mu}$.)

Definition 11 *Spatial slice*
A spatial slice $m(t)$ of space-time $\mathcal{M}$ is all points that correspond to a given time t.

Definition 12 *Pullback of a form*
Remember that we defined differential forms as maps from submanifolds to real numbers (e.g., a one-form is a map from curves to reals). Take two manifolds $\mathcal{M}$ and $\mathcal{N}$ and a smooth injective function ϕ from $\mathcal{M}$ to $\mathcal{N}$. Then, there is a natural mapping from the differential forms on $\mathcal{N}$ to the ones on $\mathcal{M}$, called the *pullback*. The pullback of a k-form a on $\mathcal{N}$ is given by the map

$$\mathcal{M}_k \to \int_{\phi(\mathcal{M}_k)} a \tag{B.19}$$

from k-dimensional submanifolds $\mathcal{M}_k$ in $\mathcal{M}$ to real numbers.

Definition 13 *Spatial isomorphism*
There is a natural isomorphism χ between all pairs of spatial slices. The isomorphism is defined in the following way: if a point X in $m(t)$ and Y in $m(t')$ is connected by an observer which stands still, then—when viewed as a map from $m(t)$ to $m(t')$—$\chi(X) = Y$.

This also provides an isomorphism between space-time differential forms and differential forms on spatial slices. The spatial slices are submanifolds of space-time $\mathcal{M}$, so there is the embedding map $m(t) \to \mathcal{M}$, which defines a pullback (see Definition 12) of forms in $\mathcal{M}$ to forms in $m(t)$.

All points in $\mathcal{M}$ are given by a doublet $(t, X(t))$. We can thus form a map from $\mathcal{M}$ to $m(t')$ which takes $(t, X(t))$ to $\chi(X(t))$, where χ now is interpreted as a map from $m(t)$ to $m(t')$. The pullback of this map provides a map from differential forms on the spatial slices to differential forms in space-time.

Definition 14 *Spatial form*
A spatial form is a form which maps back to itself when the spatial isomorphism is applied twice. With this isomorphism, these forms can both be viewed as forms on a spatial slices and on space-time.

When dealing with spatial forms, the spatial isomorphism will be kept implicit, but it should be clear when it is invoked.

Definition 15 *The spatial and temporal parts of a differential form*
With a notion of absolute rest we have a decomposition of any space-time k-form a into a two spatial forms: the spatial, $a^{sp.}$, and a temporal, a^{t}, part;

$$a = dt \wedge a^{t} + a^{sp.}. \tag{B.20}$$

Example 16 The electromagnetic field tensor
The spatial and temporal parts of the electromagnetic field tensor is the magnetic field two-form, $\mathcal{B}$, and electric field one-form $\mathcal{E}$, respectivly: $F = \mathcal{B} + dt \wedge \mathcal{E}$. The electric vector is defined by raising the index of $\mathcal{E}$ by the spatial metric h. The spatial Hodge dual of $\mathcal{B}$ is the magnetic field co-vector in $3\mathscr{d}$ and the magnetic field scalar in $2\mathscr{d}$.

Appendix C
Vector Bundles and Chern Numbers in Quantum Mechanics

Consider a subspace $\mathscr{h}(X)$ (of a Hilbert space $\mathcal{H}$) that varies continuously with the parameter X in some manifold $X \in \mathcal{M}$. The space

$$E = \bigcup_{X \in B} \{X\} \times h(X), \tag{C.1}$$

which is a subspace of $\mathcal{M} \times \mathcal{H}$, is called a *fiber bundle* and $\mathscr{h}(X)$ is called the *fiber* at X. When the fibers are complex vector spaces, as they are here, we use the more precise term: *complex vector bundle.* The mathematical structure of the fibre bundle can be described without the notion of the embedding space, $\mathcal{M} \times \mathcal{H}$, but for our purposes it is more practical to also use some of the extra structure that the embedding provides.

We now define the *Berry connection*, which we use to construct invariants that characterize complex vector bundles.

C.1 The Berry Connection

We start by picking a coordinate system $X \doteq \{x_\mu\}_{\mu=1,\ldots,d}$ of $\mathcal{M}$, and a basis $\{|X; i\rangle\}_{i=1,\ldots,N}$ that varies smoothly in some region of $\mathcal{M}$. We then consider a curve in E:

$$s \to (X(s), |s\rangle)\,; \qquad |s\rangle \in h(X(s)), \tag{C.2}$$

parametrized by s. We assume that the curve fits in the region of the bundle where the basis is defined, so we can write

$$|s\rangle = \sum_i \alpha_i(s)|X(s); i\rangle \tag{C.3}$$

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for some coefficients $\{\alpha_i\}_{i=1,\ldots,N}$. If $\hbar(X)$ is independent of X, there is no obstruction to having

$$0 = \frac{d}{ds}|s\rangle. \tag{C.4}$$

On the other hand, if $\hbar(X(s))$ varies with X, $|s\rangle$ has to change simply because $\hbar(X(s))$ does. The total Hilbert space has the usual notion of distance and provides a natural prescription for which vector in $\hbar(X(s+ds))$ is closest to $|s\rangle$, namely the linear projection of $|s\rangle$ onto $\hbar(X(s+ds))$. A curve that changes as little as possible is thus

$$|s+ds\rangle = P\left(X(s+ds)\right)|s\rangle. \tag{C.5}$$

Taking the inner product with $\langle X(s+ds;\alpha)|$ on both sides of the above equation gives

$$\begin{aligned}
\alpha_i\left(s+ds\right) &= \sum_j \alpha_j\left(s\right)\langle X(s+ds); i|X(s); j\rangle \\
&= \sum_j \alpha_j\left(s\right)\left(\delta^{ij} - ds\left\langle X(s); i\left|\frac{d}{ds}\right|X(s); j\right\rangle\right) \\
&\equiv \sum_j \left(\delta^{ij} - i\left[\mathbb{A}_\mu\right]^{ij}\frac{\partial x^\mu}{\partial s}ds\right)\alpha_j\left(s\right) ,
\end{aligned} \tag{C.6}$$

where we used $(\frac{d}{ds}\langle X(s); i|)|X(s); j\rangle = -\langle X(s); i|\frac{d}{ds}|X(s); j\rangle$. The last step defines the Berry connection,

$$[\mathbb{A}]^{ij} = -\, i\,\langle X; i|d|X; j\rangle\ , \tag{C.7}$$

where d denotes the exterior derivative in $\mathcal{M}$. (The symbol $\mathbb{A}$ (and $\mathbb{F}$ below) are matrices, and matrix multiplication is assumed. We use $[\mathbb{A}]^{ij}$ and $[\mathbb{F}]^{ij}$ to denote their components.)

C.2 The Berry Field Strength

We now want to consider basis independent properties of the Berry connection. The Berry connection relates coefficients of vectors in two different Hilbert spaces: one in $\hbar(X(s))$ and the other in $\hbar(X(s+ds))$. Thus, the matrix $\mathbb{A}$ can be taken arbitrary since we can change the basis of $\hbar(X(s))$ independently from $\hbar(X(s+ds))$.

By instead considering the smallest change along a closed curve, one gets an operator acting within a Hilbert space $\hbar(X)$. The explicit form of the matrix obtain in this way is basis dependent, but the trace of the matrix is not. We will now consider a curve that changes as little as possible around an infinitesimal closed path.

Subtracting $\alpha_j(s)$ and dividing by ds in (C.6) we can conclude that for a curve that changes as little as possible, the coefficients change according to

$$0 = \left(\frac{d}{ds} + i\mathbb{A}_\mu \frac{\partial x^\mu}{\partial s} \right) \boldsymbol{\alpha}(s), \tag{C.8}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)^T$. We now consider an infinitesimal loop gotten by moving the length dx^μ in the μ-direction, and the length dx^ν in the ν-direction, then backward in the μ-direction and finally back to where we started. For coefficients $\boldsymbol{\alpha}$ (that correspond to the smallest change along this curve) we have the equation

$$\begin{aligned} 0 &= \frac{1}{i}\Big(\left(\partial_\mu + i\mathbb{A}_\mu\right)(\partial_\nu + i\mathbb{A}_\nu) - (\partial_\nu + i\mathbb{A}_\nu)\left(\partial_\mu + i\mathbb{A}_\mu\right)\Big)\boldsymbol{\alpha}(X) \\ &= \left(\partial_\mu \mathbb{A}_\nu - \partial_\mu \mathbb{A}_\nu + i\mathbb{A}_\mu\mathbb{A}_\nu - i\mathbb{A}_\nu\mathbb{A}_\mu\right)\boldsymbol{\alpha}(X) \equiv \mathbb{F}_{\mu\nu}\boldsymbol{\alpha}(X)\,, \end{aligned} \tag{C.9}$$

where the last step defines the *Berry field strength*

$$\mathbb{F} = d\mathbb{A} + i\mathbb{A} \wedge \mathbb{A}. \tag{C.10}$$

We defined $\mathbb{F}$ specifically such that it has a basis independent trace, but it is instructive to see how this works by an explicit calculation. From the Definition (C.7) it follows, that under a coordinate transformation

$$|X; i\rangle \to \sum_j [\mathbb{U}]^{ij}\, |X; j\rangle, \tag{C.11}$$

the Berry connection transforms as

$$\mathbb{A} \to \mathbb{U}\mathbb{A}\mathbb{U}^\dagger - i\mathbb{U}^\dagger d\mathbb{U}\,, \tag{C.12}$$

while the field strength just rotates;

$$\mathbb{F} \to \mathbb{U}\mathbb{F}\mathbb{U}^\dagger. \tag{C.13}$$

Thus, because of the cyclic property of the trace, any product of the type

$$Tr\,(\mathbb{F} \wedge \mathbb{F} \cdots) \tag{C.14}$$

is basis independent.

Note that when taking the trace of the field strength, the second term,

$$\mathrm{Tr}\,(\mathbb{A} \wedge \mathbb{A})\,, \tag{C.15}$$

vanishes because of the cyclic property of the trace. We can therefore write

$$\mathrm{Tr}\left(\mathbb{F}_{\mu\nu}\right) = -i\sum_i\left(\frac{\partial}{\partial x^\mu}\left\langle X;i\left|\frac{\partial}{\partial x^\nu}\right|X;i\right\rangle - \frac{\partial}{\partial x^\nu}\left\langle X;i\left|\frac{\partial}{\partial x^\mu}\right|X;i\right\rangle\right). \tag{C.16}$$

C.3 Chern Numbers and Chern-Simons Invariants

We now have to consider the fact that there in general is no basis that varies smoothly over the entire $\mathcal{M}$. If $\mathcal{M}$ has a trivial topology (*i.e.,* that of an open ball) there is no obstruction to having a continuous basis that covers the entire $\mathcal{M}$. This is , however, not the case in general.

Let us start by studying some two-dimensional submanifold S of $\mathcal{M}$ and define

$$ch_1[\mathbb{F}](S) = \frac{1}{2\pi}\int_S \mathrm{Tr}\left(\mathbb{F}\right) \ . \tag{C.17}$$

This expression defines the first *Chern number*, ch_1, and the integrand (including the prefactor) is called the first *Chern character*. What we want to show in this section is that the first Chern number of a closed surface is an integer.

If S have a trivial topology, by using $\mathrm{Tr}(\mathbb{A}\wedge\mathbb{A}) = 0$ (which is true because of the cyclic property of the trace) we can write $\mathrm{Tr}\left(\mathbb{F}\right) = d\mathrm{Tr}\left(\mathbb{A}\right)$. This allows us to use Stokes theorem to rewrite the Chern number as

$$\frac{1}{2\pi}\int_{\partial S}\mathrm{Tr}\left(\mathbb{A}\right). \tag{C.18}$$

Let us now calculate the first Chern number of a sphere[1] S. To evaluate the integral in (C.17), we partition the sphere into one region S_{big} that covers almost the full area, and a very small region S_{small} around the south pole. We assume that we can take the region S_{small} to be arbitrarily small, so that its contribution to the integral can be neglected. That is,

$$\frac{1}{2\pi}\int_S \mathrm{Tr}\left(\mathbb{F}\right) = \frac{1}{2\pi}\int_{S_{big}}\mathrm{Tr}\left(\mathbb{F}\right) \ . \tag{C.19}$$

Since the region S_{big} has a trivial topology, we can choose a basis that is smooth in the whole region and use Stokes theorem to get,

$$\frac{1}{2\pi}\int_S \mathrm{Tr}\left(\mathbb{F}\right) = \frac{1}{2\pi}\int_{S_{big}}\mathrm{Tr}\left(\mathbb{F}\right) = \frac{1}{2\pi}\int_{\partial S_{big}}\mathrm{Tr}\left(\mathbb{A}^{big}\right), \tag{C.20}$$

[1] The statements we will prove hold for a general closed manifold. Considering e.g., the torus, one has to divide $\mathcal{M}$ into more pieces, making the proof a bit more involved. However, the arguments would be analogous.

where the superscript big means that $\mathbb{A}^{big}$ is defined with respect to a coordinate system that is continuous in the region S_{big}. We can, however, also write

$$\frac{1}{2\pi}\int_{\partial S_{small}} \mathrm{Tr}\left(\mathbb{A}^{small}\right) = -\frac{1}{2\pi}\int_{S_{small}} \mathrm{Tr}\left(\mathbb{F}\right) = 0\,. \tag{C.21}$$

In the overlap between S_{big} and S_{small}, both coordinate systems are continuous, and on ∂S_{big} they are related by some unitary transformation $\mathbb{U}$;

$$\mathrm{Tr}\left(\mathbb{A}^{big}\right) = \mathrm{Tr}\left(\mathbb{A}^{small}\right) - i\mathrm{Tr}\left(\mathbb{U}^\dagger d\mathbb{U}\right)\,. \tag{C.22}$$

Putting this together, we get

$$\frac{1}{2\pi}\int_{S} \mathrm{Tr}\left(\mathbb{F}\right) = \frac{1}{2\pi i}\int_{\partial S_{big}} \mathrm{Tr}\left(\mathbb{U}^\dagger d\mathbb{U}\right). \tag{C.23}$$

The matrix trace is the same in all bases, so we may consider it in the basis where $\mathbb{U}$ is diagonal,

$$\mathbb{U} = \mathrm{diag}\left(\xi_1, \xi_2, \dots\right), \tag{C.24}$$

and we get

$$ch_1[\mathbb{F}](S) = \frac{1}{2\pi i}\int_{\partial S_{big}} \mathrm{Tr}\left(\mathbb{U}^\dagger d\mathbb{U}\right) = \frac{1}{2\pi i}\sum_i \int_{\partial S_{big}} \xi_i^* d\xi_i\,. \tag{C.25}$$

The ith term in this sum gives the change of the complex phase ξ^i accumulated when integrating over ∂S_{big}. Since ∂S_{big} is a closed curve, the phase change has to be a multiple of 2π, which proves that $ch_1(S)$ is an integer.

Reference

1. M. Nakahara, *Geometry, Topology and Physics*, 2nd edn. Graduate student series in physics (Taylor & Francis, 2003)