Appendix A
Classical Tools from Geometry and Analysis

A.1 About "Heat Operators"

Recall first that if P is an unbounded operator, then e^{tP} cannot be defined by the series $\|\cdot\|$-$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{t^n}{n!} P^n$. What is needed is that $G : t \geq 0 \to e^{tP} \in \mathscr{B}(\mathscr{H})$ is a strongly-continuous contraction semigroup (i.e. $G(0) = \mathbb{1}$, $G(s)\, G(t) = G(s+t)$, $\|G(t)\| \leq 1$ and the function $t \geq 0 \to G(t)\,\psi$ is norm-continuous for each $\psi \in H$). Then, a closed densely defined operator P is the generator of this semigroup, i.e. by definition $G(t) = e^{tP}$, if and only if $\mathbb{R}^+$ is contained in the resolvent set of P and $\|(P-\lambda)^{-1}\| \leq \lambda^{-1}$ for all $\lambda > 0$ (see [24, Chap. 14]).

Moreover, $e^{tP} = \text{strong-lim}_{n\to\infty}(\mathbb{1} - (t/n)P)^{-n}$ and

$$(P-\lambda)^{-1} = -\int_0^\infty e^{-\lambda t}\, e^{tP}\, dt, \quad \text{for} \quad \Re(\lambda) > 0 \tag{A.1}$$

holds true in this generality since actually the right half-plane $\{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$ is in the resolvent set of P and $\|(P-\lambda)^{-1}\| \leq \Re(\lambda)^{-1}$. The generator P is upper semibounded: $\Re(\langle P\psi, \psi\rangle) \leq 0$, for all $\psi \in \operatorname{Dom} P$.

Sometimes, the generator of $G(t)$ is denoted by $-P$ like in [33, Sect. X.8].

We can rephrase previous results as a constructive way to get the exponential. Let P be an unbounded operator on the Hilbert space $\mathscr{H}$ such that $P - \lambda$ is invertible in the sector $\Lambda_\theta := \{r\, e^{i\phi} \mid r \geq 0,\ |\phi| \geq \theta\}$, $0 < \theta < \frac{\pi}{2}$ and assume there exists c with

$$\left\|(P-\lambda)^{-1}\right\| \leq c\,(1 + |\lambda|^2)^{-1/2}, \quad \forall \lambda \in \Lambda_\theta. \tag{A.2}$$

This allows to define

$$e^{-tP} := \tfrac{i}{2\pi} \int_{\mathscr{C}} e^{-t\lambda}\, (P-\lambda)^{-1}\, d\lambda, \tag{A.3}$$

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where $\mathscr{C} = \mathscr{C}_{r_0,\theta}$ (with $\theta < \frac{\pi}{2}$) is the path from ∞ along the ray $r\,e^{i\theta}$ for $r \geq r_0$ followed by a clockwise circle around zero of radius r_0 and ending at infinity along the ray $r\,e^{-i\theta}$. Since the two rays lie in the right-half plane, the exponential decay of $e^{-t\,\lambda}$ guarantees the convergence of the integral. We will need that fact for instance in (2.15) of Sect. 2.2.2 on Laplace transform.

To close these remarks on heat operators, we recall that many functions can be defined through an integral along a curve in $\mathbb{C}$. For instance, given a selfadjoint operator P, one defines (cf. [36, Sect. 10]), $P^{-2z} = \frac{1}{i2\pi} \int_{\lambda \in \mathscr{C}} \lambda^{-z}\,(\lambda - P^2)^{-1}\, d\lambda$ along the curve $\mathscr{C}$ for any $z \in \mathbb{C}$ with $\Re(z) > 0$, as in (A.3). Typically, to control the norm-convergence of the integral, one uses

$$\|(\lambda - P)^{-1}\| = \sup_{\mu \in \text{spec } P} |\mu - \lambda|^{-1} = \text{dist}(\lambda, \text{spec } P)^{-1} \leq |\Im(\lambda)|^{-1}\,, \tag{A.4}$$

$$\|P(\lambda - P)^{-1}\| = \sup_{\mu \in \text{spec } P} |\mu|\,|\mu - \lambda|^{-1}. \tag{A.5}$$

Moreover, if $P = P^*$ is positive,

$$\|P(\lambda - P)^{-1}\| \leq \begin{cases} |\lambda|\,|\Im(\lambda)|^{-1} & \text{if } \Re(\lambda) \geq 0, \\ 1 & \text{if } \Re(\lambda) < 0, \end{cases} \tag{A.6}$$

which follows from $\|P(\lambda - P)^{-1}\| = \sup_{\mu \in \text{spec } P} f(\mu)$ with $f(\mu) = \mu\ |\mu - \lambda|^{-1}$ and the computation of the maximum of f.

A.2 Definition of pdos, Sobolev Spaces and a Few Spectral Properties

There are several good textbooks on pdos: [19, 23, 25, 36, 37]. For the heat trace asymptotics of a pdo we closely follow [23, Sect. 4.2] and the nice notes [35]. See also [1, 20, 27] for the computation of heat kernel coefficients.

To study pdos on $\mathbb{R}^d$ we need a few basic definitions:

– $\langle x\rangle := (1 + \|x\|^2)^{1/2}$ and $\langle x, \eta\rangle := (1 + \|x\|^2 + |\eta|)^{1/2}$ for $x \in \mathbb{R}^n,\ n \in \mathbb{N}^*$, and $\eta \in \mathbb{C}$.

– $S^m := \{p :\ (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that $|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq c_{\alpha\beta}\langle \xi\rangle^{m-|\alpha|}\}$ is the set of *symbols of order* $m \in \mathbb{R}$. Here $\alpha,\ \beta$ are in $\mathbb{N}^d$ with $|\alpha| = \sum \alpha_i$. This yields a family of seminorms on S^m defined by $|p|_{m,\alpha,\beta} := \sup_x |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)|\langle \xi\rangle^{-m+|\alpha|}$.

The set of smoothing symbols is $S^{-\infty} := \cap_m S^m$.

– The symbol $p \in S^m$ has the expansion $p \sim \sum_{j=0}^\infty p_{m-j}$, when $p_{m-j} \in S^{m-j}$ and for each n, $p - \sum_{j=0}^n p_{m-j} \in S^{m-n}$. It is named *classical* if, moreover, for all j,

$$p_{m-j}(x, t\xi) = t^{m-j} p(x, \xi),\ \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d,\ \|\xi\| \geq 1,\ t \geq 1.$$

It is said to be *elliptic* when $p(x,\xi)$ is invertible and $|p(x,\xi)^{-1}| \le c\,\langle\xi\rangle^{-m}$ for all x and $\|\xi\| > r$ for some $r \ge 0$. A classical symbol $p \in S^m$ is elliptic when $p_m(x,\xi)$ is invertible for all $x \in \mathbb{R}^d$ and $\|\xi\| = 1$.

From now on, *we assume in this appendix that all symbols are classical.*

– Every symbol p gives rise to a pdo acting on u in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ via the inverse Fourier transform by

$$(\mathscr{O}(p)\,u)(x) := \mathfrak{F}^{-1}[(p(x,\cdot)\,\mathfrak{F}[u](\cdot)](x) = \int_{\mathbb{R}^d} e^{i\,2\pi\,x.\xi}\,p(x,\xi)\,\mathfrak{F}[u](\xi)\,d\xi.$$

This definition is compatible with the product of operators as for $p \in S^{m_1},\ q \in S^{m_2}$ there exists a symbol called the Leibniz product of p and q, denoted $p \circ q \in S^{m_1+m_2}$, such that $\mathscr{O}(p)\,\mathscr{O}(q) = \mathscr{O}(p \circ q)$ with the expansion

$$(p \circ q)(x,\xi) \sim \sum_\alpha \tfrac{(-i)^{|\alpha|}}{\alpha!}\,(\partial_\xi^\alpha p)(x,\xi)\;(\partial_x^\alpha q)(x,\xi).$$

In particular, when $p \in S^m$ is elliptic, there exists a symbol q called *the parametrix* such that $p \circ q - 1$ and $q \circ p - 1$ are both in $S^{-\infty}$.

– The Sobolev spaces read $H^s(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \langle\xi\rangle^s\,\mathfrak{F}(u) \in L^2(\mathbb{R}^d)\}$ for $s \in \mathbb{R}$, with scalar product $(u,u') := \int \mathfrak{F}(u)(\xi)\,\overline{\mathfrak{F}(v)(\xi)}\,\langle\xi\rangle^{2s}\,d\xi$ and complete for the norm $\|u\|_s^2 := (u,u)$. For instance $\delta_y \in H^s(\mathbb{R}^d)$ if $s < -d/2$.

We have, $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ and if $s > d/2$ then $H^s(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ (Sobolev embedding theorem). When $s > d/2$, any bounded operator $A : H^{-s} \to H^s$ is an integral operator with a Schwartz kernel given by $k_A(x,y) = (A\delta_x, \delta_y)$.

We now adapt previous definitions to

a compact boundaryless Riemannian manifold M of dimension d,

so we need coordinate charts (U,h) where U are open sets in M and h are diffeomorphisms from U to open sets in $\mathbb{R}^d$.

Let $P : C^\infty(M) \to C^\infty(M)$; when $\phi,\ \psi \in C_c^\infty(U)$ (smooth functions on U with compact support), the localised operator $\phi P \psi$ on $C_c^\infty(U)$ is pushed-forward as $h_*(\phi P \psi)$ on $C_c^\infty(h(U))$. The operator P is a pseudodifferential operator of order m when each such localisation is a pdo of order m on $h(U)$.

Then, one extends S^m to symbols on M as follows:

$$S^m(M) := \{p(x,\xi) \in C^\infty(T^*M) \mid h(\phi p) \in S^m(h(U)),\ \forall (U,h,\phi)\}.$$

Of course, P is said to be elliptic or smoothing if all its local symbols have such a property and the set of classical pdos of order m is denoted by $\Psi^m(M)$ which defines $\Psi(M) := \cup_m \Psi^m(M)$.

For $x \in (U,h)$ one defines the principal (or leading) symbol of $P \in \Psi^m(M)$ as $p_m(P) := h^* p_m(h_*(\phi P \phi)) \in S^m(M)/S^{m-1}(M)$, where one chooses a $\phi \in C_c^\infty(M)$ equal to 1 in vicinity of x. One checks that this principal symbol makes sense and is invariantly defined on T^*M while the total symbol is quite sensitive to a change of coordinates. Moreover, for each $p \in S^m(M)$, one constructs $P \in \Psi^m(M)$ with $p_m(P) = p$ via the partition of unity.

A new extension is possible when P acts on sections of a smooth vector bundle E of finite rank over M equipped with a smooth inner product. So, typically, a fiber is acted upon by a matrix. By local triviality, on can define $H^s(M, E)$ using a partition of unity. Hence $P : C^\infty(M, E) \to C^\infty(M, E)$ is called a pseudodifferential operator of order m if every localisation is a matrix of pdos of order m for all charts U over which E is trivial. Such operators, the symbols of which are now matrices, define $\Psi^m(M, E)$. The properties of classicality and ellipticity are generalised in a straightforward way.

In particular, P has a matrix-valued kernel K_P, which in local coordinates reads

$$k_P(x, y) = \int_{\mathbb{R}^d} e^{i\,2\pi(x-y).\xi}\, p(x, \xi)\, d\xi. \tag{A.7}$$

Similarly, the Sobolev space $H^s(M)$ is defined as the set of distributions u on M which, in a given local patch U, satisfy $u \in \mathscr{D}'(U)$ with $\psi u \in H^s(\mathbb{R}^d)$ for all $\psi \in C_c^\infty(U)$. By the Rellich theorem, the inclusion $H^s(M) \hookrightarrow H^t$ is compact for any $t < s$ and even trace-class when $t + d < s$.

From the beginning, classical symbols can be seen as objects defined up to $S^{-\infty}$. It has the following consequence: $P \in \Psi(M, E)$ is smoothing (i.e. all of its local symbols are smoothing) if and only if P has a Schwartz kernel k_P which is smooth on $M \times M$. For instance, $\psi P \phi$ is smoothing if $\psi, \phi \in C_c^\infty(M)$ with disjoint supports.

We now recall a few classical results on pdos — see loc. cit. at the beginning of this section. They provide links between a pdo and the same object, but viewed as an operator which has eventually several closed extensions on a Hilbert space. Recall that a bounded operator between Banach spaces is Fredholm if it has a finite dimensional kernel and cokernel and a closed range.

Theorem A.1 *Let $P \in \Psi^m(M, E)$. Then:*

(i) The extension of $P : H^s(M, E) \to H^{s-m}(M, E)$ is bounded for all $s \in \mathbb{R}$.
(ii) If P is elliptic, all previous extensions are Fredholm operators, which means that there exists a Fredholm inverse which is a pdo of order $-m$.
In particular, when $m > 0$, $P : C^\infty(M, E) \to C^\infty(M, E)$ acting on the Hilbert space $H^0(M, E) = L^2(M, E)$ has only one closed extension with the domains $H^m(M, E)$ and a spectrum either equal to $\mathbb{C}$ or discrete without accumulation points except 0.
(iii) When $P : H^s(M, E) \to H^{s-m}(M, E)$ is invertible for some s, then we have $P^{-1} \in \Psi^{-m}(M, E)$.
(iv) The space $\Psi^0(M, E)$ is an algebra.
(v) If $P \in \Psi^m(M, E)$ with $m < -d$, then P has a continuous kernel and its extension P on $L^2(M, E)$ is trace-class with $\mathrm{Tr}_{L^2(M,E)} P = \int_M \mathrm{tr}_E\, k_P(x, x)\, dx$.

The inverse in *(ii)* is obtained by the construction of a local parametrix over a local chart, which, after being patched, gives rise to two pdos Q and Q' of order $-m$

such that $R = QP - \mathbb{1}$ and $R' = PQ' - \mathbb{1}$ are smoothing pdos. Moreover, $Q - Q'$ is a smoothing pdo. So, modulo smoothing pdos, Q is the left and right inverse of P.

We now present some details from the constructive proof of this theorem for parameter-dependent symbols.

A.3 Complex Parameter-Dependent Symbols and Parametrix

Let us be given an elliptic pdo $P \in \Psi^m(M, E)$ of order $m > 0$ with the matrix symbol $p \sim \sum_{j=0}^{\infty} p_{m-j}$. Despite the nice unique L^2-extension of P provided by Theorem A.1, it is still interesting to look at e^{-tP} not only as an operator on $L^2(M, E)$ but as a smoothing pdo (or, similarly, to regard the complex power P^s as a pdo of order $m\,\Re(s)$).

The main idea, which we expound in some detail below, is to replace the resolvent $(P - \lambda)^{-1}$ of P by a parameter-dependent parametrix, the symbol of which is under control. Since we want to control the integrand of (A.3), we assume the following (uniform) parameter-ellipticity of the principal symbol of P:

Hypothesis A.1 The operator $(P - \lambda)^{-1}$ exists in the left keyhole region $V_{r_0,\theta}$ defined by $\mathscr{C}_{r_0,\theta}$ for $\theta < \frac{\pi}{2}$. Moreover, we have the resolvent growth condition: The matrices $p_m(x, \xi) - \lambda$ are invertible for all x, ξ when $\lambda \in V_{r_0,\theta}$ and

$$\left\| (p_m(x, \xi) - \lambda)^{-1} \right\| \leq (1 + \|\xi\|^2 + |\lambda|^{2/m})^{-m/2} = \langle \xi, \lambda^{1/m} \rangle^{-m}.$$

For the principal symbol, let us introduce the strictly homogeneous symbol p_m^h:

$$p_m^h(x, \xi) := \|\xi\|^m \; p_m(x, \xi / \|\xi\|)$$

(which coincides with p_m for $\|\xi\| \geq 1$, but is now homogeneous of degree m for all $\xi \neq 0$) and we can rephrase the hypothesis as: $p_m^h(x, \xi)$ has no eigenvalues in $V_{r_0,\theta}$ for all $\xi \neq 0$ (see [23, Lemma 1.5.4]). Recall that $p_m(x, \xi)$ is homogeneous of degree m only for $\|\xi\| \geq 1$ and we have to control the integral in ξ, as in (A.7).

Let (U, h) be a fixed coordinate chart. For $x \in U$, $\xi \in \mathbb{R}^d$, $\lambda = \eta^m \in V_{r_0,\theta}$ and $j \in \mathbb{N}$, we want to generate a parametrix by an inductive sequence (see [28])

$$q_{-m}(x, \xi, \eta) := (p_m(x, \xi) - \eta^m)^{-1},$$
$$q_{-m-j}(x, \xi, \eta) := - \sum_{k=1}^{j-1} \sum_{\alpha, \ell} \tfrac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha q_{-m-k}(x, \xi, \eta)\, \partial_x^\alpha p_{m-\ell}(x, \xi) (p_m(x, \xi) - \eta^m)^{-1},$$

where the second sum is over $\alpha \in \mathbb{N}^d$, $\ell \in \mathbb{N}$ such that $k + \ell + |\alpha| = j$.

In the scalar case (i.e. the fibers of E are one-dimensional),

$$q_{-m-j}(x,\xi,\eta) = \sum_{k=1}^{2j} p_{j,k}(x,\xi)(p_m(x,\xi)-\eta^m)^{-k-1}, \tag{A.8}$$

where the $p_{j,k}$ are symbols of order $mk-j$ obtained from $p_m,\dots,p_{m-j}$.

Using $\partial((p_m-\eta^m)^{-1}) = -(p_m-\eta^m)^{-1}(\partial p_m)(p_m-\eta^m)^{-1}$, with $\partial = \partial_x$ or ∂_ξ, one checks that $\partial_\xi^\alpha \partial_x^\beta q_{-m-j}$ is a sum of terms of the form

$$(p_m-\eta^m)^{-1}\partial_\xi^{\alpha_1}\partial_x^{\beta_1}p_{m-k_1}(p_m-\eta^m)^{-1}\cdots\partial_\xi^{\alpha_r}\partial_x^{\beta_r}p_{m-k_r}(p_m-\eta^m)^{-1}, \tag{A.9}$$

where $\sum_{\ell=1}^r k_\ell + |\alpha_\ell| = j+|\alpha|$.

Moreover, $\partial_\xi^\alpha\partial_x^\beta q_{-m-j}(x,t\xi,t\eta) = t^{-m-j}\partial_\xi^\alpha\partial_x^\beta q_{-m-j}(x,\xi,\eta)$ for $\|\xi\|\geq 1$, $t\geq 1$ and there are at least two factors $(p_m(x,\xi)-\eta^m)^{-1}$ if either $j>0$ or $|\alpha|+|\beta|>0$.

This implies the following estimates:

$$\|\partial_\xi^\alpha\partial_x^\beta q_{-m-j}(x,\xi,\eta)\| = \begin{cases} \mathcal{O}_{\|\xi\|\to\infty}(\langle\xi,\eta\rangle^{-m}\langle\xi\rangle^{-j-|\alpha|}) & \text{for } j\in\mathbb{N}, \\ \mathcal{O}_{\|\xi\|\to\infty}(\langle\xi,\eta\rangle^{-2m}\langle\xi\rangle^{m-j-|\alpha|}) & \text{if } j+|\alpha|+|\beta|>0. \end{cases} \tag{A.10}$$

Now, defining q such that $q\sim\sum_{j\in\mathbb{N}} q_{-m-j}$, we get

$$\partial_\xi^\alpha\partial_x^\beta\Big[q(x,\xi,\eta)-\sum_{j<J}q_{-m-j}(x,\xi,\eta)\Big] = \mathcal{O}_{\|\xi\|\to\infty}(\langle\xi,\eta\rangle^{-2m}\langle\xi\rangle^{m-J-|\alpha|}). \tag{A.11}$$

We claim that: *If* $r(x,\xi,\lambda) := q(x,\xi,\lambda^{1/m})\circ(p(x,\xi)-\lambda)-\mathrm{id}$*, then for any* N*, all seminorms in* S^{-N} *of the symbol* r *are* $\mathcal{O}_\infty(\langle\lambda\rangle^{-1})$.

The proof is based on the decomposition of the series defining p, q in finite sums and remainders and their Leibniz products via the above estimates.

Then, one gets rid of the local chart U by patching previous parametrices to get a parameter-dependent pdo $Q(\lambda)$ (associated to the symbol $q(x,\xi,\lambda^{1/m})$), such that $R(\lambda) = Q(\lambda)\,(P-\lambda)-\mathbb{1}$ (with the symbol $r(x,\xi,\lambda)$) is a smoothing pdo, where the seminorms $\|\cdot\|_{-N,\alpha,\beta}$ of its symbol are $\mathcal{O}_\infty(\langle\lambda\rangle^{-1})$ for each N.

Since formally $[Q(\lambda)\,(P-\lambda)]^{-1} = [\mathbb{1}+R(\lambda)]^{-1} = \sum_{j=0}^\infty(-R(\lambda))^j$, we deduce that

$$(P-\lambda)^{-1}-Q(\lambda) = \big([Q(\lambda)(P-\lambda)]^{-1}-\mathbb{1}\big)\,Q(\lambda) = \sum_{j=1}^\infty(-R(\lambda))^j\,Q(\lambda)$$

has a norm which is $\mathcal{O}_\infty(\langle\lambda\rangle^{-2})$ since $\|Q(\lambda)\| = \mathcal{O}_\infty(\langle\lambda\rangle^{-1})$ so that:

Proposition A.2 *We have* $\big\|(P-\lambda)^{-1}-Q(\lambda)\big\| = \mathcal{O}_\infty(\langle\lambda\rangle^{-2})$ *for all* $\lambda\in\Lambda_\theta$.

Remark that we can define similarly $Q'(\lambda)$ such that $(P-\lambda)\,Q'(\lambda)-\mathbb{1} = R'(\lambda)$ for another smoothing pdo $R'(\lambda)$.

Thus $Q(\lambda) - Q'(\lambda) = R(\lambda)Q'(\lambda) - Q(\lambda)R'(\lambda)$ is a smoothing pdo. Moreover, the operator $(P-\lambda)^{-1}$ can be seen as an elliptic pdo of order $-m$ since, by (A.10)–(A.11), the operator $(P-\lambda)^{-1} - \sum_{j<J} q_{-m-j}(x,\xi,\lambda^{1/m})$ is a pdo of order $m-J$, the seminorms of which are $\mathcal{O}_\infty(\langle\lambda\rangle^{-2})$.

This explains why the operator $(P-\lambda)^{-1}$ seen as a pdo is nothing else than $Q(\lambda)$. Consequently, $G(t) := e^{-tP}$ defined by (A.3) is also equal to $\frac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda} Q(\lambda)\, d\lambda$.

Another consequence is that $(P-\lambda)^{-1}$ is compact by Theorem A.1(v) so P has a discrete spectrum without accumulation points (compare with Theorem A.1(ii)).

A.4 About e^{-tP} as a pdo and About its Kernel

Let $P \in \Psi^m(M, E)$ be elliptic with $m > 0$ and let its principal symbol satisfy (A.1).

Theorem A.3 *For $t > 0$, $G(t) := e^{-tP}$ is a pdo of order zero, the symbol $g(x,\xi,t)$ of which has the expansion $g(x,\xi,t) \sim \sum_{j=0}^\infty g_{-j}(x,\xi,t)$ with*

$$g_{-j}(x,\xi,t) := \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}\, q_{-m-j}(x,\xi,\lambda^{1/m})\, d\lambda, \quad \textit{for } j \in \mathbb{N}. \tag{A.12}$$

Moreover, $g_0(x,\xi,0) = 1$, while $g_{-j}(x,\xi,0) = 0$ for $j \in \mathbb{N}^*$.

Proof We follow [23, Theorem 4.2.2].

In a coordinate chart, we have

$$e^{-tP} = \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}\mathscr{O}[q(x,\xi,\lambda^{1/m})]\, d\lambda = \mathscr{O}[\tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}(q(x,\xi,\lambda^{1/m})\, d\lambda]$$
$$\sim \sum_{j=0}^\infty \mathscr{O}[g_{-j}(x,\xi,t)].$$

First, one checks that $g_0 = e^{-tp_m}$ (by residue calculus) and the homogeneity property

$$g_{-j}(x, r\xi, r^{-m}t) = r^{-j} g_{-j}(x,\xi,t), \quad \text{for } \|\xi\| \geq 1,\ r \geq 1.$$

We now want to prove the following estimates: There exists $c > 0$ such that

$$\|\partial_\xi^\alpha \partial_x^\beta g_{-j}(x,\xi,t)\| \leq \langle\xi\rangle^{-j-|\alpha|}\, (t^{1/m}\langle\xi\rangle)^a\, e^{-c\,\langle\xi\rangle^m t},\ \forall a \leq \min(m, j+|\alpha|). \tag{A.13}$$

These hold true for $j = 0$ and for $j \geq 1$ we begin with the scalar case, cf. (A.8):

$$g_{-j} = \sum_{k=1}^{2j} p_{j,k}\, \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}(p_m - \lambda)^{-k-1}\, d\lambda = \sum_{k=1}^{2j} p_{j,k}\, \tfrac{1}{k!}\, t^k\, e^{-tp_m}. \tag{A.14}$$

We get $\|P_{j,k}\, t^k\| \le \langle\xi\rangle^{mk-j}\, t^k = \langle\xi\rangle^{m-j}\, t\, (\langle\xi\rangle^d\, t)^{k-1}$ with $(\langle\xi\rangle^m\, t)^{k-1} \le 1$ if $\langle\xi\rangle^d\, t \le 1$. Moreover, $\|(\langle\xi\rangle^d\, t)^{k-1} e^{-tp_m/2}\| \le 1$ if $\langle\xi\rangle^m\, t \ge 1$. Thus, the estimates are proved when $\alpha = \beta = 0$ with $a = m$. As a consequence, we cover the situation where $a < m$ and for $\alpha,\ \beta$ non-zero one differentiates under the integral of (A.12) until the estimate for $\alpha = \beta = 0$ applies. For the non-scalar case, one proceeds as in the proof of (A.10) thanks to the expansion (A.9) for the derivatives.

The equality $g_{-j}(x, \xi, 0) = 0$ is a consequence of (A.13).

We can now conclude the proof thanks to the following argument: The symbol $g(x, \xi, t) \sim \sum_{j=0}^{\infty} g_{-j}(x, \xi, t)$ can be chosen in such a way that

$$\|\partial_\xi^\alpha \partial_x^\beta (g - \sum_{j<J} g_{-j})\| \le \langle\xi\rangle^{-J-|\alpha|}\, (t^{1/\mu}\langle\xi\rangle)^a\, e^{-c\,\langle\xi\rangle^m t}, \quad \forall a \le \min(m, j + |\alpha|).$$

So, for any integer J, $\frac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}\, \mathscr{O}[q(x, \xi, \lambda^{1/m}) - \sum_{j<J} q_{-m-j}(x, \xi, \lambda^{1/m})]\, d\lambda$ is a pdo of order zero and the asymptotics of the symbol for $G(t) = e^{-tP}$ is justified. □

But since we are interested in Tr e^{-tP} it is worthwhile to control the kernel of $G(t)$ as a function of t and to give an alternative proof of the previous theorem. On the way, it is shown that e^{-tP} is a smoothing pdo.

Let $G_{-j}(t)$ be the pdo defined locally by $g_{-j}(x, \xi, t)$ after patching local charts.

Lemma A.4 *For any $t > 0$:*
(i) The kernels of $G_{-j}(t)$ satisfy the estimates

$$\|K_{G_0}(x, y, t)\| \le t^{-d/m}\, e^{-c'\,t}, \quad \|K_{G_{-j}}(x, y, t)\| \le t^{(j-d)/m}\, e^{-c'\,t}, \quad \textit{for } 0 < j < m + d;$$
$$\|K_{G_{-m-d}}(x, y, t)\| \le t(1 + |\log t|)\, e^{-c'\,t}, \quad \|K_{G_{-j}}(x, y, t)\| \le t\, e^{-c'\,t}, \qquad \textit{for } j > m + d.$$

Moreover, on the diagonal we have, with $c_j(G, x) := \int_{\mathbb{R}^d} g^h_{-j}(x, \xi, 1)\, d\xi$,

$$K_{G_{-j}}(x, x, t) = c_j(G, x)\, t^{(j-d)/m} + \mathcal{O}_0(t), \quad \textit{for } 0 \le j < m + d.$$

(ii) The remainder $G^r_J(t) := G(t) - \sum_{j<J} G_{-j}(t)$ satisfies

$$\|K_{G^r_J}(x, y, t)\| \le t(1 + |\log t|)\, e^{-c'\,t}, \qquad \textit{for } J > m + d,$$
$$K_{G^r_J}(x, y, 0) = 0, \qquad \textit{for } J > d.$$

Proof (*i*) We begin with the kernel of G_0: $K_{G_0}(x, y, t) = \int_{\mathbb{R}^d} e^{i2\pi(x-y).\xi}\, e^{-tp_m(x,\xi)}\, d\xi$. Since $\|e^{-tp_m} - e^{-tp^h_m}\| = \mathcal{O}_0(t)$ when $\|\xi\| \le 1$, we get

$$\|K_{G_0}(x,y,t)\| \le c_1 \int_{\mathbb{R}^d} \|e^{-t\,p_m^h(x,\xi)}\|\,d\xi + c_1 \int_{\|\xi\|\le 1} \|e^{-tp_m(x,\xi)} - e^{-tp_m^h(x,\xi)}\|\,d\xi$$
$$\le c_1\, t^{-d/m} \int_{\mathbb{R}^d} e^{-c\|\eta\|^m}\,d\eta + c_2\,t \le c_3\,t^{-d/m} + c_2\,t,$$

and for the diagonal

$$K_{G_0}(x,x,t) = t^{-d/m} \int_{\mathbb{R}^d} e^{-p_m^h(x,\eta)}\,d\eta + \int_{\|\xi\|\le 1} e^{-tp_m(x,\xi)} - e^{-tp_m^h(x,\xi)}\,d\xi$$
$$=: c_0(G,x)\,t^{-d/m} + \mathcal{O}_0(t).$$

When $j \neq 0$, the estimate $\|g^h_{-j}(x,\xi,t)\| \le \|\xi\|^{m-j}\; t\,e^{-c\,t\|\xi\|^m}$ follows from (A.13). Since the last function is ξ-integrable when $m - j > -d$, we deduce that the kernel $K_{G_{-j}}(x,y,t) = \int_{\mathbb{R}^d} e^{i2\pi(x-y).\xi}\, g_{-j}(x,\xi,t)\,d\xi$ is $\mathcal{O}_\infty(e^{-c\,t/2})$, while for $t > 0$,

$$\|K_{G_{-j}}(x,y,t)\| \le c_1 \int_{\mathbb{R}^d} \|g^h_{-j}\|\,d\xi + c_1 \int_{\|\xi\|\le 1} (\|g_{-j}\| + \|g^h_{-j}\|)\,d\xi$$
$$\le c_2\,t \int_{\mathbb{R}^d} \|\xi\|^{m-j}\; e^{-c\,t\,\|\xi\|^m}\,d\xi + c_3\,t = c_4\,t^{(j-d)/m} + c_3\,t.$$

As above, still with $0 < j < d + m$, one gets

$$K_{G_{-j}}(x,x,t) = \int_{\mathbb{R}^d} g^h_{-j}(x,\xi,t)\,d\xi + c_1 \int_{\|\xi\|\le 1} (g_{-j} - g^h_{-j})(x,\xi,t)\,d\xi$$
$$= c_j(G,x)\,t^{(j-d)/m} + \mathcal{O}_0(t). \tag{A.15}$$

Moreover, using (A.13), we obtain

$$\|K_{G_{-j}}(x,y,t)\| \le t\left(\int_{\|\xi\|\le 1} + \int_{\|\xi\|\ge 1}\right) \langle\xi\rangle^{m-j}\, e^{-c\,t\langle\xi\rangle^m}\,d\xi$$
$$\le t\Big(1 + \int_1^\infty r^{m-j-d-1}\, e^{-c\,t\,r^m}\,dr\Big)$$
$$\le \begin{cases} t(c_1 + c_2\,|\log t|), & \text{for } j = m + d, \\ t(c_1 + c_2\,t^{-1+(j-d)/m)}), & \text{for } j > m + d. \end{cases}$$

This completes the proof of (i).

(ii) The remaining symbol

$$q^r_J(x,\xi,\lambda) := q(x,\xi,\lambda^{1/m}) - \sum_{j<J} q_{-m-j}(x,\xi,\lambda^{1/m})$$

gives $G^r_J(t) = \frac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}\,\mathscr{O}(q^r_J)(\lambda)\,d\lambda$, which is a pdo of order $-m - J < -2m - d$, so has a continuous kernel (and is trace-class) by Theorem A.1(v). Moreover,

$$K_{G_J^r}(x,y,t) = \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda} K_{\mathscr{O}(q_J^r)}(x,y,\lambda)\,d\lambda,$$

with $\|K_{\mathcal{O}(q_J^r)}(x,y,\lambda)\| = \mathcal{O}_\infty(\langle\lambda\rangle^{-2})$ as in the proof of Proposition A.2. Thus, the integral over $\mathscr{C}$ converges uniformly for all $t > 0$ since $|e^{-t\lambda}| \le e^{-ct}$ for some $c > 0$. Consequently, $K_{G_J^r}(x,y,t)$ is $\mathcal{O}_\infty(e^{-ct})$ and as such, it has a continuous extension at zero since $K_{G_J^r}(x,y,0) = 0$, because in $\frac{i}{2\pi}\int_{\mathscr{C}} K_{\mathscr{O}(q_J^r)}(x,y,\lambda)\,d\lambda$, $\mathscr{C}$ can be deformed into a closed contour around zero.

We already know from Sect. A.1 that $G_J^r(t) \in C^\infty((0,\infty),\mathscr{B}(\mathscr{H}))$ and, for $t \le 1$,

$$\begin{aligned}\|\partial_t K_{G_J^r}(x,y,t)\| &= \tfrac{1}{2\pi}\Big\|\int_{\mathscr{C}} \lambda e^{-t\lambda} K_{G_J^r}(x,y,\lambda)d\lambda\Big\| \le c_1 e^{-ct} + c_3\int_{r_0}^\infty e^{-c_2|\lambda|t}\langle\lambda\rangle^{-1}d\lambda \\ &\le c_4 + c_5\int_{r_0 t}^\infty e^{-c_2 a}\,a^{-1}\,da \le c_6 + c_7\,|\log t|\,. \end{aligned} \tag{A.16}$$

(See the definition of r_0 after (A.3).)

The Taylor series in t gives $|K_{G_J^r}(x,y,t)| \le c\,t(1+|\log t|)$ when $t \in (0,1]$ and hence the announced estimate. □

Technically, it is useful to use $e^{-tP} = \frac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}\,\lambda^{-k}\,P^k\,Q(\lambda)\,d\lambda,\ \forall k \in \mathbb{N}$ which follows from $Q(\lambda) = \lambda^{-1}(\lambda - P + P)Q(\lambda) = -\lambda^{-1} + \lambda^{-1}P\,Q(\lambda)$ which after iteration gives $Q(\lambda) = -\sum_{j=1}^k \lambda^{-j}P^{j-1} + \lambda^{-k}P^k Q(\lambda)$ and $\int_{\mathscr{C}} e^{-t\lambda}\,\lambda^{-j}\,d\lambda = 0$.

We denote: $Q^{(k)}(\lambda) := P^k\,Q(\lambda)$, seen as a pdo of order $(k-1)m$ with the symbol $q^{(k)} \sim \sum_{j\in\mathbb{N}} q^{(k)}_{(k-1)m-j}$ and, preserving the notation, $Q^{(k)}_{(k-1)m-j} := \mathscr{O}[q^{(k)}_{(k-1)m-j}]$ and $Q^{(k)}_J := Q^{(k)} - \sum_{j<J} Q^{(k)}_{(k-1)m-j}$. As an example,

$$\begin{aligned} q &= -\lambda^{-1} - \lambda^{-2}p + \lambda^{-2}q^{(2)} \\ &= \lambda^{-1} - \lambda^{-2}\big(p_m + \cdots + p_{-m} + \mathcal{O}_{\|\xi\|\to\infty}(\langle\xi\rangle^{-m-1})\big) \\ &\qquad + \lambda^{-2}\big(q^{(2)}_m + \cdots + q^{(2)}_{-m} + \mathcal{O}_{\|\xi\|\to\infty}(\langle\xi\rangle^{-1}\langle\xi,\lambda^{1/m}\rangle^{-m})\big). \end{aligned}$$

By iteration, $q_{-m-j} = -\lambda^{-2}p_{m-j} - \cdots - \lambda^{-k}p^{(k-1)}_{(k-1)m-j} + \lambda^{-k}q^{(k)}_{(k-1)m-j}$, where $p^{(\ell)}$ is the symbol of P^ℓ. Since these symbols are independent of λ, we get

$$\begin{aligned} g_{-j}(x,\xi,t) &= \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}\,q_{-m-j}(x,\xi,\lambda^{1/m})\,d\lambda \\ &= \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-t\lambda}\,\lambda^{-k}\,q^{(k)}_{(k-1)m-j}(x,\xi,\lambda^{1/m})\,d\lambda. \end{aligned} \tag{A.17}$$

In particular, for $k \in \mathbb{N}$ we rewrite the $Q^{(k)}$'s as

$$\begin{aligned} G(t) &= \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-\lambda t}\,\lambda^{-k}\,Q^{(k)}(\lambda)\,d\lambda, \\ G_{-j}(t) &= \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-\lambda t}\,\lambda^{-k}\,Q^{(k)}_{(k-1)m-j}(x,\xi,\lambda)\,d\lambda, \end{aligned}$$

$$G^r_J(t) = \tfrac{i}{2\pi} \int_{\mathscr{C}} e^{-\lambda t}\, \lambda^{-k}\, Q^{(k)}_J(\lambda)\, d\lambda. \tag{A.18}$$

With the help of this representation, we can improve the key estimate (A.13) along with the estimates from Lemma A.4:

Lemma A.5 *For any $k \in \mathbb{N}$,*

$$\|\partial^\alpha_\xi \partial^\beta_x \partial^k_t\, g_{-j}(x, \xi, t)\| \leq c'(x)\, \langle\xi\rangle^{km-j-|\alpha|}\, e^{-c\langle\xi\rangle^m t}. \tag{A.19}$$

For the kernel of $G_{-j} = \mathscr{O}(g_{-j})$ we have

$$\|\partial^k_t K_{G_{-j}}(x, y, t)\| \leq \begin{cases} (1 + |\log t|)\, e^{-ct}, & \text{for } j - km = d, \\ (1 + t^{(j-km-d)/m})\, e^{-ct}, & \text{for } j - km \neq d. \end{cases} \tag{A.20}$$

On the diagonal we get, with $c_{j,k}(x) := \int_{\mathbb{R}^d} \partial^k_t g^h_{-j}(x, \xi, 1)\, d\xi$,

$$\partial^k_t K_{G_{-j}}(x, x, t) = c_{j,k}(x)\, t^{(j-km-d)/m} + \mathcal{O}_0(t^0), \quad j < (k-1)m + d. \tag{A.21}$$

For the kernel of G_{-j} we have, with $j > km + d$ for some $k \in \mathbb{N}$,

$$K_{G_{-j}}(x, y, t) = \sum_{\ell=1}^{k-1} \tfrac{t^\ell}{\ell!}\, \partial^\ell_t K_{G_{-j}}(x, y, 0) + t^k R(x, y, t) \tag{A.22}$$

and $\partial^\ell_t K_{G_{-j}}(x, y, 0)$ and $R(x, y, t)$ are continuous in x, y and in $t \geq 0$.

When $J > (k+1)m + d$ for some $n \in \mathbb{N}$, the kernel of the remainder G^r_J satisfies

$$\|\partial^k_t K_{G^r_J}(x, y, t)\| \leq e^{-ct},$$
$$K_{G^r_J}(x, y, t) = \sum_{\ell=1}^{k-1} \tfrac{t^\ell}{\ell!}\, \partial^\ell_t K_{G^r_J}(x, y, 0) + t^k R(x, y, t), \tag{A.23}$$

where $\partial^\ell_t K_{G^r_J}(x, y, 0)$ and $R(x, y, t)$ are continuous in x, y and in $t \geq 0$.

Proof For $\ell \in \mathbb{N}^*$, $\partial^\ell_t g_{-j}(x, \xi, t) = \frac{i(-1)^\ell}{2\pi} \int_{\mathscr{C}} e^{-t\lambda}\, q^{(\ell)}_{(\ell-1)m-j}(x, \xi, \lambda^{1/m})\, d\lambda$ is derived from (A.17) and the estimate (A.19) is proved in the same way as (A.13).

Since $\partial^{\ell+1}_t g_{-j}$ is bounded when $t \to 0$ by the above formula, the function $\partial^\ell_t g_{-j}$ is continuous at $t = 0$. Since $g_{-j}(x, \xi, 0) = 0$ by Theorem A.3, we have the Taylor expansion $g_{-j}(x, \xi, t) = \sum_{\ell=1}^k \frac{1}{\ell!} \partial^\ell_t g_{-j}(x, \xi, 0)\, t^\ell + t^k r_{-j,k}(x, \xi, t)$ where $\partial^\ell_t g_{-j}(x, \xi, 0)$ are pdos of order $\ell m - j$.

The operators $\partial^k_t G_{-j}(t) = \mathscr{O}(\partial^k_t g_{-j})$ have kernels, for which we can repeat the same arguments used for the proof of Lemma A.4(*i*) in order to get (A.20).

If $j > d$, $K_{G_{-j}}(x, y, 0) = 0$ by Lemma A.4 and by a Taylor expansion, we get (A.22).

We need to control the remainder. For $k \geq 1$, choose $J > km + d$ and the presentation (A.18) for the kernel of $K_{G^r_J}$. Since G^r_J is a pdo of order $(k-1)m - M$,

$\|q_J^r(x,\xi,\lambda)\| \le \langle\xi\rangle^{km-J}\,\langle\xi,\lambda^{1/m}\rangle^{-m}$, thus, after a ξ-integration, we get the estimate of the kernel $\|K_{Q_J^{(k)}}(x,y,\lambda)\| \le \langle\lambda\rangle^{-1}$ and

$$K_{G_J^r}(x,y,t) = \tfrac{i}{2\pi}\int_{\mathscr{C}} e^{-\lambda t}\,\lambda^{-k}\,Q_J^{(k)}(\lambda)\,d\lambda$$

because all $Q^{(k)}$ are holomorphic in λ. Thus, for $\ell \le k-1$,

$$\|\partial_t^\ell K_{G_J^r}(x,y,t)\| = \tfrac{1}{2\pi}\|\int_{\mathscr{C}} e^{-\lambda t}\lambda^{\ell-k}K_{Q_J^{(k)}}(\lambda)d\lambda\| \le \tfrac{1}{2\pi}|\int_{\mathscr{C}} e^{-\lambda t}\langle\lambda\rangle^{\ell-k-1}d\lambda| \le e^{-ct}.$$

As a consequence we get: $K_{G_J^r}(x,y,t) = \sum_{\ell=1}^{k-2}\frac{t^\ell}{\ell!}\,\partial_t^\ell K_{G_J^r}(x,y,0) + t^{k-1}R(x,y,t)$. This Taylor expansion begins at $\ell = 1$ since $K_{G_J^r}(x,y,0) = 0$ by Lemma A.4. Swapping $m-1$ to m completes the proof of the lemma. □

A.5 The Small-t Asymptotics of e^{-tP}

The above estimates can be used to prove that e^{-tP} has a smooth Schwartz kernel for any $t > 0$. Therefore, e^{-tP} is a smoothing pdo, and hence is trace-class.

Theorem A.6 *Let $P \in \Psi^m(M,E)$ be elliptic with $m > 0$ and let its principal symbol satisfy (A.1). Then, $G(t) = e^{-tP}$ is a smoothing pdo and its kernel has the following asymptotics on the diagonal:*

$$K_G(x,x,t) \underset{t\downarrow 0}{\sim} \sum_{\substack{n\in\mathbb{N}\\ n-d\notin m\mathbb{N}}} c_{n-d}(x,G)\,t^{(n-d)/m} + \sum_{\substack{n\in\mathbb{N}\\ n-d\in m\mathbb{N}}} c_{n-d}(x,G)\,t^{(n-d)/m}\log t + \sum_{\ell\in\mathbb{N}} r_\ell(x,G)\,t^\ell,$$

where the coefficient $c_{n-d}(x,G) \in C^\infty(M)$ depends only on $p_m,\ldots,p_{m-n}$, while $r_\ell(x,G) \in C^\infty(M)$ depends globally on the operator P.

Proof The smoothness of the kernel $K_{-j}(x,y,t)$ in x, y follows from (A.19) and it remains to control the remainder. In fact, $\|(x-y)^\gamma\,\partial_x^\alpha\,\partial_y^\beta K_{G_J^r}(x,y,t)\| \le e^{-ct}$ when $J > (k+1)m - |\gamma| + |\alpha| + |\beta| + d$ which follows, as in the previous lemma, from the estimate $\|(x-y)^\gamma\,\partial_x^\alpha\,\partial_y^\beta K_{Q_J^{(k+1)}}(x,y,\lambda)\| \le \langle\lambda\rangle^{-1}$. Thus, $G(t)$ is a smoothing pdo.

Now, let us choose a large $k \in \mathbb{N}$. By a successive integration of (A.21) and using (A.15) with $j = d$, we get, for $j < (k-1)m + d$,

$$K_{G_{-j}}(x,x,t) = \begin{cases} c'_{j,k}(x)\,t^{(j-d)/m} + p_{j,k}(x,t) + \mathcal{O}_0(t^k), & \text{for } j-d \notin m\mathbb{Z},\\ c'_{j,k}(x)\,t^{(j-d)/m}\,\log t + p'_{j,k}(x,t) + \mathcal{O}_0(t^k), & \text{for } j-d \in m\mathbb{Z}, \end{cases}$$

where $c'_{j,k}(x)$ depends only on $c_{j,k}(x)$ of Lemma A.5, $p_{j,k}(x,t)$ and $p'_{j,k}(x,t)$ are polynomials of degree k in t and are continuous in x with $p_{j,k}(x,0) = p'_{j,k}(x,0) = 0$.

Moreover, the remainder in (A.23) for $J = (k-1)m + d > ((k-3)+1)m + d$ is $K_{G_J^r}(x, y, t) = \sum_{\ell=1}^{k-4} \frac{t^\ell}{\ell!} \partial_t^\ell K_{G_{-J}^r}(x, y, 0) + \mathcal{O}_0(t^{k-3})$. Thus, for the full integral kernel,

$$K_G(x,x,t) = \sum_{\substack{0 \leq j < J=(k-1)m+d \\ j-d \notin m\mathbb{N}}} c'_{j,k}(x)\, t^{(jd)/m} + \sum_{\substack{j=m\ell+d \\ 1\ leq \ell < k-1}} c'_{j,k}(x)\, t^\ell \,\log t \\ + p_k(x,t) + \mathcal{O}_0(t^{k-3}),$$

where the $p_k(x,t)$ are polynomials in t such that $p_k(x,0) = 0$. Sending k to infinity, we get, after a relabeling, the announced asymptotics, because $K_G(x,x,t)$ minus the sum of terms up to $(n-d)/m = N$ and $\ell = N$ is $\mathcal{O}_0(t^{N+1/d})$. The coefficients $c'_{j,k}$ depend only on G_{-j}, thus locally on the symbols of P of orders from m to $m-j$. The p_k's are not easy to characterise, but they are smooth in x: The smoothness of $c_{j,k}$ (and so of $c'_{j,k}$) is clear from its definition in Lemma A.5, while the smoothness of $p_{j,k}$ or $p'_{j,k}$ can be checked at each step of the above integrations in t with $t = 1$. □

By taking the trace and relabeling, we immediately get the celebrated expansion:

Corollary A.7 *Let (M, g) be a compact Riemannian manifold of dimension d and let $P \in \Psi^m(M, E)$ be an elliptic pdo with $m > 0$, the principal symbol of which satisfies (A.1). Then,*

$$\mathrm{Tr}\ e^{-tP} \underset{t\downarrow 0}{\sim} \sum_{k=0}^{\infty} a_k(P)\, t^{(k-d)/m} + \sum_{\ell=0}^{\infty} b_\ell(P)\, t^\ell \,\log t,$$

$$\textit{with} \begin{cases} a_k(P) = \int_M \mathrm{tr}\ c_{k-d}(x,P)\sqrt{g}\, d^dx, & \textit{for } k-d \notin m\mathbb{N}, \\ a_k(P) = \int_M \mathrm{tr}\ r_{(k-d)/m}(x,P)\sqrt{g}\, d^dx, & \textit{for } k-d \in m\mathbb{N}, \\ b_\ell(P) = \int_M \mathrm{tr}\ c_{m\ell}(x,P)\sqrt{g}\, d^dx, & \textit{for } \ell \in \mathbb{N}. \end{cases}$$ □

It can be of interest to recall a few links between the resolvent, complex powers and heat operators (with $s \in \mathbb{C},\ k \in \mathbb{N}, c > 0$)

$$e^{-tP} = t^{-k} \tfrac{i}{2\pi} \int_{\mathscr{C}} e^{-t\lambda}\, \partial_\lambda^k (P-\lambda)^{-1}\, d\lambda = \tfrac{1}{i2\pi} \int_{\Re(s)=c} t^{-s}\, \Gamma(s)\, P^{-s}\, ds, \tag{A.24}$$

$$P^{-s} = \tfrac{1}{(s-1)\cdots(s-k)}\, \tfrac{i}{2\pi} \int_{\mathscr{C}} \lambda^{k-s}\, \partial_\lambda^k (P-\lambda)^{-1}\, d\lambda = \tfrac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-tP}\, dt.$$

A.6 Meromorphic Extensions of Certain Series and their Residues

We gather below some results on meromorphic extensions of certain series. These will allow us for an extension of Proposition 2.26 and provide tools for the computation of the dimension spectrum of the noncommutative torus in Sect. B.3.1. On the way, a 'Diophantine condition' will pop up guaranteeing a control on the commutation between the residues and the series. For complete proofs, see [15].

In the following, $\sum'$ means that we omit the division by zero in the summand.

Theorem A.8 *Let P be a polynomial* $P(x) = \sum_{j=0}^{d} P_j(x) \in \mathbb{C}[x_1, \ldots, x_n]$*, where* P_j *is the homogeneous part of P of degree j and d is fixed.*

Then, the function $f(s) := \sum'_{k\in\mathbb{Z}^n} P(k)\, \|k\|^{-s}$ *has a meromorphic extension to* $\mathbb{C}$.

Moreover, f is not entire iff $\mathscr{P}_P := \{j \mid \int_{u\in S^{n-1}} P_j(u)\, ds(u) \neq 0\} \neq \varnothing$ *and f has only simple poles at* $j+n$, $j \in \mathscr{P}_P$*, with* $\operatorname*{Res}_{s=j+n} f(s) = \int_{u\in S^{n-1}} P_j(u)\, ds(u)$.

Here ds is the Lebesgue measure on S^{n-1}. The proof is based on the fact that the function $\sum'_{k\in\mathbb{Z}^n} P(k)\, \|k\|^{-s} - \int_{\mathbb{R}^n\setminus B^n} P(x)\|x\|^{-s}\, dx$, with B^n – the unit ball in $\mathbb{R}^n$, originally defined for $\Re(s) > d+n$, extends holomorphically to $\mathbb{C}$.

This result can be seen as an extension of Proposition 2.26: Let D be a self-adjoint invertible operator with only discrete spectrum equal to $\mathbb{Z}^n$ such that each eigenvalue $\pm k \in \mathbb{Z}^n$ has multiplicity $p(k)$ for a given polynomial $P \in \mathbb{N}[x_1, \ldots, x_n]$. Then, $\operatorname*{Res}_{s=j+n} \zeta_D(s) = \int_{u\in S^{n-1}} P_j(u)\, ds(u)$.

Example A.9 Let us consider the functions $\zeta_{q_1,\ldots,q_n}(s) := \sum'_{k\in\mathbb{Z}^n} k_1^{q_1} \cdots k_n^{q_n}\, \|k\|^{-s}$, for $q_i \in \mathbb{N}^*$. By the symmetry $k \mapsto -k$, the functions $\zeta_{q_1,\ldots,q_n}$ vanish if any q_i is odd.

Assume that all of q_i's are even, then $\zeta_{q_1,\ldots,q_n}(s)$ is a nonzero sum of terms $P(k)\|k\|^{-s}$, where P is a homogeneous polynomial of degree $q_1 + \cdots + q_n$. Theorem A.8 yields the following: $\zeta_{p_1,\ldots,p_n}$ has a meromorphic extension to $\mathbb{C}$ with a unique pole at $n + q_1 + \cdots + q_n$. This pole is simple and the residue at this pole is

$$\operatorname*{Res}_{s=n+q_1+\cdots+q_n} \zeta_{q_1,\ldots,q_n}(s) = 2\, \frac{\Gamma[(q_1+1)/2]\cdots\Gamma[(q_n+1)/2]}{\Gamma[(n+q_1+\cdots+q_n)/2]}. \qquad \blacksquare \qquad \text{(A.25)}$$

We now recall few notions from the Diophantine approximation theory.

Definition A.10 (i) Let $\delta > 0$. A vector $a \in \mathbb{R}^n$ is said to be δ-badly approximable if the Diophantine condition holds true:

There exists $c > 0$ such that $|q.a - m| \geq c\, |q|^{-\delta}$, $\forall q \in \mathbb{Z}^n \setminus \{0\}$ and $\forall m \in \mathbb{Z}$.

We denote by $\mathscr{BV}_\delta$ the set of δ-badly approximable vectors and $\mathscr{BV} := \cup_{\delta>0} \mathscr{BV}_\delta$ the set of badly approximable vectors.

(ii) A matrix $\Theta \in \mathscr{M}_n(\mathbb{R})$ (real $n \times n$ matrices) will be called *badly approximable* if there exists $u \in \mathbb{Z}^n$ such that ${}^t\Theta(u)$ is a badly approximable vector of $\mathbb{R}^n$.

It is known that for $\delta > n$ the Lebesgue measure of $\mathbb{R}^n \setminus \mathscr{BV}_\delta$ is zero (i.e almost any element of $\mathbb{R}^n$ is δ-badly approximable) and, consequently, almost any matrix in $\mathscr{M}_n(\mathbb{R})$ is badly approximable.

We store below a rather technical result [15, Theorem 2.6], omitting the proof. As compared with the previous theorem, it takes care of the possible oscillations $e^{i\,2\pi k.a}$, where a is a vector in $\mathbb{R}^n$, which will be allowed to vary later on.

Theorem A.11 *Let $P \in \mathbb{C}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d and let $b \in \mathcal{S}((\mathbb{Z}^n)^q)$ for $q \in \mathbb{N}^*$. Then,*

(i) For $a \in \mathbb{R}^n$, define $f_a(s) := \sum'_{k\in\mathbb{Z}^n} P(k)\, \|k\|^{-s}\; e^{i\,2\pi k.a}$.

1. If $a \in \mathbb{Z}^n$, then f_a has a meromorphic extension to the whole space $\mathbb{C}$.

Moreover, f_a is not entire if and only if $\int_{u\in S^{n-1}} P(u)\, ds(u) \neq 0$. In that case, f_a has a single simple pole at the point $d+n$, with $\underset{s=d+n}{\mathrm{Res}}\; f_a(s) = \int_{u\in S^{n-1}} P(u)\, ds(u)$.

2. If $a \in \mathbb{R}^n \setminus \mathbb{Z}^n$, then $f_a(s)$ extends holomorphically to $\mathbb{C}$.

(ii) Suppose that $\Theta \in \mathscr{M}_n(\mathbb{R})$ is badly approximable. For any $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$, the function

$$g(s) := \sum\nolimits_{l\in(\mathbb{Z}^n)^q} b(l)\, f_{\Theta\, \sum_i \varepsilon_i l_i}(s)$$

extends meromorphically to $\mathbb{C}$ with only one possible pole at $s = d + n$.

Moreover, if we set $\mathscr{Z} := \{l \in (\mathbb{Z}^n)^q \mid \sum_{i=1}^q \varepsilon_i l_i = 0\}$ and $V := \sum_{l\in\mathscr{Z}} b(l)$, then

1. If $V \int_{S^{n-1}} P(u)\, ds(u) \neq 0$, then $s = d + n$ is a simple pole of $g(s)$ and

$$\underset{s=d+n}{\mathrm{Res}}\; g(s) = V \int_{u\in S^{n-1}} P(u)\, ds(u).$$

2. If $V \int_{S^{n-1}} P(u)\, ds(u) = 0$, then $g(s)$ extends holomorphically to $\mathbb{C}$.

(iii) Suppose that $\Theta \in \mathscr{M}_n(\mathbb{R})$ is badly approximable. For any $(\varepsilon_i)_i \in \{-1, 0, 1\}^q$, the function

$$g_0(s) := \sum\nolimits_{l\in(\mathbb{Z}^n)^q\setminus\mathscr{Z}} b(l)\, f_{\Theta\, \sum_{i=1}^q \varepsilon_i\, l_i}(s),$$

with $\mathscr{Z} := \{l \in (\mathbb{Z}^n)^q \mid \sum_{i=1}^q \varepsilon_i l_i = 0\}$, extends holomorphically to $\mathbb{C}$.

Is is unknown whether the Diophantine condition, which is sufficient to get the results of (*iii*), is also necessary — see nevertheless [15, Remark 2.9].

In the study of the dimension spectrum of the noncommutative torus we will need the following result [15, Theorem 2.18(i)].

It requires some notations: Fix $q \in \mathbb{N}$, $q \geq 2$ and $r = (r_1, \ldots, r_{q-1}) \in (\mathbb{N}^*)^{q-1}$.

When $(x_1, \ldots, x_{2q})$, we set $\widetilde{x}_j := x_1 + \cdots + x_j + x_{q+1} + \cdots + x_{q+j}$ for any j with $1 \leq j \leq q$ and we let $P \in \mathbb{R}[x_1, \ldots, x_n]$ and $d = \deg P$.

Theorem A.12 *Let $\frac{1}{2\pi}\Theta$ be a badly approximable matrix, and $a \in \mathscr{S}\big((\mathbb{Z}^n)^{2q}\big)$. Then,*

$$s \mapsto f(s) := \sum_{l\in[(\mathbb{Z}^n)^q]^2} a_l \sum_{k\in\mathbb{Z}^n} \prod_{i=1}^{q-1} |k + \widetilde{l}_i|^{r_i}\;\; \|k\|^{-s}\; P(k)\, e^{ik.\Theta \sum_1^q l_j}$$

has a meromorphic extension to $\mathbb{C}$ with at most simple possible poles at the points $s = n + d + |r_1| + \cdots + \left|r_{q-1}\right| - m$ where $m \in \mathbb{N}$.

An explicit formula for the residues of f is given in [15, Theorem 2.18(ii)].

Appendix B
Examples of Spectral Triples

B.1 Spheres

A particularly illustrative example of a commutative spectral triple (recall Example 1.2) is provided by the d-dimensional unit spheres S^d.

On S^1 there are two possible spin structures, where the nontrivial one is associated to functions with antiperiodic boundary conditions. When $d \geq 2$ there is only one spin structure available since S^d is simply connected. Let us equip S^d with the standard round metric and cook up the standard Dirac operator $\mathscr{D}$ acting on the chosen spinor bundle $\mathscr{S}$. Then, $(C^\infty(S^d), L^2(S^d, \mathscr{S}), \mathscr{D})$ is a d-dimensional regular spectral triple with a simple dimension spectrum $d - \mathbb{N}$ (cf. Example 1.25), for any $d \geq 1$ and any spinor bundle $\mathscr{S}$.

The spectrum of $\mathscr{D}$ turns out to be very simple [2, 21]: For the trivial spin structure on S^1 we have $\lambda_n(\mathscr{D}) = n$ for $n \in \mathbb{Z}$ and all of the eigenspaces are one-dimensional. In particular, we have $\dim \ker \mathscr{D} = 1$. In the non-trivial case, the spectrum of the Dirac operator agrees with the general pattern for S^d and for $d \geq 1$:

$$\lambda_n(\mathscr{D}) = \operatorname{sign}(n)(n + \tfrac{d}{2}), \qquad M_n(\mathscr{D}) = 2^{\lfloor \frac{d}{2} \rfloor} \tbinom{|n|+d-1}{d-1}, \quad \text{with } n \in \mathbb{Z}. \tag{B.1}$$

Hence, $\mu_n(\mathscr{D}) = n + \frac{d}{2}$ with $M_n(|\mathscr{D}|) = 2^{\lfloor \frac{d}{2} \rfloor + 1} \binom{n+d-1}{d-1}$, with $n \in \mathbb{N}$.

B.2 Tori

Another commutative spectral triple is given by the flat tori $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and, as above, $(C^\infty(\mathbb{T}^d), L^2(\mathbb{T}^d, \mathscr{S}), \mathscr{D})$ is a d-dimensional regular spectral triple with a simple dimension spectrum $d - \mathbb{N}$. There are 2^d different spin structures on $\mathbb{T}^d$ classified by the twisting of each coordinates of the lattice $\mathbb{Z}^d$: Given a basis $e_1, \ldots, e_d$ of $\mathbb{Z}^d$, this is realised by choosing $s_1, \ldots, s_d \in \{0, 1\}$, so that we have the group

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homomorphism: $e_k \in \mathbb{Z}^d \to (-1)^{s_k} \in \{+1, -1\}$ and all spin structures are given this way by $(s_1, \dots, s_d)$. The eigenvalues of the Dirac operator $\mathscr{D}$ (endowed with the induced flat metric) depend on the chosen spin structures and are given by (see [2, 21])

$$\{\pm 2\pi \, \|k + \tfrac{1}{2} \sum_{j=1}^{d} s_j \, e_j^* \| \;\mid\; k \in \mathbb{Z}^{d^*}\}. \tag{B.2}$$

where $\mathbb{Z}^{d^*}$ is the dual lattice and $(e_1^*, \dots, e_d^*)$ is the dual basis of $(e_1, \dots, e_d)$.

The multiplicity of the eigenvalue 0 (given by $s_j = 0, \ \forall j$) is $2^{\lfloor d/2 \rfloor}$ while the non-zero eigenvalues have multiplicity $2^{\lfloor d/2 \rfloor - 1}$.

B.3 Noncommutative Tori

The noncommutative d-tori were introduced by Rieffel [34] and Connes [5] as deformations of $\mathbb{T}^d$ characterised a by non-zero skew-symmetric matrix $\Theta \in M_d(\mathbb{R})$.

Denote by $C^\infty(\mathbb{T}^d_\Theta)$ the algebra generated by d unitaries $u_i, i = 1, \dots, d$ satisfying

$$u_\ell \, u_j = e^{i \, \Theta_{\ell j}} \, u_j \, u_\ell, \tag{B.3}$$

and with Schwartz coefficients. So, $a \in C^\infty(\mathbb{T}^d_\Theta)$ can be written as $a = \sum_{k \in \mathbb{Z}^d} a_k \, U_k$, where $\{a_k\} \in \mathcal{S}(\mathbb{Z}^d)$ (i.e. $\sup_{k \in \mathbb{Z}^d} |k_1|^{n_1} \cdots |k_d|^{n_d} \, |a_k| < \infty, \ \forall n_i \in \mathbb{N}$) and

$$U_k := e^{-\frac{i}{2} k.\Theta' k} \, u_1^{k_1} \cdots u_n^{k_d}, \quad k \in \mathbb{Z}^d,$$

where Θ' is the restriction of Θ to its upper triangular part. Relation (B.3) reads

$$U_k U_q = e^{-\frac{i}{2} k.\Theta q} \, U_{k+q} \ \text{ or } \ U_k U_q = e^{-ik.\Theta q} \, U_q U_k \, . \tag{B.4}$$

Thus, the unitary operators U_k satisfy

$$U_k^* = U_{-k} \ \text{ and } [U_k, U_l] = -2i \, \sin(\tfrac{1}{2} k.\Theta l) \, U_{k+l}.$$

Let τ be the trace on $C^\infty(\mathbb{T}^d_\Theta)$ defined by

$$\tau\big(\textstyle\sum_{k \in \mathbb{Z}^d} a_k \, U_k\big) := a_0$$

and $\mathscr{H}_\tau$ be the GNS Hilbert space obtained by completion of $C^\infty(\mathbb{T}^d_\Theta)$ with respect to the norm induced by the scalar product $\langle a, b \rangle := \tau(a^* b)$.

On $\mathscr{H}_\tau = \{\, \sum_{k \in \mathbb{Z}^d} a_k \, U_k \mid \{a_k\}_k \in \ell^2(\mathbb{Z}^d) \,\}$, let δ_μ, for $\mu \in \{\, 1, \dots, d \,\}$, be the pairwise commuting canonical derivations, given by

$$\delta_\mu(U_k) := i k_\mu U_k \, . \tag{B.5}$$

Define now

$$\mathscr{A}_\Theta := C^\infty(\mathbb{T}^d_\Theta) \text{ acting on } \mathscr{H} := \mathscr{H}_\tau \otimes \mathbb{C}^{2^m}, \text{ with } m := \lfloor d/2 \rfloor.$$

Each element of $\mathscr{A}_\Theta$ is represented on $\mathscr{H}$ as $\pi(a) := L(a) \otimes 1_{2^m}$ where $L(\cdot)$ (and $R(\cdot)$) is the left (right) regular representation of $C^\infty(\mathbb{T}^d_\Theta)$. The Tomita conjugation $J_0(a) := a^*$ satisfies $J_0 L(a) = R(a^*) J_0$ and $[J_0, \delta_\mu] = 0$ and we define $J := J_0 \otimes C_0$, where C_0 is an operator on $\mathbb{C}^{2^m}$ such that $C_0^2 = \pm 1_{2^m}$, depending on the parity of m. The (flat) Dirac-like operator is given by

$$\mathscr{D} := -i\,\delta_\mu \otimes \gamma^\mu, \tag{B.6}$$

with hermitian Dirac matrices γ satisfying $C_0\gamma^\alpha = \pm\gamma^\alpha C_0$ (see [6, 22] for details about the signs). The operator $\mathscr{D}$ is defined and symmetric on the dense subset of $\mathscr{H}$ given by $C^\infty(\mathbb{T}^d_\Theta) \otimes \mathbb{C}^{2^m}$ and we still denote by $\mathscr{D}$ its selfadjoint extension. Thus

$$\mathscr{D}\, U_k \otimes e_i = k_\mu U_k \otimes \gamma^\mu e_i,$$

where (e_i) is the canonical basis of $\mathbb{C}^{2^m}$.

Finally, in the even case, the chirality operator reads: $\gamma := \mathrm{id} \otimes (-i)^m \gamma^1 \cdots \gamma^d$.

The operator $\mathscr{D}$ is not invertible: $\ker \mathscr{D} = U_0 \otimes \mathbb{C}^{2^m}$ has dimension 2^m because if $\psi = \sum_{k,j} c_{k,j} U_k \otimes e_j$, then $0 = \mathscr{D}^2\psi = \sum_{k,j} c_{k,j} \, \|k\|^2 \, U_k \otimes e_j$. Thus,

$$P_0 = |U_0\rangle\langle U_0| \otimes \mathbb{1}_{\mathbb{C}^{2^m}}. \tag{B.7}$$

This yields a spectral triple:

Theorem B.1 *The tuple $(\mathscr{A}_\Theta, \mathscr{H}, \mathscr{D}, J, \gamma)$ is a real regular spectral triple of dimension d. Its KO-dimension is also d.*

Most of the arguments will be revisited in the computation of the dimension spectrum — see Theorem B.2. For a complete proof see [6, 22].

We remark that the torus actions on C^*-algebras lead to interesting nonunital spectral triples, see [4, Chap. 5].

B.3.1 Dimension Spectrum

Theorem B.2 *(i) If $\frac{1}{2\pi}\Theta$ is badly approximable, the spectrum dimension of the triple $\big(C^\infty(\mathbb{T}^d_\Theta), \mathscr{H}, \mathscr{D}\big)$ is equal to the set $\{\, d - k \;:\; k \in \mathbb{N} \,\}$ and all of the poles are simple.*

(ii) $\zeta_D(0) = 0$.

Proof (*i*) Let $B \in \mathscr{P}(\mathscr{A})$ and $p \in \mathbb{N}$. Suppose that B is of the form

$$B = a_r b_r \mathscr{D}^{q_{r-1}} |\mathscr{D}|^{p_{r-1}} a_{r-1} b_{r-1} \cdots \mathscr{D}^{q_1} |\mathscr{D}|^{p_1} a_1 b_1,$$

where $r \in \mathbb{N}$, $a_i \in \mathscr{A}$, $b_i \in J\mathscr{A}J^{-1}$, $q_i, p_i \in \mathbb{N}$.

We decompose $a_i =: \sum_{\ell\in\mathbb{Z}^d} a_{i,\ell}\, U_\ell$ and $b_i =: \sum_{\ell'\in\mathbb{Z}^d} b_{i,\ell'}\, U_{\ell'}$.

With the shorthands $k_{\mu_1,\mu_{q_i}} := k_{\mu_1} \cdots k_{\mu_{q_i}}$ and $\gamma^{\mu_1,\mu_{q_i}} := \gamma^{\mu_1} \cdots \gamma^{\mu_{q_i}}$, we get

$$\begin{aligned}
&\mathscr{D}^{q_1} |\mathscr{D}|^{p_1} a_1 b_1\, U_k \otimes e_j \\
&\qquad = \sum_{\ell_1,\,\ell'_1} a_{1,\ell_1} b_{1,\ell'_1} U_{\ell_1} U_k U_{\ell'_1}\, |k+\ell_1+\ell'_1|^{p_1}\, (k+\ell_1+\ell'_1)_{\mu_1,\mu_{q_1}} \otimes \gamma^{\mu_1,\mu_{q_1}}\, e_j,
\end{aligned}$$

which gives, after r iterations,

$$\begin{aligned}
B(U_k \otimes e_j) = &\sum_{\ell,\ell'\in\mathbb{Z}^d} \widetilde{a}_\ell \widetilde{b}_\ell\, U_{\ell_r} \cdots U_{\ell_1} U_k U_{\ell'_1} \cdots U_{\ell'_r} \prod_{i=1}^{r-1} |k+\widehat{\ell}_i+\widehat{\ell'}_i|^{p_i} (k+\widehat{\ell}_i+\widehat{\ell'}_i)_{\mu^i_1,\mu^i_{q_i}} \\
&\otimes \gamma^{\mu^{r-1}_1,\mu^{r-1}_{q_{r-1}}} \cdots \gamma^{\mu^1_1,\mu^1_{q_1}}\, e_j, \\
\text{where}\quad &\widetilde{a}_\ell := a_{1,\ell_1} \cdots a_{r,\ell_r}, \qquad \widetilde{b}_{\ell'} := b_{1,\ell'_1} \cdots b_{r,\ell'_r}, \\
&\widehat{\ell}_i := \ell_1 + \cdots + \ell_i, \qquad \gamma^\mu := \gamma^{\mu^{r-1}_1,\mu^{r-1}_{q_{r-1}}} \cdots \gamma^{\mu^1_1,\mu^1_{q_1}}.
\end{aligned}$$

Let us denote $F_\mu(k,\ell,\ell') := \prod_{i=1}^{r-1} |k+\widehat{\ell}_i+\widehat{\ell'}_i|^{p_i}\, (k+\widehat{\ell}_i+\widehat{\ell'}_i)_{\mu^i_1,\mu^i_{q_i}}$.

With the shortcut $\sim_c$ meaning equality modulo a constant function in the variable s, we have

$$\operatorname{Tr}\big(B|D|^{-p-s}\big) \sim_c \sum_{k\in\mathbb{Z}^d}{}' \sum_{\ell,\ell'\in\mathbb{Z}^d} \widetilde{a}_\ell \widetilde{b}_{\ell'}\, \tau\big(U_{-k} U_{\ell_r} \cdots U_{\ell_1} U_k U_{\ell'_1} \cdots U_{\ell'_r}\big) \tfrac{F_\mu(k,\ell,\ell')}{\|k\|^{s+p}} \operatorname{tr}(\gamma^\mu).$$

Since $U_{\ell_r} \cdots U_{\ell_1} U_k = U_k U_{\ell_r} \cdots U_{\ell_1}\, e^{-i\sum_{i=1}^r \ell_i.\Theta k}$, we get

$$\tau\big(U_{-k} U_{\ell_r} \cdots U_{\ell_1} U_k U_{\ell'_1} \cdots U_{\ell'_r}\big) = \delta_{\sum_{i=1}^r \ell_i+\ell'_i,0}\, e^{i\phi(\ell,\ell')}\, e^{-i\sum_{i=1}^r \ell_i.\Theta k},$$

where ϕ is a real valued function. Thus,

$$\begin{aligned}
\operatorname{Tr}\big(B|D|^{-p-s}\big) &\sim_c \sum_{k\in\mathbb{Z}^d}{}' \sum_{\ell,\ell'\in\mathbb{Z}^d} e^{i\phi(\ell,\ell')}\, \delta_{\sum_{i=1}^r \ell_i+\ell'_i,0}\, \widetilde{a}_\ell \widetilde{b}_{\ell'}\, \frac{F_\mu(k,\ell,\ell')\, e^{-i\sum_{i=1}^r \ell_i.\Theta k}}{\|k\|^{s+p}} \operatorname{tr}(\gamma^\mu) \\
&\sim_c f_\mu(s) \operatorname{tr}(\gamma^\mu).
\end{aligned}$$

The function $f_\mu(s)$ can be decomposed as a linear combination of zeta functions of the type described in Theorem A.12 (or, if $r = 1$ or if all p_i's are zero, in Theorem A.11). Thus, by linearity, $s \mapsto \operatorname{Tr}\big(B|D|^{-p-s}\big)$ has a meromorphic extension to $\mathbb{C}$ with simple poles located exclusively in $\mathbb{Z} \subset \mathbb{C}$.

Moreover, if $B \in \Psi^0(\mathscr{A})$ and $q \in \mathbb{N}$ is such that $q > d$, then $B\,|D|^{-s} \in \mathrm{OP}^{-\Re(s)}$, so it is trace-class around q and hence $\zeta_{B,D}(s) = \operatorname{Tr} B\,|D|^{-s}$ is regular around q.

(*ii*) Let $Z_d(s) := \sum'_{k\in\mathbb{Z}^d} \|k\|^{-s}$ be the Epstein zeta function associated to the quadratic form $q(x) := x_1^2 + \cdots + x_d^2$. Then Z_d enjoys the functional equation:

$$Z_d(s) = \pi^{s-d/2}\Gamma(d/2 - s/2)\Gamma(s/2)^{-1} Z_d(d-s). \tag{B.8}$$

Since $\pi^{s-d/2}\Gamma(d/2-s/2)\,\Gamma(s/2)^{-1} = 0$ for any negative even integer d and $Z_d(s)$ is meromorphic on $\mathbb{C}$ with only one pole at $s = d$ (with residue $2\pi^{d/2}\Gamma(d/2)^{-1}$ according to (A.25)), we get $Z_d(0) = -1$.

By definition, $\zeta_D(s) = \sum_{k\in\mathbb{Z}^d}\sum_{j=1}^{2^m}\langle U_k\otimes e_j, |D|^{-s}U_k\otimes e_j\rangle$, so that

$$\zeta_D(s) = 2^{\lfloor d/2\rfloor}(Z_d(s)+1) \tag{B.9}$$

and the conclusion $\zeta_D(0) = 0$ follows. □

B.3.2 Heat Kernel Expansion

Proposition B.3 *The heat trace asymptotics is* Tr $e^{-t\mathscr{D}^2} \underset{t\downarrow 0}{\sim} 2^{\lfloor d/2\rfloor}\pi^{d/2}\,t^{-d/2}$.

Proof Since $\mathscr{D}^2\,U_k\otimes e_i = \|k\|^2\,U_k\otimes e_i$, we know by Formula (2.22) that

$$\text{Tr } e^{-t\mathscr{D}^2} = 2^m\sum_{k\in\mathbb{Z}^d} e^{-t\|k\|^2} \underset{t\downarrow 0}{\sim} 2^{\lfloor d/2\rfloor}\pi^{d/2}\,t^{-d/2}.$$ □

This result can also be obtained (in an admittedly circuitous way) from Theorem 3.6: We have $\mathcal{Z}_{D^2}(s) = 2^m\Gamma(s)(Z_d(2s)+1)$ and the Epstein zeta function Z_d is meromorphic on $\mathbb{C}$ with a single simple pole at $s = d$. Moreover, the Epstein zeta function enjoys a polynomial growth on the verticals — see the non trivial estimates demonstrated in [14, 30].

Remark that we recover the classical result: For the torus $\mathbb{T}^d$ with the usual scalar Laplacian $\Delta = -g^{\mu\nu}\partial_\mu\partial_\nu$, $\text{Tr } e^{-t\Delta} = \text{Vol}(\mathbb{T}^d)(4\pi)^{-d/2}\,t^{-d/2} + \mathcal{O}_0(t^{-d/2}e^{-1/4t})$ and $\text{Vol}(\mathbb{T}^d) = (2\pi)^d$ so that $\text{Vol}(\mathbb{T}^d)(4\pi)^{-d/2} = \pi^{d/2} = a_{d,0}$ (mod 2^m).

B.3.3 Spectral Action for Noncommutative Tori

We consider the d-dimensional noncommutative torus $(\mathscr{A}_\Theta, \mathscr{H}, \mathscr{D})$ of Theorem B.1 with $\mathscr{A}_\Theta = C^\infty(\mathbb{T}^d_\Theta)$, a one-form $A = A^* \in \Omega^1_{\mathscr{D}}(\mathscr{A}_\Theta)$ and

$$\mathbb{A} := A + \varepsilon JAJ^*. \tag{B.10}$$

Thus, the constraint $\mathbb{A} = \mathbb{A}^* \in \Psi^0(\mathscr{A})$ is satisfied.

Remark that A can be written as $A =: L(-iA_\alpha) \otimes \gamma^\alpha$, where $A_\alpha = -A_\alpha^* \in \mathscr{A}_\Theta$, so that

$$\mathscr{D}_{\mathbb{A}} = -i[\delta_\alpha + L(A_\alpha) - R(A_\alpha)] \otimes \gamma^\alpha. \tag{B.11}$$

As for the commutative torus $\mathbb{T}^d$, we get

$$\mathscr{D}_{\mathbb{A}}^2 = -\delta^{\alpha_1\alpha_2}(\delta_{\alpha_1} + \mathbb{A}_{\alpha_1})(\delta_{\alpha_2} + \mathbb{A}_{\alpha_2}) \otimes 1_{2^m} - \tfrac{1}{2}\Omega_{\alpha_1\alpha_2} \otimes \gamma^{\alpha_1\alpha_2},$$

where $\delta^{\alpha\beta}$ is the Kronecker symbol and

$$\begin{aligned} &\mathbb{A}_\alpha := L(A_\alpha) - R(A_\alpha), \quad \gamma^{\alpha_1\alpha_2} := \tfrac{1}{2}(\gamma^{\alpha_1}\gamma^{\alpha_2} - \gamma^{\alpha_2}\gamma^{\alpha_1}),\\ &\Omega_{\alpha_1\alpha_2} := [\delta_{\alpha_1} + \mathbb{A}_{\alpha_1}, \delta_{\alpha_2} + \mathbb{A}_{\alpha_2}] = L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2}),\\ &F_{\alpha_1\alpha_2} := \delta_{\alpha_1}(A_{\alpha_2}) - \delta_{\alpha_2}(A_{\alpha_1}) + [A_{\alpha_1}, A_{\alpha_2}] \in \mathscr{A}_\Theta. \end{aligned} \tag{B.12}$$

Thus,

$$\begin{aligned} \mathscr{D}_{\mathbb{A}}^2 = -\delta^{\alpha_1\alpha_2}[\delta_{\alpha_1} + L(A_{\alpha_1}) - R(A_{\alpha_1})]\,[\delta_{\alpha_2} + L(A_{\alpha_2}) - R(A_{\alpha_2})] \otimes 1_{2^m}\\ -\tfrac{1}{2}\big(L(F_{\alpha_1\alpha_2}) - R(F_{\alpha_1\alpha_2})\big) \otimes \gamma^{\alpha_1\alpha_2}. \end{aligned} \tag{B.13}$$

We now prove the existence of the asymptotics of the fluctuated heat trace on the noncommutative torus using the one for the 'bare' one given in Proposition B.3 (cf. also Problem 6(e) in Chap. 5). To this end, we employ the pseudodifferential calculus introduced by Connes for C^*-dynamical system $(A, \mathbb{R}^d, \alpha)$ (see [5, 7, 8]) and follow the arguments given in the proof of [29, Theorem 4.2]. The idea is essentially to mimic the classical pdo calculus on a manifold — cf. Appendix A.3 — improving the Proposition 2.27 to gain control on the series defined by Tr $e^{-t\mathscr{D}_{\mathbb{A}}^2}$.

We first quickly summarise this symbolic calculus — see [5, 7, 8] (and especially the complete approach by Lesch and Moscovici [29]) for details.

Let A_θ be the universal C^*-algebra generated by the U_k and let $\mathscr{A}_\theta$ consist of those elements in A_θ for which $a \to \alpha_s(a)$ is C^∞ for each $s \in \mathbb{R}^d$, with the definition $\alpha_s(U_k) := e^{-i2\pi\, s.k}\,U_k$.

A smooth map $\rho : C^\infty(\mathbb{R}^d) \to \mathscr{A}_\theta$ is named a symbol of order $m \in \mathbb{Z}$ if for any $k, \ell \in \mathbb{N}^d$, $\|\delta^k \partial_\xi^\ell \rho(\xi)\| \le c_{k,\ell}(1 + \|\xi\|)^{m-|\ell|}$ for some constants $c_{k,\ell}$, where we define $\delta^k := \delta_1^{k_1} \cdots \delta_d^{k_d}$ for $k \in \mathbb{N}^d$ and $\partial_\xi^\ell := \partial_{\xi_1}^{\ell_1} \cdots \partial_{\xi_d}^{\ell_d}$, and if there exists $\sigma \in C^\infty(\mathbb{R}^d, A_\theta^\infty)$ such that $\lim_{\lambda\to\infty} \lambda^{-m}\rho(\lambda\xi) = \sigma(\xi)$. Such a symbol is elliptic when $\rho(\xi)^{-1}$ exists and the estimate $\|\rho(\xi)^{-1}\| \le c(1 + \|\xi\|)^{-m}$ holds for $\|\xi\|$ large enough.

Given a symbol ρ, let us define the pdo $P_\rho : \mathcal{S}(\mathbb{R}^d, \mathscr{A}_\theta) \to \mathcal{S}(\mathbb{R}^d, \mathscr{A}_\theta)$ by

$$P_\rho(u) := \int \alpha_{-x}(\mathfrak{F}^{-1}[\rho](x-y))\, u(y)\, dy = \int \alpha_{-x}(\mathfrak{F}^{-1}[\rho](y))\, u(x-y)\, dy.$$

An action of the pre-C^*-algebra $\mathscr{A}_\theta \rtimes_\alpha \mathbb{R}^d$ on the pre-C^*-module $\mathcal{S}(\mathbb{R}^d, \mathscr{A}_\theta)$ is given, for $a \in \mathscr{A}_\theta$, by $a\,u(x) = \alpha_{-x}(a)\,u(x)$ and $U_y(u)(x) := u(x-y)$.

Then, $P_\rho = \int \mathfrak{F}^{-1}[\rho](y)\,U_y\,dy$ can be seen as an element of the multiplier space of $A_\theta \rtimes_\alpha \mathbb{R}^d$. The GNS representation of A_θ given by the trace τ can be extended to $A_\theta \rtimes_\alpha \mathbb{R}^d$, with the maps $a \in A_\theta \to a$ and $U_x \to \alpha_x$, so that $P_\rho = \int \mathfrak{F}^{-1}[\rho](y)\,\alpha_y\,dy$ can be seen as an element of the multiplier of A_θ.

Since $P_\rho\,U_k = \int \mathfrak{F}^{-1}[\rho](y)e^{-i2\pi\,y.k}\,dy\,U_k = \rho(k)\,U_k$, we get, for any element $a = \sum_{k\in\mathbb{Z}^d} a_k\,U_k$, $P_\rho(a) = \sum_{k\in\mathbb{Z}^d} a_k\,\rho(k)\,U_k$.

Thus, $\operatorname{Tr} P_\rho = \sum_{k\in\mathbb{Z}^d}\langle U_k, P_\rho U_k\rangle = \sum_{k\in\mathbb{Z}^d}\tau(U_k^*\rho(k)U_k) = \sum_{k\in\mathbb{Z}^d}\tau[\rho(k)]$ is finite if $m < -d$, because $\|\rho(k)\| \le c_{0,0}(1 + (\sum_{i=1}^d k_i^2)^{1/2})^m$.

A parametric symbol $\rho(\xi,\lambda)$ with $\lambda \in V$ – a region in $\mathbb{C}$ is defined similarly:

$$\|\delta^k \partial_\xi^\ell \partial_\lambda^r \rho(\xi,\lambda)\| \le c_{k,\ell}(1 + \|\xi\| + |\lambda|)^{m-|\ell|-|r|}. \tag{B.14}$$

Cf. [19, Sect. 1.7.1] and the (slightly different) Hypothesis A.1 of Appendix A.3.

We now adapt the Proposition 2.27:

Proposition B.4 *If $\rho \in S^m(\mathbb{R}^d \times V, A_\theta^\infty)$ is a parametric symbol of order $m < -d$, then $\sum_{k\in\mathbb{Z}^d}\rho(k,\lambda) = \int_{\mathbb{R}^d}\rho(\xi,\lambda)\,d\xi + \mathcal{O}_\infty(|\lambda|^{-\infty})$.*

Proof Using (2.19), $\sum_{k\in\mathbb{Z}^d}\rho(k,\lambda) = \int_{\mathbb{R}^d}\rho(\xi,\lambda) + \sum_{k\in\mathbb{Z}^d\setminus\{0\}}\mathfrak{F}[\rho(\cdot,\lambda)](k)$.

Via the Fourier transform of a derivative and (B.14), we get, for each $N \in \mathbb{N}^*$, $\|\mathfrak{F}[\rho(\cdot,\lambda)]\| \le \|\xi\|^{-N}\,|\lambda|^{d+m-N}$ and the conclusion follows from: For any $q \in \mathbb{N}^d$,

$$\Big\| \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \mathfrak{F}[\rho(\cdot,\lambda)](k) \Big\|_q \le c_{q,N}\Big(\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\|k\|^{-N}\Big)\,|\lambda|^{d+m-N}.$$

□

Lemma B.5 *For $\mathscr{D}_\mathbb{A}$ as in (B.11), we have $\operatorname{Tr} e^{-t\mathscr{D}_\mathbb{A}^2} \underset{t\downarrow 0}{\sim} \sum_{k=0}^\infty a_k(\mathscr{D}_\mathbb{A})\,t^{(k-d)/2}$. Moreover, for $f(\sqrt{\cdot}) \in \mathcal{C}_0^r$ with $r > d/2$, the spectral action has the asymptotics*

$$S(\mathscr{D}_\mathbb{A}, f, \Lambda) = \sum_{k=1}^d \Lambda^k \int_0^\infty f(t)\,t^{k-1}\,dt \;⨍\, |D_\mathbb{A}|^{-k} + f(0)\,\zeta_{D_\mathbb{A}}(0) + \mathcal{O}_\infty(\Lambda^{-1}). \tag{B.15}$$

Proof The operator $\mathscr{D}_\mathbb{A}$ given in (B.11) can be seen as a differential multiplier of order 1. Namely, $\mathscr{D}_\mathbb{A} \in \operatorname{Diff}_\sigma^1(\mathbb{R}^d, \mathscr{B}_\theta \otimes \mathbb{C}^{2^m})$, where $\mathscr{B}_\theta$ is the algebra generated by $\mathscr{A}_\theta$ and $J\mathscr{A}_\theta J^{-1}$, following a slight extension of [29, Definition 3.5]. The symbol of $\mathscr{D}_A$ is $\sigma_{\mathscr{D}_A}(\xi) = (\xi_\alpha - iA_\alpha - iJA_\alpha J^{-1}) \otimes \gamma^\alpha$. Thus, $\mathscr{D}_\mathbb{A}^2$ given in (B.13) has the form $\sum_{\beta\in\mathbb{N}^d,\,|\beta|\le 2} b_\beta \xi^\beta$, where $b_\beta \in \mathscr{B}_\theta \otimes \mathbb{C}^{2^m}$ with $\xi^\beta := \xi_1^{\beta_1}\cdots\xi_d^{\beta_d}$, and $\mathscr{D}_\mathbb{A}^2$ is an elliptic differential multiplier of order 2. As such, its symbol can be decomposed into a sum of monomials $a_j(\xi)$ of order $j = 2,\ 1,\ 0$: $\sigma_{\mathscr{D}_\mathbb{A}^2}(\xi) = a_2(\xi) + a_1(\xi) + a_0$.

If λ is the resolvent parameter for $(\mathscr{D}_A^2 - \lambda)^{-1}$, in the search for the resolvent parametrix B_λ, we need to solve $1 = \sigma_\lambda \star \sigma_{B_\lambda}$ for the parameter dependent symbol $\sigma_\lambda(\xi) := \sigma_{\mathscr{D}_\mathbb{A}^2}(\xi) - \lambda = a_2'(\xi) + a_1(\xi) + a_0$ with $a_2'(\xi) = a_2(\xi) - \lambda$. Since

$\sigma_{B_\lambda} = b_0 + b_1 + b_2 + \dots$, where $b_j(\xi;\lambda)$ is a symbol of order $-2-j$, one can compute recursively the symbols $b_0,\ b_1,\ b_2, \dots$ of the parameter-dependent pseudodifferential multiplier B_λ (see [7, 8, 16, 17, 29] for examples of such computations).

Then, thanks to (A.24), $\mathrm{Tr}\, e^{-t\mathscr{D}_{\mathbb{A}}^2} = t^{-k+1} \frac{i}{2\pi} \int_{\mathscr{C}} e^{-t\lambda}\ \mathrm{Tr}\,[(\mathscr{D}_{\mathbb{A}}^2 - \lambda)^{-k}]\, d\lambda$ (recall that $\mathrm{Tr}\,[(\mathscr{D}_{\mathbb{A}}^2-\lambda)^{-k}] < \infty$ for $k \geq \lfloor d/2 \rfloor + 1$), one implements the resolvent expansion into the integral to get the asymptotics $\mathrm{Tr}\, e^{-t\mathscr{D}_{\mathbb{A}}^2} \underset{t\downarrow 0}{\sim} \sum_{k=0}^{\infty} a_k(\mathscr{D}_{\mathbb{A}}^2)\, t^{(k-d)/2}$ exactly as in [19, Sects. 1.7 and 1.8] using the Proposition B.4.

Thus, $a_k(\mathscr{D}_{\mathbb{A}}^2) = \frac{i}{2\pi} \int_{\mathbb{R}^d} d\xi \int_{\mathscr{C}} e^{-\lambda}\, \tau[b_k(\xi,\lambda)]\, d\lambda$.

Finally, we use Formula (3.43), which becomes (B.15). □

While there is no Diophantine condition to get the asymptotics of the fluctuated heat trace, the computation of the coefficients seems to need one — this is due to a commutation between a series and a residue.

Theorem B.6 *Assume that $\frac{1}{2\pi}\Theta$ is badly approximable. Then, the spectral action (B.15) fluctuated by $\mathbb{A}$ as in (B.10) reads, for $f\left(\sqrt{\cdot}\right) \in \mathcal{C}_0^r$ with $r > d/2$,*

$$S(\mathscr{D}_{\mathbb{A}}, f, \Lambda) = \begin{cases} 4\pi f_2\, \Lambda^2 + \mathcal{O}_\infty(\Lambda^{-2}), & \text{for } d = 2, \\ 8\pi^2 f_4\, \Lambda^4 - \frac{4\pi^2}{3} f(0)\, \tau(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}_\infty(\Lambda^{-2}), & \text{for } d = 4. \end{cases}$$

For arbitrary d, $S(\mathscr{D}_{\mathbb{A}}, f, \Lambda) = \sum_{k=0}^{d-1} f_{d-k}\, c_{d-k}(A)\, \Lambda^{d-k} + \mathcal{O}_\infty(\Lambda^{-1})$, where we have $c_{d-2}(A) = 0$ and $c_{d-k}(A) = 0$ for k odd. In particular, $c_0(A) = 0$ when d is odd.

We do not know if, without the Diophantine condition, the above spectral action would stay the same, see [18, Appendix B].

The proof goes through several steps and the first one is to identify the noncommutative integrals in (B.15):

Proposition B.7 *Assume that $\frac{1}{2\pi}\Theta$ is a badly approximable matrix. Then, we have $\int\!\!\!\!\!- |D_{\mathbb{A}}|^{-d} = 2^{\lfloor \frac{d+2}{2} \rfloor} \pi^{d/2} \Gamma(\frac{d}{2})^{-1}$, $\int\!\!\!\!\!- |D_{\mathbb{A}}|^{-d+k} = 0$ for k odd and $\int\!\!\!\!\!- |D_{\mathbb{A}}|^{-d+2} = 0$.*

These equalities follow from the explicit computation of (4.15)–(4.24) and the fact that $\int\!\!\!\!\!- P\,|D|^{-d+q} = 0$ for any $P \in \Psi'(\mathscr{A})$ and any odd integer q.

In a similar way, $\int\!\!\!\!\!- A^p D^{-q} = \int\!\!\!\!\!- (\varepsilon JAJ^*)^p D^{-q} = 0$ and $\int\!\!\!\!\!- P\, D^{-d+q} = 0$ for $p \geq 0$, $1 \leq q < d$ and P in the algebra generated by $\mathscr{A}$, $[\mathscr{D}, \mathscr{A}]$, $J\mathscr{A}J^*$, $J[\mathscr{D}, \mathscr{A}]J^*$. All these equalities can be proved using the deep results stored in Appendix A.6. The case $p = q = 1$ corresponds to the fact that there is no tadpole for the noncommutative torus — see Definition 4.12. The main point is again to be able to commute an infinite series, like those defining an element of $\mathscr{A}_\Theta$, and a residue given by $\int\!\!\!\!\!-$ and this is where the hypothesis on $\frac{1}{2\pi}\Theta$ is used.

In the second step, we face explicit computations like, inter alia, the following:

Lemma B.8 *Under same hypothesis (recall that $A = L(-iA_\alpha) \otimes \gamma^\alpha$),*

$$\int\!\!\!\!-\, A^q D^{-q} = \begin{cases} -\delta_{q,2} 4\pi \, \tau(A_\alpha A^\alpha) & \textit{for } d = 2, \\ \delta_{q,4} \frac{\pi^2}{12} \, \tau(A_{\alpha_1} \cdots A_{\alpha_4}) \, \mathrm{tr}\,(\gamma^{\alpha_1} \cdots \gamma^{\alpha_4} \gamma^{\mu_1} \cdots \gamma^{\mu_4}) \delta_{\mu_1,\mu_2,\mu_3,\mu_4} & \textit{for } d = 4, \end{cases}$$

where $\delta_{\mu_1,\mu_2,\mu_3,\mu_4} = \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4} + \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4} + \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}$.

With $A = L(-iA_\alpha) \otimes \gamma^\alpha = -i \sum_{\ell \in \mathbb{Z}^d} a_{\alpha,\ell} U_\ell \otimes \gamma^\alpha$ *and* $c := \frac{4\pi^2}{3}$, *we have*

$$\tfrac{1}{2} \int\!\!\!\!-\, (AD^{-1})^2 = \tfrac{1}{2} \int\!\!\!\!-\, (\varepsilon JAJ^* D^{-1})^2 = c \sum_{\ell \in \mathbb{Z}^d} a_{\alpha_1,\ell}\, a_{\alpha_2,-\ell} \left(l^{\alpha_1} \ell^{\alpha_2} - \delta^{\alpha_1\alpha_2} \|\ell\|^2 \right).$$

$$-\tfrac{1}{3} \int\!\!\!\!-\, (AD^{-1})^3 = -\tfrac{1}{3} \int\!\!\!\!-\, (\varepsilon JAJ^* D^{-1})^3 = 4c \sum_{\ell_i \in \mathbb{Z}^d} a_{\alpha_3,-\ell_1-\ell_2}\, a_{\alpha_1,\ell_2}\, a_{\alpha_1,\ell_1} \sin \tfrac{\ell_1.\Theta\ell_2}{2}\, \ell_1^{\alpha_3}.$$

$$\tfrac{1}{4} \int\!\!\!\!-\, (AD^{-1})^4 = \tfrac{1}{4} \int\!\!\!\!-\, (\varepsilon JAJ^* D^{-1})^4$$
$$= 2c \sum_{\ell_i \in \mathbb{Z}^d} a_{\alpha_1,-\ell_1-\ell_2-\ell_3}\, a_{\alpha_2,\ell_3}\, a_{\alpha_1,\ell_2}\, a_{\alpha_2,\ell_1} \sin \tfrac{\ell_1.\Theta(\ell_2+\ell_3)}{2} \sin \tfrac{\ell_2.\Theta\ell_3}{2}.$$

These equalities follows from tedious computations (cf. [15] for the details). They are necessary to get $\zeta_{D_{\mathbb{A}}}(0)$ from (4.32), using $\zeta_D(0) = 0$ obtained in Theorem B.2. This gives $\zeta_{D_{\mathbb{A}}}(0) = -c\,\tau(F_{\mu\nu}F^{\mu\nu})$ in dimension $d = 4$, while $\zeta_{D_{\mathbb{A}}}(0) = 0$ for $d = 2$ or when d is odd. Gathering all these computations brings us to Theorem B.6.

B.4 Podleś Sphere

Podleś has introduced the eponym standard quantum spheres in [32] as homogeneous spaces under the action of the quantum deformation of the $SU(2)$ group. They fit into the picture of noncommutative geometry à la Connes and concrete spectral triples were constructed in [11, 12], see also [10].

The algebra $\bar{\mathscr{A}_q}$ introduced in [32] is a complex $*$-algebra generated, for a parameter $0 < q < 1$, by $A = A^*$, B, B^* subject to the relations

$$AB = q^2 BA, \quad AB^* = q^{-2} B^* A, \quad BB^* = q^{-2} A(1 - A), \quad B^* B = A(1 - q^2 A).$$

As a C^*-algebra, it is isomorphic to the minimal unitisation of the algebra of compact operators on a separable Hilbert space and is an invariant subalgebra of the quantum group $SU_q(2)$ under a circle action. We will use only the polynomial algebra $\mathscr{A}_q$ in the above generators, which is a dense subalgebra of $\bar{\mathscr{A}_q}$.

In the following we will employ the *q-numbers* defined as

$$[n] := \tfrac{q^{-n} - q^n}{q^{-1} - q}, \quad \text{for } q \in (0, 1) \text{ and } n \in \mathbb{C}. \tag{B.16}$$

Observe that $\lim_{q \to 1}[n] = n$ for any $n \in \mathbb{C}$.

Let $\mathcal{H}_{1/2}$ be the separable Hilbert space with an orthonormal basis $|l, m\rangle$ for $m \in \{-l, -l+1, \dots, l\}$ and $l \in \frac{1}{2} + \mathbb{N}$. It is suitable for a representation of $\mathcal{A}_q$, equivariant under the action of the *-Hopf algebra $\mathcal{U}_q(\mathfrak{su}(2))$. There exist two non-equivalent $\mathcal{U}_q(\mathfrak{su}(2))$-equivariant representations of $\mathcal{A}_q$ on $\mathcal{H}_{1/2}$ [12]:

$$\begin{aligned}
\pi_\pm(A)|l,m\rangle_\pm &:= A^+_{l,m,\pm}\,|l+1,m\rangle_\pm + A^0_{l,m,\pm}\,|l,m\rangle_\pm + A^-_{l,m,\pm}\,|l-1,m\rangle_\pm\,, \qquad \text{(B.17)}\\
\pi_\pm(B)|l,m\rangle_\pm &:= B^+_{l,m,\pm}\,|l+1,m+1\rangle_\pm + B^0_{l,m,\pm}\,|l,m+1\rangle_\pm + B^-_{l,m,\pm}\,|l-1,m+1\rangle_\pm,\\
\pi_\pm(B^*)|l,m\rangle_\pm &:= \widetilde{B}^+_{l,m,\pm}|l+1,m-1\rangle_\pm + \widetilde{B}^0_{l,m+1,\pm}|l,m-1\rangle_\pm + \widetilde{B}^-_{l,m,\pm}|l-1,m-1\rangle_\pm,
\end{aligned}$$

$$\begin{aligned}
A^+_{l,m} &:= -q^{m+l+\frac{1}{2}}\sqrt{[l-m+1][l+m+1]}\;\alpha^+_l, \qquad \text{(B.18)}\\
A^0_{l,m} &:= q^{-\frac{1}{2}}\tfrac{1}{1+q^2}\big([l-m+1][l+m] - q^2[l-m][l+m+1]\big)\,\alpha^0_l + \tfrac{1}{1+q^2}\,,\\
A^-_{l,m} &:= q^{m-l-\frac{1}{2}}\sqrt{[l-m][l+m]}\;\alpha^-_l\,,
\end{aligned}$$

$$\begin{aligned}
B^+_{l,m} &:= q^m\sqrt{[l+m+1][l+m+2]}\;\alpha^+_l\,, \qquad \text{(B.19)}\\
B^0_{l,m} &:= q^m\sqrt{[l+m+1][l-m]}\;\alpha^0_l\,,\\
B^-_{l,m} &:= q^m\sqrt{[l-m][l-m-1]}\;\alpha^-_l\,,
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}^+_{l,m} &:= q^{m-1}\sqrt{[l-m+2][l-m+1]}\;\alpha^-_{l+1}\,, \qquad \text{(B.20)}\\
\widetilde{B}^0_{l,m} &:= q^{m-1}\sqrt{[l+m][l-m+1]}\;\alpha^0_l\,,\\
\widetilde{B}^-_{l,m} &:= q^{m-1}\sqrt{[l+m][l+m-1]}\;\alpha^+_{l-1}\,.
\end{aligned}$$

The coefficients α_l read: $\alpha^-_l := -q^{2l+2}\,\alpha^+_l$ and

$$\text{for } \pi_+: \qquad \alpha^0_l := \frac{1}{\sqrt{q}}\,\frac{(q-\frac{1}{q})[l-\frac{1}{2}][l+\frac{3}{2}]+q}{[2l][2l+2]}\,, \qquad \text{(B.21)}$$

$$\alpha^+_l := q^{-l-2}\,\frac{1}{\sqrt{[2l+2]([4l+4]+[2][2l+2])}}\;; \qquad \text{(B.22)}$$

$$\text{for } \pi_-: \qquad \alpha^0_l := \frac{1}{\sqrt{q}}\,\frac{(q-\frac{1}{q})[l-\frac{1}{2}][l+\frac{3}{2}]-q^{-1}}{[2l][2l+2]}\,, \qquad \text{(B.23)}$$

$$\alpha^+_l := q^{-l-1}\,\frac{1}{\sqrt{[2l+2]([4l+4]+[2][2l+2])}}\,. \qquad \text{(B.24)}$$

Since $\pi^\pm$ are faithful, the algebra $\mathcal{A}_q$ is dense in $\bar{\mathcal{A}_q}$ in the operator norm.

Let now $\mathscr{H}_q := \mathscr{H}_+ \oplus \mathscr{H}_-$, where $\mathscr{H}_\pm := \mathscr{H}_{1/2}$, with the representation of $\mathscr{A}_q$:

$$\pi(a) := \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix}. \tag{B.25}$$

On $\mathscr{H}$ we define an unbounded selfadjoint operator given by

$$\mathscr{D}_q := \begin{pmatrix} 0 & \bar{w}\, T_q \\ w\, T_q & 0 \end{pmatrix}, \qquad T_q : |l, m\rangle \in \mathscr{H}_{1/2} \mapsto [l + \tfrac{1}{2}]\, |l, m\rangle \in \mathscr{H}_{1/2}, \tag{B.26}$$

for an arbitrary constant $w \in \mathbb{C} \setminus \{0\}$.

We note that $\ker \mathscr{D}_q = \{\, 0\, \}$ and the polar decomposition of $\mathscr{D}_q$ reads

$$\mathscr{D}_q = F|\mathscr{D}_q|, \quad \text{with} \quad |\mathscr{D}_q| = |w| \begin{pmatrix} T_q & 0 \\ 0 & T_q \end{pmatrix} \quad \text{and} \quad F := \tfrac{1}{|w|}\begin{pmatrix} 0 & \bar{w} \\ w & 0 \end{pmatrix}. \tag{B.27}$$

The phase operator satisfies $[F,\, \pi(a)] = 0$ for $a \in \mathscr{A}_q$. We have

$$|\mathscr{D}_q|\, (|l, m\rangle_+ \oplus |l, m\rangle_-) = |w|[l + \tfrac{1}{2}]\, (|l, m\rangle_+ \oplus |l, m\rangle_-)\,,$$

which gives

$$\mu_n(\mathscr{D}_q) = \lambda_n(|\mathscr{D}_q|) = |w|[n + 1] \quad \text{and} \quad M_n(|\mathscr{D}_q|) = 4(n + 1), \quad n \in \mathbb{N}. \tag{B.28}$$

Let us recall (B.16) and observe that the singular values of $\mathscr{D}_q$ grow exponentially.

In [12] it is proven that $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q)$ is a spectral triple, which moreover is even for $\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and real for the antiunitary operator J on $\mathscr{H}_q$ defined by

$$J\, |l, m\rangle_\pm := i^{2m} p^m\, |l, -m\rangle_\mp \quad \text{with}\; p \in \mathbb{R}^+.$$

In particular, $J^2 = -1$, $J\gamma = -\gamma J$, JaJ^{-1} commutes with $\mathscr{A}_q$ and $[\mathscr{D}_q, J] = 0$, so the spectral triple is of *KO*-dimension 2 [9]. The operator $\mathscr{D}$ is the unique $\mathscr{U}_q(\mathfrak{su}(2))$-equivariant operator fulfilling the first-order condition, which makes the spectral triple real [12].

In the limit $q \to 1$ one recovers the commutative spectral triple on the 2-dimensional sphere — cf. [32, Remark 2] and [12, p. 8].

In [32] yet another Dirac operator for the standard Podleś sphere was introduced

$$\mathscr{D}_q^S := \begin{pmatrix} 0 & \bar{w}\, T_q^S \\ w\, T_q^S & 0 \end{pmatrix}, \qquad T_q^S : |l, m\rangle \in \mathscr{H}_{1/2} \mapsto \tfrac{q^{-(l+1/2)}}{q^{-1} - q}\, |l, m\rangle \in \mathscr{H}_{1/2}.$$

which shares the property of the exponential growth of singular values with $\mathscr{D}_q$.

The operators $\mathscr{D}_q$ and $\mathscr{D}_q^S$ commute and are related by $\mathscr{D}_q = \mathscr{D}_q^S - (\tfrac{|w|\, q}{1-q^2})^2 (\mathscr{D}_q^S)^{-1}$. Remark that $(\mathscr{D}_q^S)^{-1}$ is actually trace-class. It turns out that by taking $\mathscr{D}_q^S$ instead of $\mathscr{D}_q$ we preserve most of the properties of the spectral triple with the exception of the first order condition. Moreover, we have

$$|\mathscr{D}_q| = |\mathscr{D}_q^S| - (\tfrac{|w|\,q}{1-q^2})^2 |\mathscr{D}_q^S|^{-1}.$$

Proposition B.9 *The triples* $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q)$ *and* $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q^S)$ *are* 0*-dimensional.*

Proof This is a direct consequence of the exponential growth of singular values of $\mathscr{D}_q$ and $\mathscr{D}_q^S$. Indeed, $\mu_n(\mathscr{D}_q) = \mathcal{O}_\infty(q^{-n})$ and $\mu_n(\mathscr{D}_q^S) = \mathcal{O}_\infty(q^{-n})$, whereas we have $M_n(|\mathscr{D}_q|) = M_n(|\mathscr{D}_q^S|) = \mathcal{O}_\infty(n)$, which means that for any $\varepsilon > 0$ both Tr $|\mathscr{D}_q|^{-\varepsilon}$ and Tr $|\mathscr{D}_q^S|^{-\varepsilon}$ are finite. □

Observe that, whereas the limit $q \to 1$ of the spectral triple $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q)$ is well defined and yields the standard round geometry on S^2, the spectral dimension jumps abruptly from 0 to 2. This phenomenon, known as the 'dimension drop', has also its impact on (co)homologies of $\bar{\mathscr{A}_q}$ [31] and provided inspiration for 'twisted' noncommutative geometries (see Problem 3 in Chap. 5). The potential physical implications of the dimension drop in the context of quantum gravity are the subject of an intensive study — see, for instance, [3].

Another drastic consequence of the exponential growth of singular values is that neither of the spectral triples $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q)$, $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q^S)$ is regular. In fact, already $\big[|\mathscr{D}|, [|\mathscr{D}|, a]\big]$ is an unbounded operator and more generally $\delta^n(a)$ is an operator of order $n-1$ for a generic element $a \in \mathscr{A}$. This could be seen by computing a fixed matrix element of the operator $\delta^n(A)$, for instance:

$$_\pm\langle l+1,\, m|\, \delta^n(A) |l, m\rangle_\pm = \big(\big[l + \tfrac{3}{2}\big] - \big[l + \tfrac{1}{2}\big]\big)^n A^+_{l,m}\,.$$

The behaviour of $A^+_{l,m}$ for large l is $\mathcal{O}_\infty(q^l)$, which can be read from the explicit Formula (B.18), while $\big([l + \tfrac{3}{2}] - [l + \tfrac{1}{2}]\big)^n = \mathcal{O}_\infty(q^{-nl})$, and the above expression is unbounded for $n \geq 2$, and, generally, $\delta^n(A)$ is in OP^{n-1}.

Note that, in general, $\delta^0(a)$ and $\delta^1(a)$ are bounded for any $a \in \mathscr{A}_q$, but not $\delta^2(a)$.

Nevertheless, the spectral triple $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q)$ is quasi-regular (recall Sect. 1.7). Within this extended framework one discovers that [13, Corollary 3.10]:

Theorem B.10 *For any* $0 < q < 1$*, the dimension spectrum of* $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q)$ *is of the order 2 and equals to* $-\mathbb{N} + i\frac{2\pi}{\log q}\mathbb{Z}$.

Proof (sketch) Firstly, one shows (using a simple summation of geometric series [13, Proposition 3.2]) that the basic zeta function reads

$$\zeta_{\mathscr{D}_q}(s) = \mathrm{Tr}\ |\mathscr{D}_q|^{-s} = 4(\tfrac{1-q^2}{|w|})^s \sum_{n=0}^{\infty} \tfrac{\Gamma(s+n)}{n!\,\Gamma(s)}\, \tfrac{q^{2n}}{(1-q^{s+2n})^2}\,, \quad \text{for} \quad \Re(s) > 0.$$

This formula leads us to the realm of 'q-zeta functions' (see [26] and other references on [13, p. 633]). It yields a meromorphic extension of $\zeta_{\mathscr{D}_q}$ to the whole complex plane with second order poles in $-2\mathbb{N} + i\frac{2\pi}{\log q}\mathbb{Z} \subset \mathbb{C}$.

Secondly, one shall consider general functions $\zeta_{P,\mathscr{D}_q}$ with $P \in \widetilde{\Psi}^0(\mathscr{A}_q)$ (cf. [13, Proposition 3.8]). To start, one notices that it is sufficient to consider elements of the form $P = T|\mathscr{D}_q|^{-p}$, with $T \in \mathscr{P}(\mathscr{A}_q) \cap \mathrm{op}^n$ for some $n \in \mathbb{N}$ and $n \leq p \in \mathbb{N}$ and uses the quasi-regularity [13, Lemmas 3.4 and 3.6] to commute all of the operators $\mathscr{D}_q$ and $|\mathscr{D}_q|$ from T to the right. Then, one observes that the generators of $\mathscr{A}_q$ are represented via π in terms of weighted shift operators on $\mathscr{H}_q$. Moreover, the weights are analytic functions of (bounded) variables q^{l+m}, q^{l-m} and q^l. Rewriting these in terms of infinite convergent Taylor series one arrives at a formula for $\zeta_{T,\mathscr{D}_q}$, which involves a (multiple) infinite series in q^l. Finally, a resummation over $l \in \mathbb{N} + 1/2$ yields the desired meromorphic extension of $\zeta_{P,\mathscr{D}_q}$ to $\mathbb{C}$ — cf. [13, Eq. (30)]. □

The zeta function associated with the simplified operator $\mathscr{D}_q^S$ was presented in Example 3.13. Although the spectral triple $(\mathscr{A}_q, \mathscr{H}_q, \mathscr{D}_q^S)$ has not been studied extensively in [13], one can show along the same lines that it is quasi-regular and has a dimension spectrum of second order equal to $-\mathbb{N} + i\frac{2\pi}{\log q}\mathbb{Z}$.

The exponential growth of singular values of $\mathscr{D}_q$ has also some pros: It leads to the following spectacular result highlighted at the beginning of Sect. 2.6:

Theorem B.11 *Let $f \in C_0^r$ for some $r > 0$ and denote $\kappa := \frac{2\pi i}{\log q}$. Then, for any $\Lambda > 0$,*

$$S(\mathscr{D}_q, f, \Lambda) = \sum_{k=0}^{\infty} \sum_{j \in \mathbb{Z}} \sum_{n=0}^{2} a_{-2k+\kappa j,n} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} f_{-2k+\kappa j,m} (\log \Lambda)^{n-m} \Lambda^{-2k+\kappa j}.$$

Proof (sketch) The theorem is proven in detail in [13], via the heat trace expansion. It uses the full force of Theorems 3.12, 3.17, 3.24 and Corollary 3.25. The estimation of the contour integrals is rather subtle and arduous — cf. [13, Proposition 4.3]. □

This result is remarkable for two reasons: Firstly, the formula for the spectral action contains terms proportional to $\log^2 \Lambda$ and oscillating with Λ, which is a sign that the geometry of Podleś sphere lies outside of the kingdom of classical pdos (recall Example 2.37). Secondly, the formula is exact for all $\Lambda > 0$ and for a fairly general class of cut-off functions. Recall that in the classical (pseudo)differential geometry one is bound to use the asymptotic expansion, which might obscure some important information (see p. 30).

As for the fluctuations, these are much more tedious to control in the quasi-regular case. In [13, Theorem 5.6] it was shown that the leading term of $S(\mathscr{D}_q + \mathbb{A}, f, \Lambda)$ does not depend on $\mathbb{A}$, when the fluctuation is 'small', but a deeper understanding of the problem is missing. In particular, it is not clear whether an explicit exact formula for the fluctuated action is available at all.

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