Appendix A
Technical Points and Summary of Equations

A.1 Geometric and Planck Units

In the code GRChombo, and NR in general, one uses geometrised units, whereas in most other areas of physics, including cosmology, one uses Planck units. This section aims to briefly summarise the two systems and their conversions. The reader is referred to the Appendix of Wald [1], which has a good introduction to geometrised units. Most physics textbooks contain an introduction to Planck units.

Note that where we refer to the Planck mass throughout this thesis, we mean $M_{\rm Pl} = 2.17651 \times 10^{-8}$ kg, i.e. $M_{\rm Pl}^2 = \hbar c/G$, rather than the reduced planck mass $M_{\rm Pl}^2 = \hbar c/8\pi G$. The reduced Planck mass, which is heavily used in cosmology, has an additional factor of 8π. In GRChombo, factors of 8π are included explicitly and not set to one, so we prefer to work with the non-reduced Planck mass. Future versions of the code will enable the user to set G so as to eliminate the factors of 8π if desired.

A.1.1 Geometric Units

These are units in which $G = c = 1$, which means that lengths, masses and times all have the same unit (recall that the SI units of G are $\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ so setting length equal to time from $c = 1$ leads to mass and length having the same dimension). For our purposes in GRChombo, we usually work with a Mass M as the standard geometrised unit (this is equivalent to using a length L, which is often preferred). Then all lengths and all time intervals are expressed in these units of mass M.

To convert to real SI units one must multiply by the appropriate factors of G and c, and then by kg to the appropriate geometrised mass dimension, to give the correct units. For example, the length in meters is obtained by multiplying the length in geometrised (mass) units by G/c^2 (which has units m/kg), and then by the value of M in kg. Notice that the geometrised mass unit M can be chosen freely. Thus if we simulate a black hole of geometrised mass one, $M = 1$, we can describe the

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spacetime for any mass of black hole simply by scaling the results according to what we want this unit mass to physically represent in kg. However, once one physical value has been fixed - for example, in inflation, the energy density scale associated with the field - everything else must be consistent with this scale.

The geometrised unit of any other dimensionful quantity in the simulation can be expressed by considering its real physical dimension and replacing any L and/or T with M. The value in real SI units must again be calculated using the appropriate conversion factors of G and c, and the geometrised mass dimension. Sometimes one wishes to find the geometrised quantity from a real physical value, which is the reverse process.

Two useful examples in this thesis are:

1. *Energy Density* The energy density has real dimensions $ML^{-1}T^{-2}$ so in geometrised units it has dimension M^{-2}. The geometric quantity must be multiplied by a factor of c^8/G^3 and then by the value of $1/M^2$ in kg^{-2} to obtain the SI value.
2. *Scalar Field* Using the fact that the action has dimensions of $\hbar$ we can show that the scalar field has units of $L^{1/2}M^{1/2}T^{-1}$ (alternatively $(\partial_x\phi)^2$ has the same units as energy density), thus in geometrised units it is dimensionless. The geometric quantity must be multiplied by a factor of $c^2/G^{\frac{1}{2}}$ to obtain its SI value (note that the value of M does not affect this).

For convenience we also usually choose M to be $M_{\rm Pl}$, which makes the conversion to Planck units more obvious, as we will discuss below. However, in some cases we prefer to choose that $M = qM_{\rm Pl}$ where q gives the fraction or multiple of $M_{\rm Pl}$ that is represented by one geometrised unit in our code. This can lead to more manageable numbers in our simulations, for example, we can make the length of our grid of order 10.

A.1.2 Planck Units

The aim of Planck units is to express all quantities as multiples of a set of base units defined by appropriate dimensionful combinations of the fundamental constants $\hbar$, c and G, which are the Planck length l_p, the Planck mass $M_{\rm Pl}$ and the Planck time t_p.

Any other dimensionful quantities are expressed as multiples of the appropriate combination of l_p, $M_{\rm Pl}$ and t_p, given their real physical dimension, for example, one "Planck force" F_p is equal to $M_{\rm Pl}l_p/t_p^{-2}$.

Quantities are then expressed in dimensionless form in equations, e.g. one can write Newton's law of gravitation

$$F = \frac{Gm_1m_2}{r^2}\,, \tag{A.1}$$

as

$$F = \frac{m_1m_2}{r^2}\,, \tag{A.2}$$

where this is understood to mean

$$\frac{F}{F_p} = \frac{(m_1/M_{\rm Pl})(m_2/M_{\rm Pl})}{(r/l_p)^2} \,. \tag{A.3}$$

However, what conventionally happens when using Planck units is that c and $\hbar$ are set to one, and then, since all powers of l_p and t_p can be expressed in terms of $M_{\rm Pl}$, all figures in Planck units are expressed as a multiple of $M_{\rm Pl}$. This is to contain the uncertainty in G, which is not as well measured as the other constants. What this means in effect is that $M_{\rm Pl}$ replaces G (as $G = M_{\rm Pl}^{-2}$) in all the equations, thus the above Newton's equation would be written

$$F = M_{\rm Pl}^{-2}\frac{m_1 m_2}{r^2} \,. \tag{A.4}$$

In this formulation of Planck units, lengths and times have units of inverse mass $M_{\rm Pl}^{-1}$.

To convert back to SI units one must multiply the value in Planck units by the appropriate factors of $\hbar$ and c to get a result in the SI units, and then by $M_{\rm Pl}$ in kg to the appropriate mass dimension. For example, an energy in Planck units has mass dimension 1 (it is expressed as a multiple of $M_{\rm Pl}$), so its value in joules is found by multiplying the value by c^2 to obtain the correct SI unit, and then by $M_{\rm Pl}$ in kg.

A.1.3 Conversion between units

Using Planck Units Directly

If one uses the geometrised unit $M = M_{\rm Pl}$ in simulations, one can extract directly the values in Planck units of other quantities - the numerical value in geometrised units is the same as that in Planck units.

This can be shown by converting the value first into SI units using appropriate factors of G and c, and then into Planck units by adding factors of $\hbar$ and c. The factors of $\hbar$, c and G combine into some multiple of the Planck mass such that the correct units are given but the numerical value is the same.

That is, if one models a black hole of (geometrised) mass $2M$, where $M = M_{\rm Pl}$, then the lengths are Planck lengths and the times are Planck times. To be totally explicit, this means that if the radius is $4M$, one finds that the radius in SI units is $4l_p$ (with l_p expressed in meters) and thus equal to $4M_{\rm Pl}^{-1}$ in Planck units where $\hbar = c = 1$. Initially this may seem a bit counterintuitive as one unit contains an inverse $M_{\rm Pl}$ whilst the other contains $M_{\rm Pl}$, but one can show that this is correct by first converting to SI units from geometrised ones and then on to Planck units as above. In addition, if $M_{\rm Pl}$ is set to one, which is equivalent to setting $G = 1$, both units agree on the numerical value, as we would expect.

Measures of time behave in exactly the same way, with $1M$ equal to t_p and thus $1M_{\rm Pl}^{-}1$ in Planck units, where $\hbar = c = 1$. Once we know how to treat mass, length and time, other quantities that are combinations of these follow in the natural way.

A useful example to consider is the scalar field ϕ, which is dimensionless in geometrised units as above. This means that a change in ϕ of 1 in our simulation is equivalent to a change of $1M_{\rm Pl}$ in Planck units, where $\hbar = c = 1$. This is invariant whatever we choose our geometrised mass unit to represent, which initially seems surprising, but can be explained by the fact that the physical meaning of the scalar field is effectively absorbed into the energy density $V(\phi)$, and its spatial and temporal gradients.

Similarly, an energy density of $1M^{-2}$, where $M = M_{\rm Pl}$, corresponds to an energy density of $1M_{\rm Pl}^4$ in Planck units. However, care needs to be taken (especially in this case) when $M \neq M_{\rm Pl}$, as explained below.

Using Scaled Planck Units

As noted above, in some cases we prefer to choose that $M = qM_{\rm Pl}$ where q gives the fraction or multiple of $M_{\rm Pl}$ that is represented by one geometrised unit in our code. This can lead to more manageable numbers in our simulations, for example, we can make the length of our grid of order 10, which tends to be easier to work with.

The strategy in this case is to first recover the case above, where the geometrised unit is $M_{\rm Pl}$. Then, whatever number you have is again the correct number in Planck units. This is best seen by an example:

One chooses a geometrised unit M to represent a physical mass of $10M_{\rm Pl}$.

Now for a length of, say, $2M$, this corresponds to $2 \times 10M_{\rm Pl} = 20M_{\rm Pl}$, and therefore it represents a length of $20l_p$ or $20M_{\rm Pl}^{-1}$ in Planck units. The same thing happens with time, and other dimensionful quantities follow in the same way.

For another example, consider an energy density of $\rho = 2$ in geometric units. This is actually $\rho = 2M^{-2}$ because energy density has (geometrised) mass dimension of -2 as discussed above. Therefore the correct conversion when one has chosen $M = 10M_{\rm Pl}$ is to first say that (still in geometrised units)

$$\rho = 2(10M_{\rm Pl})^{-2} = 0.02M_{\rm Pl}^{-2} \,. \tag{A.5}$$

Therefore the value in Planck units is $0.02\, M_{\rm Pl}^4$.

Note that this conversion can also be thought of in terms of an effective mass m_{eff}, which is used to "undimensionalise" the geometric quantities, so that all masses, lengths and times are divided by m_{eff} to obtain their value in dimensionless Planck units. In the example above, $m_{eff} = M/q = 0.1M$, so a geometrised quantity of $2M$ becomes $2M/m_{eff} = 20$ in dimensionless (and thus Planck) units. We are essentially using the freedom inherent in geometric units to choose the mass scaling which achieves a convenient simulation unit. Converting from Planck units into scaled geometrised units is simply the reverse of this process.

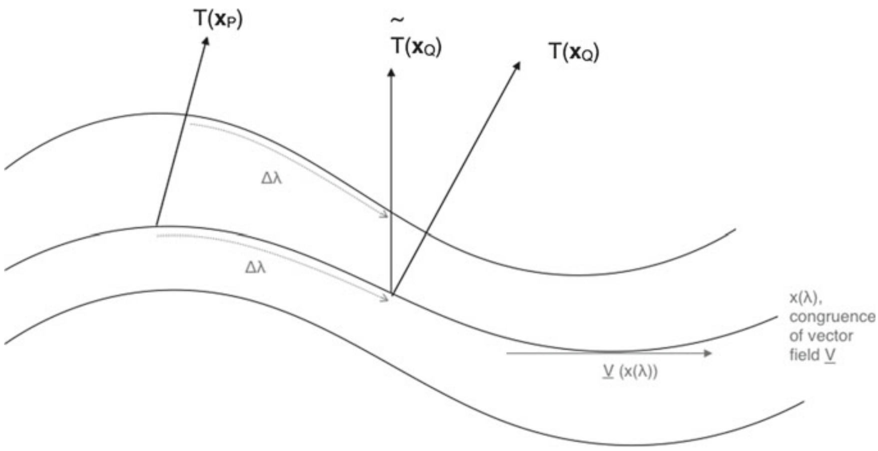

Fig. A.1 The Lie Derivative is a measure of how much a tensor T^a_b changes as one moves infinitesimally along the congruence of a vector field $\vec{V}$

A.2 Lie Derivatives - A Brief Description and Notes on Use

The concept of a Lie derivative is in many respects more fundamental than the concept of a covariant derivative since it does not require an affine connection to be defined on the manifold. A full discussion of Lie derivatives can be found in Schutz [2], or other standard texts on GR and differential geometry. Here we will summarise the key ideas and show how they are used in the derivation of the extrinsic curvature and in the BSSN equations. We assume a coordinate basis throughout.

A.2.1 Lie Derivatives

If one has a vector field $\vec{V}$ on a manifold, it is possible to define the *integral curves* $\mathbf{x}(\lambda)$ by integrating the relation for the coordinates

$$\frac{dx^a}{d\lambda} = V^a(\mathbf{x}(\lambda)) \ , \tag{A.6}$$

which simply insists that that tangent to the curve at each point is the vector $\vec{V}$ at that point. This forms a *congruence*, a family of such curves which fill the manifold, with affine parameter λ. The integral curves are thus like streamlines in an (ideal) fluid flow, with the vector $\vec{V}$ being the fluid velocity at each point; see Fig. A.1.

The Lie derivative of a tensor T^a_b with respect to $\vec{V}$, written as $\pounds_{\vec{V}} T^a_b$, is a measure of how much that tensor changes as one moves infinitesimally along the congruence. We have already emphasised in Chap. 1 that comparing tensorial objects (like vectors) at different points on a manifold is an ambiguous concept. This is because the manifold may be curved and we must define how to transport the object from point to point.

When we define the covariant derivative, we introduce the affine connection and the concept of parallel transport in order to make a comparison. We require the covariant derivative to be metric compatible, i.e.

$$\nabla_c g_{ab} = 0\,, \tag{A.7}$$

which means that it preserves the scalar product of two vectors as they are transported infinitesimally along the manifold. The covariant derivative can then be expressed in terms of the partial derivatives and the Christoffel symbols for a general tensor T^a_b as

$$\nabla_c T^a_b = \partial_c T^b_a + T^d_b \Gamma^a_{cd} - T^a_d \Gamma^d_{bc}\,. \tag{A.8}$$

The procedure for taking the Lie derivative is slightly different - again we want to somehow compare our tensors at two different points, but this time we use an infinitesimal coordinate transform, generated by the vector field, to drag the tensor at one point an infinitesimal distance along the congruence, and compare it to the "actual" value of the tensor at the new location. Consider the points P and Q on an integral curve of $\vec{V}$, separated by $\Delta\lambda$, as in Fig. A.1. What we called the "actual" value of the tensor at Q is given by Taylor expanding from the point P

$$T^a_b(\mathbf{x}_Q) = T^a_b(\mathbf{x}_P + \Delta\lambda\, \vec{V}) = T^a_b(\mathbf{x}_P) + \Delta\lambda\, V^c \partial_c T^a_b + O(\Delta\lambda^2)\,. \tag{A.9}$$

We want to compare this to the tensor at P which has been "dragged along" the congruence to Q using the infinitesimal coordinate transform

$$\frac{\partial x^{a'}}{\partial x^b} = \delta^a_b + \Delta\lambda\, \partial_b V^a\,, \tag{A.10}$$

where the primed indices relate to the new transformed coordinate system. We will call this object $\tilde{T}^{a'}_{b'}(\mathbf{x_Q})$, and it is given by

$$\tilde{T}^{a'}_{b'}(\mathbf{x}_Q) = \frac{\partial x^{a'}}{\partial x^c} \frac{\partial x^d}{\partial x^{b'}}\, T^c_d(\mathbf{x}_P)\,. \tag{A.11}$$

We have the relation we need to transform the raised index via Eq. (A.10), and the second comes from inverting it

$$\frac{\partial x^a}{\partial x^{b'}} = \delta^a_b - \Delta\lambda\, \partial_b V^a + O(\Delta\lambda^2)\,. \tag{A.12}$$

We define the Lie derivative as

$$\pounds_{\vec{V}} T^a_b \equiv \lim_{\Delta\lambda \to 0} \left[\frac{\tilde{T}^{a'}_{b'}(\mathbf{x}_Q) - T^a_b(\mathbf{x}_Q)}{\Delta\lambda} \right] , \tag{A.13}$$

which using the above relations is

$$\pounds_{\vec{V}} T^a_b = V^c \partial_c T^a_b - T^c_b \partial_c V^a + T^a_c \partial_b V^c . \tag{A.14}$$

This generalises to tensors of higher orders in the usual way, with upper indices generating additional terms like the second, and lower indices generating additional terms like the third. Comparing this to the covariant derivative in Eq. (A.8), we see that the Lie derivatives require the derivatives of $\vec{V}$ at each point, and it is this piece of additional structure that replaces the chosen connection (the Christoffel symbols) from the covariant derivative. The result is that the Lie derivative does not require a connection and thus (in the case of the Levi-Civita connection) it is independent of the metric.

Note that one can show that the partial derivatives in Eq. (A.14) can be replaced by covariant ones with the same result, that is

$$\pounds_{\vec{V}} T^a_b = V^c \nabla_c T^a_b - T^c_b \nabla_c V^a + T^a_c \nabla_b V^c . \tag{A.15}$$

We will use this result in the following section.

A.2.2 Lie Derivatives and the Extrinsic Curvature

We now use Eq. (A.15) to show that the two definitions of the extrinsic curvature K_{ab} are equivalent using the results above, as was stated in Sect. 2.2.1. Starting from the definition of K_{ab} in terms of the Lie derivative along the normal direction of the spatial metric, per Eq. (2.47)

$$K_{ab} \equiv -\frac{1}{2} \pounds_{\vec{n}} \gamma_{ab} = -\frac{1}{2} \left(n^c \nabla_c \gamma_{ab} + \gamma_{ac} \nabla_b n^c + \gamma_{cb} \nabla_a n^c \right) . \tag{A.16}$$

Expanding out the spatial metric as $g_{ab} + n_a n_b$ and using the fact that the normal vector is orthogonal to its gradient $n_a \nabla_b n^a = 0$ this becomes

$$K_{ab} = -\frac{1}{2} \left(n^c n_a \nabla_c n_b + n^c n_b \nabla_c n_a + \nabla_a n_b + \nabla_b n_a \right) . \tag{A.17}$$

Reversing the trick to replace the normal vectors with $n^a n_b = \gamma^a_b - g^a_b$ and using the fact that the 4-metric commutes with the covariant derivative we find

$$K_{ab} = -\frac{1}{2} \left(\gamma^c_a \nabla_c n_b + \gamma^c_b \nabla_c n_a \right) = -P^c_a \nabla_c n_b , \tag{A.18}$$

which is the alternative definition as per Eq. (2.46).

A.2.3 Lie Derivatives and the Evolution Equations

ADM Equations

The ADM evolution equations are derived by finding the change of each tensor quantity T along the normal direction to the spatial hyperslice, which is in a sense the most "natural" direction to consider. This must then be re-expressed as the change with respect to the coordinate time in terms of the gauge variables α and β^i, i.e.

$$\pounds_{\vec{n}} T = \frac{1}{\alpha} \pounds_{\alpha \vec{n}} T = \frac{1}{\alpha} \left(\pounds_{\vec{t}} T - \pounds_{\vec{\beta}} T \right) , \tag{A.19}$$

which is rearranged to give the evolution in coordinate time (recognising that the Lie derivative along $\vec{t}$ reduces to the partial derivative with respect to the coordinate time t) such that

$$\partial_t T = \alpha \pounds_{\vec{n}} T + \pounds_{\vec{\beta}} T . \tag{A.20}$$

The first term will be some combination of the evolution variables per the derivation of the change along the normal direction, to which one then adds the Lie derivative of T along the shift vector. This is what gives rise to the terms like $\beta^i \partial_i T$ in each of the expanded ADM equations.

BSSN Equations

In the BSSN evolution equations, the decomposition of the evolution variables into conformal quantities means that they are no longer tensors but *tensor densities*. A tensor density $\tilde{T}$ of "weight" w is a tensor T multiplied by the determinant of the spatial metric γ to the power $w/2$, i.e.

$$\tilde{T} = \gamma^{w/2} T . \tag{A.21}$$

The Lie derivative then becomes

$$\pounds_{\vec{V}} \tilde{T} = \left[\pounds_{\vec{V}} \tilde{T} \right]_{w=0} + w \tilde{T} \partial_i V^i , \tag{A.22}$$

where the first term is the expression arising from Eq. (A.15) as if $\tilde{T}$ were a normal tensor, and the second is the correction for the non zero tensor density. The conformal factor χ as defined as $\gamma_{ij} = \frac{1}{\chi^2} \tilde{\gamma}_{ij}$ has weight $-1/3$, and the conformal metric and the traceless part of the extrinsic curvature have weight $-2/3$ according to their definitions. K is a normal tensor. This leads to additional terms in the BSSN evolution equations, compared to the ADM versions.

The evolution of $\tilde{\Gamma}^i$ is further complicated by the fact that it is not a true vector density either. This is clear from the fact that the Christoffel symbols Γ^i_{jk} are not tensors, therefore neither is their contraction. One thus obtains second derivatives of the shift in the Lie derivative, in addition to the term to account for the tensor density weight of $2/3$, as can be seen in the second line of Eq. (B.15).

References

1. R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984), https://books.google.co.uk/books?id=ibSdQgAACAAJ
2. B. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge University Press, Cambridge, 1980), https://books.google.co.uk/books?id=Akr4mgEACAAJ

Appendix B
Summary of Equations

B.1 Summary of the ADM Equations

For the standard 3 + 1 ADM decomposition per York of the spacetime metric

$$ds^2 = -\alpha^2\,dt^2 + \gamma_{ij}(dx^i + \beta^i\,dt)(dx^j + \beta^j\,dt) \tag{B.1}$$

γ_{ij} is the induced metric on the spatial slices with timelike unit normal

$$n^\mu = \frac{1}{\alpha}\left(\partial_t^\mu - \beta^i\,\partial_i^\mu\right) \tag{B.2}$$

The extrinsic curvature is defined as

$$K_{ij} = -\frac{1}{2}\,(\pounds_{\vec{n}}\gamma_{ij}) \tag{B.3}$$

The Hamiltonian constraint

$$\mathcal{H} = R + K^2 - K_{ij}K^{ij} - 16\pi\rho \tag{B.4}$$

The Momentum constraint

$$\mathcal{M}_i = D^j(\gamma_{ij}K - K_{ij}) - 8\pi S_i \tag{B.5}$$

The definition of the extrinsic curvature (and evolution equation for γ_{ij})

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \tag{B.6}$$

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The evolution equation for K_{ij}

$$\partial_t K_{ij} = \beta^k \partial_k K_{ij} + K_{ki}\partial_j \beta^k + K_{kj}\partial_i \beta^k - D_i D_j \alpha + \alpha \left({}^{(3)}R_{ij} + K K_{ij} - 2K_{ik}K^k_j\right) + 4\pi\alpha\left(\gamma_{ij}(S-\rho) - 2S_{ij}\right) \tag{B.7}$$

where the various components of the matter stress tensor are defined as

$$\rho = n_a n_b T^{ab}\,, \quad S_i = -\gamma_{ia} n_b T^{ab}\,, \quad S_{ij} = \gamma_{ia}\gamma_{jb} T^{ab}\,, \quad S = \gamma^{ij} S_{ij} \tag{B.8}$$

B.2 Summary of the BSSN Equations

In the BSSN formalism used in our `GRChombo` simulations, the induced metric is decomposed as

$$\gamma_{ij} = \frac{1}{\chi^2}\tilde{\gamma}_{ij} \quad \det \tilde{\gamma}_{ij} = 1 \quad \chi = \left(\det \gamma_{ij}\right)^{-\frac{1}{6}} \tag{B.9}$$

The extrinsic curvature is decomposed into its trace, $K = \gamma^{ij} K_{ij}$, and its traceless part $\tilde{\gamma}^{ij}\tilde{A}_{ij} = 0$ as

$$K_{ij} = \frac{1}{\chi^2}\left(\tilde{A}_{ij} + \frac{1}{3} K \tilde{\gamma}_{ij}\right) \tag{B.10}$$

The conformal connections $\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk}$ where $\tilde{\Gamma}^i{}_{jk}$ are the Christoffel symbols associated with the conformal metric $\tilde{\gamma}_{ij}$.
The evolution equations for BSSN are

$$\partial_t \chi = \frac{1}{3}\alpha \chi K - \frac{1}{3}\chi \partial_k \beta^k + \beta^k \partial_k \chi \tag{B.11}$$

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k + \tilde{\gamma}_{jk}\partial_i \beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k + \beta^k \partial_k \tilde{\gamma}_{ij} \tag{B.12}$$

$$\partial_t K = -\gamma^{ij} D_i D_j \alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) + \beta^i \partial_i K + 4\pi\,\alpha(\rho + S) \tag{B.13}$$

$$\partial_t \tilde{A}_{ij} = \chi^2\left[-D_i D_j \alpha + \alpha\left(R_{ij} - 8\pi\,\alpha S_{ij}\right)\right]^{\mathrm{TF}} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j) + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{jk}\partial_i \beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k \beta^k + \beta^k \partial_k \tilde{A}_{ij} \tag{B.14}$$

$$\partial_t \tilde{\Gamma}^i = -2\,\tilde{A}^{ij}\,\partial_j\alpha + 2\,\alpha\left(\tilde{\Gamma}^i_{jk}\,\tilde{A}^{jk} - \frac{2}{3}\,\tilde{\gamma}^{ij}\partial_j K - 3\,\tilde{A}^{ij}\frac{\partial_j\chi}{\chi}\right)$$
$$+\,\beta^k\partial_k\tilde{\Gamma}^i + \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \frac{1}{3}\,\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k$$
$$+\,\frac{2}{3}\,\tilde{\Gamma}^i\,\partial_k\beta^k - \tilde{\Gamma}^k\partial_k\beta^i - 16\pi\,\alpha\,\tilde{\gamma}^{ij}\,S_j \tag{B.15}$$

The covariant derivative of the lapse in the term $D_iD_j\alpha$ is calculated with reference to the full spatial metric, and not the covariant one, i.e.

$$D_iD_j\alpha = \partial_i\partial_j\alpha - \Gamma^k_{ij}\partial_k\alpha \tag{B.16}$$

where

$$\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij} - \frac{1}{\chi}\left(\delta^k_i\partial_j\chi + \delta^k_j\partial_i\chi - \tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\partial_l\chi\right) \tag{B.17}$$

The Ricci tensor R_{ij} is split into conformal and non conformal parts $R_{ij} = \tilde{R}_{ij} + R^{\chi}_{ij}$ which are calculated as

$$\tilde{R}_{ij} = -\frac{1}{2}\tilde{\gamma}^{lm}\partial_m\partial_l\tilde{\gamma}_{ij} + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm}(2\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}) \tag{B.18}$$

and

$$R^{\chi}_{ij} = \frac{1}{\chi}(\tilde{D}_i\tilde{D}_j\chi + \tilde{\gamma}_{ij}\tilde{D}^l\tilde{D}_l\chi) - \frac{2}{\chi^2}\tilde{\gamma}_{ij}\tilde{D}^l\chi\tilde{D}_l\chi. \tag{B.19}$$

The scalar field matter evolution equations are

$$\partial_t\phi = \alpha\Pi_M + \beta^i\partial_i\phi \tag{B.20}$$

$$\partial_t\Pi_M = \beta^i\partial_i\Pi_M + \alpha\partial_i\partial^i\phi + \partial_i\phi\partial^i\alpha + \alpha\left(K\Pi_M - \gamma^{ij}\Gamma^k_{ij}\partial_k\phi - \frac{dV}{d\phi}\right) \tag{B.21}$$

About the Author

Katy Clough studied Engineering Science at Oxford University, before embarking on a short-lived career in finance. An Open University degree in Physics inspired her to return to academia and she subsequently completed her PhD at King's College London under the supervision of Dr Eugene Lim.

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