Comments

Chapter 1 Sections 1.1–1.4 give a concise summary of standard convex analysis duality theory, which is pioneered by Fenchel [18], Moreau [41], and Rockafellar [45]. Our exposition follows [9, 20] emphasizing the variational approach by focusing on convex programming. We also highlight the role of subdifferential of the optimal value function as the set of Lagrange multipliers and the set of dual solutions.

Generalized convexity, conjugacy and related duality discussed in Section 1.5 can be traced back to Moreau. It gained more attention recently due to diverse applications and also due to its role in mass transport theory [59]. Our main references here are [16, 30, 39]. Their applications in hedging with contingent claims are discussed in Section 4.4.

Chapter 2 Section 2.1 provides a unified treatment of the classical Markowitz portfolio theory [38], CAPM model [50], and Sharpe ratio [51]. Following [64] we emphasize that the underlying mathematical tools for all these applications are minimizing a quadratic function with linear constraint, a simplest form of convex programming. Convex duality is essential in revealing the structure of the solutions with a practical financial meaning.

Section 2.2 deals with the portfolio problem from the perspective of utility optimization. Utility function has a long history that goes back to the work of Daniel Bernoulli [4] who in 1738 related to the St. Petersburg paradox proposed earlier by his cousin Nicolas Bernoulli. The relevance to financial problem comes in as optimizing the utility of a portfolio simultaneously accounts for investors pursuing capital growth and risk aversion. The concavity of utility functions means convex analysis is essential. Different agents have different degree of risk aversion. They can be measured by using either absolute risk aversion coefficients or relative risk aversion coefficients [1, 44]. Interestingly, utility functions with those risk aversion coefficients bounded at a given level can be characterized by generalized convexity discussed in Section 1.5. These new characterizations are included in Section 2.2.2.

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Growth optimal portfolio theory [32] and Kelly'es criterion [27, 34, 35, 55–57] as a money management tool in investment general and in games in particular are discussed as an illustration of such utility optimization problems. In particular, following [27, 49, 63] we highlight that optimizing the expected log utility for a portfolio of cash and a given investment strategy on historical performance data amounts to measure the useful information implied by the investment strategy and can be used as a measure to compare different investment strategies. In practice the growth optimal portfolio and its special case the Kelly criterion are often too risky as illustrated in Example 2.2.11. Various fractional Kelly money management schemes, often ad hoc, were proposed to limiting the risk. Recently Vince and Zhu [60] and Lopez de Prado, Vince and Zhu [33] provided theoretical justification for such more conservative betting strategies. They use more realistic finite investment horizon and select betting size based on risk adjusted returns. The analysis involves, however, nonconvex functions.

Fundamental theorem of asset pricing (FTAP) relates no arbitrage to the existence of a martingale measure that can be used to price assets in a financial market. Cox, Ross, and Rubinstein observed such a principle in their classical work related to option pricing in complete markets [12, 13]. General FTAPs were discussed in [15, 21, 22, 29] with progressing generality, usually with a proof based on separation arguments. Dybvig and Ross [17] observed that in an incomplete market the martingale measures are related to the risk aversion of market agent. In Section 2.3 we approach the FTAP from the perspective of convex duality. We show that in an incomplete market, a martingale measure is, in fact, a scaling of the dual solution to a portfolio utility maximization problem. We also illustrate with example that this relationship helps us to understand that in an incomplete market, a martingale measure provides a reference price for a certain agent to improve their utility rather than arbitrage. In a finite dimensional space, the linear programming duality approach in Section 2.3.4 (see e.g. [28]) is equivalent to the Krep-Yan cone separation theorem which is used by Harrison and Kreps [21], Harrison and Pliska [22], Delbaen and Schachermayer [15], and many others in their proofs of FTAP in different settings.

Section 2.4 deals with risk measures, a concept that plays important roles for both financial institutions and regulatory agencies. Diversification reduces risk which implies the convexity of risk measures. We focus on coherent risk measures proposed by Artzner, Delbaen, Eber, and Heath in [2]. Coherent risk measures are sublinear, a particular type of convex function. Duality is involved in providing a dual characterization of a coherent risk measure as the conjugate of an indicator function of a cone, called acceptance cone. Interestingly, the generating set for the acceptance cone is closely related to the practice of stress tests. Convex duality also provides several equivalent description of the coherent risk measures in terms of linear preference and value bonds. Moreover, the same argument is at the core of the discussion of good deal in financial markets as explained in Jaschke and Küchler [24]. Beside providing a framework to understand risk measures and their relationship with other important financial concepts, convex duality methods also help to amend widely used nonconvex risk measure value at risk [25] to the convex conditional value at risk proposed by Rockafellar and Uryasev in [46, 47].

Chapter 3 Sections 3.1–3.3 demonstrate that many of the results in the previous chapter also persist in the more general setting of a multiperiod economy. We use the general model laid out in S. Roman's textbook [48].

Section 3.4 discusses super hedging (and symmetrically subhedging) bounds in incomplete markets. This is a classical topic in financial mathematics (see [22, 23, 26]). We emphasize that the super hedging bound of a given contingent claim is a linear programming problem. Linear programming duality allows us to view the super hedging bound in two different perspectives. On one hand it is the supremum of all the prices derived through martingale measure and on the other hand it can be represented as the cost of the smallest super hedging portfolio. When the sample space is finite, the super hedging portfolio in the second representation can be derived by solving a linear programming problem. The linear programming duality can also be used to analyze narrowing the gap between the super and sub-hedging bounds by adding contingent claims with known prices. When discussing contingent claims related to currency spread, incomplete markets may arise from complete markets. Considering supper hedging bounds in this kind of problems, in general, leads to a Kantorovich mass transportation problem [59]. We illustrate the solution process with an example on a finite sample space using linear programming duality.

Section 3.5 discusses a model for financial markets with bid and ask spread. The main difference with a simplified one price financial market is that the attainable payoff set due to trading is, in general, a convex cone rather than a subspace. This leads to the title conic finance as coined by Madan in [36, 37]. Besides a concise representation of the basic conic finance model, we also discuss new refined fundamental theorem of asset pricing as well as super and sub-hedging price bounds. These results are taken from [58] emphasizing the role of convex duality.

Chapter 4 Section 4.1 summarizes facts on continuous models that we need later. To be concise we are satisfied with a heuristic description of most of the material. Readers interested in further details may consult [5, 42, 52–54]. The dual Itô formula is a first taste of the role of duality in continuous model. It develops the generalized Itô formula using quadratic covariance in [19].

Section 4.2 discusses convexity and generalized convexity emerged in Bachelier [3] and Black–Scholes [6, 40] formulae. The importance of these convexity properties is highlighted in applying them in the computation of Greeks and in illustrating the delta hedging is, in fact, the Fenchel-Legendra transform of the pricing formula. This is the observation in Carr [10] for more general settings and discussed in greater detail in Section 4.3.

It turns out that if one hedges using a contingent claim rather than the underlying itself, similar duality still persists in the sense of generalized duality that we discuss in Section 4.4. The general principles are summarized in Sections 4.4.1 and 4.4.2. A number of examples are included to illustrate their applications in financial practice. How to hedge with the popular multiple ETFs of indices is discussed in detail in

Section 4.4.3. What are also discussed in this section are examples of generalized convexity of Leland's model of stock price as contingent claims of company's assets [31] and the general convexity of the normal kernel. The common theme here is that they all follow from characterizations of the generalized convexity using the relative risk aversion coefficient and the absolute risk aversion coefficient. Hedging with derivatives can help to reduce the risk and to expand the range of volatility trading which is proposed in [11]. These are discussed in Sections 4.4.4 and 4.4.5, respectively. Much of the materials regarding these duality and generalized duality relationships appear here for the first time. We believe that this is an area that is worthy of further attention.

In addition, survey papers [14, 43, 61, 64] have also been valuable references.

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