Technical Appendix

This appendix gives a quick refresher of the topics in college algebra and basic calculus, and their extension to optimization theory, that are used in the models of the text. To see the different concepts in action, we have included EXAMPLES FROM THE TEXT as each topic is reviewed. More advanced mathematical methods are found in the chapter appendices and references.

A.1. Two Useful Functions

We use two types of functions frequently in the text.

Power Functions

A power function has the general form

$$y = f(x) = Ax^a,$$

where x is a nonnegative variable and a and $A$are positive constants, or *parameters*. In words, the function says that y is an increasing function of x, but the relationship between the two variables can have a variety of characteristics depending on the precise value of a.

For,

$0 < a < 1$ $f(x)$ is a concave function of x

$a = 1$ $f(x)$ is a linear function of x

$a > 1$ $f(x)$ is a convex function of x

If these shapes are not familiar, set $A = 1$ and plot $f(x)$ for different values of x, given a value of a that satisfies each of the three different cases above.

When dealing with power functions you need to remember some of the algebra associated with expressions that are raised to a power. Here are some important algebraic results for such expressions, where a and b are positive parameters.

S. Das et al., *Economic Growth and Development*, Springer Texts in Business and Economics, https://doi.org/10.1007/978-3-319-89755-4

(i) $x^a x^b = x^{a+b}$
(ii) $(x^a)^b = x^{ab}$
(iii) $(x^a)^{1/b} = x^{a/b}$
(iv) $x^{-a} = \frac{1}{x^a}$
(v) $\frac{x^a}{x^b} = x^a x^{-b} = x^{a-b}$
(vi) $(xz)^a = x^a z^a$, where z is another nonnegative variable

We also need to remember the rules for differentiating a power function with respect to x.

(i) First derivative

$$f'(x) = aAx^{a-1}$$

(ii) Second Derivative

$$f''(x) = (a-1)aAx^{a-2}$$

Note that, given our assumptions, $f'(x) > 0$. However, the sign of $f''(x)$ depends on the precise value of a. The sign of the second derivative is important because it offers a way of identifying the shape of the function without the need to form plots.

For,

$0 < a < 1$	$f''(x) < 0 \Rightarrow f(x)$ is a concave function of x
$a = 1$	$f''(x) = 0 \Rightarrow f(x)$ is a linear function of x
$a > 1$	$f''(x) > 0 \Rightarrow f(x)$ is a convex function of x

A way of understanding the connection between the second derivative and the shape of $f(x)$ is to note that the second derivative tells us what is happening to the *slope* of $f(x)$, i.e. it gives us the change in the first derivative when there is an increase in x.

For,

$0 < a < 1$	$f'(x)$ is *falling* as x increases, so the graph becomes flatter
$a = 1$	$f'(x)$ is *constant* as x increases, so the graph remains linear
$a > 1$	$f'(x)$ is *increasing* as x increases, so the graph becomes steeper

EXAMPLES FROM THE TEXT

In Sect. 2.1 of Chap. 2, we find the following power function representing total production,

$$Y_t = AK_t^{\alpha}M_t^{1-\alpha}.$$

Using the algebra associated with variables raised to exponents, production can be written on a per worker basis, *worker productivity*,

$$y_t \equiv \frac{Y_t}{M_t} = \frac{AK_t^{\alpha}M_t^{1-\alpha}}{M_t} = \frac{AK_t^{\alpha}M_tM_t^{-\alpha}}{M_t} = \frac{AK_t^{\alpha}}{M_t^{\alpha}} = Ak_t^{\alpha},$$

where $k_t \equiv \frac{K_t}{M_t}$.

The first and second derivatives of the worker productivity function with respect to k_t are $\alpha Ak_t^{\alpha-1} > 0$ and $\alpha(\alpha-1)Ak_t^{\alpha-2} < 0$. Worker productivity is an increasing concave function of k_t.

(Natural) Logarithmic Function

Our other special function is the natural logarithmic function, which we refer to as just the log function. The log function is an increasing concave function of the form,

$$y = f(x) = A\ln x,$$

where x is a positive variable and A is a positive parameter. As with the power function, if you are not familiar with the shape of the log function you should set $A = 1$ and plot the function for different values of x.

Alternatively, we can learn about its shape by recalling the rules of differentiation for log functions,

(i) First Derivative

$$f'(\mathrm{x}) = \frac{A}{x} > 0$$

(ii) Second Derivative

$$f''(\mathrm{x}) = -\frac{A}{x^2} < 0$$

As with the power function when $a < 1$, the derivative of the logarithmic function is positive and decreasing as x increases, i.e. its slope becomes flatter at higher values of x.

The following results will be useful when doing algebra with expressions involving logs. The parameter a and the variable z are both positive values.

(i) $\ln(xz) = \ln x + \ln z$
(ii) $\ln\left(\frac{x}{z}\right) = \ln x - \ln z$
(iii) $\ln(x^a) = a \ln x$

EXAMPLES FROM THE TEXT

The single period utility function we use throughout the text takes the natural log form, $u = \ln c$. The marginal utility of consumption is the derivative of u with respect to c, $1/c > 0$. The marginal utility of consumption is clearly decreasing in c. This can also be verified by taking the second derivative with respect to c, $-1/c^2$, which tells us how the marginal utility of consumption changes with c.

In Sect. 4.1 of Chap. 4, deriving the optimal choices for consumption, fertility (n_{t+1}), and schooling (e_t) is simplified by using the algebraic rules for taking the natural log of a product. The extended utility function in Chap. 4 is

$$U_t = \ln c_{1t} + \beta \ln c_{2t+1} + \psi \ln\left(n_{t+1}h_{t+1}w_{t+1}D_{t+1}\right),$$

where $h_{t+1} = e_t^\theta$ is the human capital production function relating parent's choice of schooling time for children to the resulting human capital when the child becomes an adult. The expression for the parent's lifetime utility can be written as

$$\begin{aligned}&\ln c_{1t} + \beta \ln c_{2t+1} + \psi \ln n_{t+1} + \psi \ln e_t^\theta + \psi \ln\left(w_{t+1}D_{t+1}\right) = \\ &\ln c_{1t} + \beta \ln c_{2t+1} + \psi \ln n_{t+1} + \psi\theta \ln e_t + \psi \ln\left(w_{t+1}D_{t+1}\right).\end{aligned}$$

Note, when maximizing U_t to find the optimal household behavior, that the last term above is unaffected by household choice. The derivatives representing the marginal utility of household choices are $1/c_{1t}$, β/c_{2t+1}, ψ/n_{t+1} and $\psi\theta/e_t$.

A.2. Optimization

Single Choice Variable

The two special functions discussed in the previous section are increasing in x. This means that they have no maximum value. In economic terms, if these functions represent output or utility, as x increases there is always a marginal benefit. However, because of scarcity, there is typically also a cost to increasing x. For simplicity, suppose the scarcity is reflected in the fact that sellers of x charge a price, p, for its use. Also assume the market for x is competitive, so individual agents take the value of p as given (unaffected by their actions)

The *rationality* assumption in neoclassical economics says that agents will assess both the benefits and costs of making a decision and make choices that do not systematically deviate from the choice that maximizes the net benefit. To illustrate how this assumption works, we create a new function that reflects both the benefit and the cost of choosing x. The simplest function that illustrates this idea is the *profit function*. Let the profit function be defined as,

$$\tilde{f}(x) = Ax^a - px,$$

where $A > 0$, $p > 0$, and $0 < a < 1$. The first and second derivatives of the profit function are

$$\tilde{f}'(x) = aAx^{a-1} - p$$

$$\tilde{f}''(x) = (a-1)aAx^{a-2} < 0.$$

Note that the second derivative is negative, so the profit function is concave. This also tells us that the first derivative is decreasing. However, the first derivative can have any sign. When x is low it is more likely to be positive. A positive derivative indicates that total profit increases as x increases. As x increases the value of the first derivative falls, the marginal profit becomes smaller, until it reaches zero. At this point, further increases in x will lower total profit. So, the rule for finding the highest profit is to choose x such that the first derivative is zero.

The previous paragraph exemplifies a general and very important result for economics, known in mathematics as *Fermat's Theorem*. For a strictly concave function, $\tilde{f}(x)$, the value of x that maximizes $\tilde{f}(x)$, satisfies the *first order condition*, $\tilde{f}'(x) = 0$. In the profit function example above, we can find the profit maximizing value of x explicitly by solving,

$$\tilde{f}'(x) = aAx^{a-1} - p = 0, \text{for } x \text{ to get } x = (aA/p)^{1/(1-\mathrm{a})}.$$

EXAMPLES FROM THE TEXT

In Sect. 2.2 from Chap. 2, we can use the budget constraint to write the life-time utility of the household as a function of a single unconstrained choice variable, c_{1t},

$$U = \tilde{f}(c_{1t}) = \ln(c_{1t}) + \beta \ln(w_t - c_{1t}) + \beta \ln(R_t).$$

The first and second derivatives taken with respect to c_{1t} are

$$\tilde{f}'(c_{1t}) = -\frac{1}{c_{1t}} - \beta \frac{1}{w_t - c_{1t}}$$

$$\tilde{f}''(c_{1t}) = -\frac{1}{c_{1t}^2} - \beta \frac{1}{(w_t - c_{1t})^2} < 0.$$

Solving the first order condition for $c_{1t}, \tilde{f}'(c_{1t}) = 0$, gives the utility maximizing choice,

$c_{1t} = \frac{w_t}{1+\beta}$. Substituting the optimal choice of c_{1t} into the first and second period household budget constraints allows one to find the optimal choice of saving and second period consumption.

Multiple Choice Variables

Often economic agents are modelled as attempting to "do the best they can," more formally as maximizing some objective function, by choosing more than one variable. The basic approach when there is more than one choice variable is analogous to the one variable case. We illustrate the approach in the situation where there are two choice variables. In this case, the net benefit function has two arguments, x_1 and x_2, and is written as $\tilde{f}(x_1, x_2)$. The derivative of $\tilde{f}(x_1, x_2)$ with respect to *each* choice variable can be taken one at a time. These types of derivatives are called *partial derivatives*—they give the change in the function due to a change in one of the arguments, *holding all other arguments constant*.

One way of reinforcing the notion and the mechanics of taking a partial derivative is to think of a function with a *single argument* created from $\tilde{f}(x_1, x_2)$. This is done by holding x_2 constant. When x_2 is fixed at a certain value, it simply becomes a constant part of the newly defined function. For example, if we think of x_2 as fixed at the value $\bar{x}_2$, we can define the new function $h(x_1) \equiv \tilde{f}\left(x_1, \bar{x}_2\right)$. The partial derivative of $\tilde{f}(x_1, x_2)$ with respect to x_1 is then defined as $\tilde{f}_{x_1} \equiv h'(x_1)$ or, using a different notation, as $\frac{\partial \tilde{f}}{\partial x_1} \equiv h'(x_1)$. The second notation is a bit clumsy, but it is clearer in dynamic models where subscripts are used to denote time periods. Both types of notation are frequently used. Of course, the same procedure can be used to define the partial derivative with respect to x_2.

The partial derivatives are themselves typically functions of x_1 and x_2 and so they can be differentiated to get the *second partial derivatives*. There is a way of checking for the concavity of $\tilde{f}(x_1, x_2)$ that involves the second partial derivatives. This check is a bit complicated, so you need to trust that when we do maximization problems in the text, that we are using concave functions. However, if you build your own original models, you need to research the different ways of checking for concavity of functions with multiple choice variables.

If you are sure that $\tilde{f}(x_1, x_2)$ is a strictly concave function of x_1 and x_2, then you can identify the maximizing choices of x_1 and x_2 using the first order conditions in a manner perfectly analogous to the case with a function of just one variable. The first order conditions simply set the partial derivatives equal to zero,

$$\frac{\partial \tilde{f}}{\partial x_1} = 0 \text{ and } \frac{\partial \tilde{f}}{\partial x_2} = 0.$$

EXAMPLES FROM THE TEXT

In Chap. 2, the Cobb-Douglas production function is introduced,

$$Y_t = AK_t^{\alpha} M_t^{1-\alpha},$$

where Y denotes output, K denotes the capital stock rented, M denotes the hours of work hired, and where $A > 0$ and $0 < \alpha < 1$ are technological parameters.

The marginal product of an input is the increase in output that results from an increase in the use of an input. Formally, it is the partial derivative of the production function with respect to a particular input, holding other inputs constant. For a Cobb-Douglas production function, the marginal product of labor and the marginal product of capital are $\dfrac{\partial Y_t}{\partial M_t} = (1-\alpha)AK_t^{\alpha}M_t^{-\alpha}$ and $\dfrac{\partial Y_t}{\partial K_t} = \alpha AK_t^{\alpha-1}M_t^{1-\alpha}$ (see the rules for differentiating power functions given above). These expressions can be simplified somewhat by using algebra to write them in terms of the *capital intensity*, $k_t \equiv K_t/M_t$. The simplified expressions for the marginal products are, $\dfrac{\partial Y_t}{\partial M_t} = (1-\alpha)Ak_t^{\alpha}$ and $\dfrac{\partial Y_t}{\partial K_t} = \alpha Ak_t^{\alpha-1}$ (see the algebra rules for manipulating expressions with exponents given above).

We assume that markets are perfectly competitive in our production economy. As discussed in elementary economics, the notion of competitive markets applies not only to the markets for goods but also to the factor markets for labor and capital. The competitive assumption applied to the factor markets means that firms demand inputs to maximize profits taking as given the market prices of the inputs: the wage rate paid to labor (w) and rental rate on physical capital (r). No single firm is large enough to be able to influence market prices when they unilaterally change their production or input levels. The price of the economy's single output good is taken to be one. So we can think of output and revenue as being the same.

Given the competitive assumptions, the profit function can then be written as $Y_t - w_tM_t - r_tK_t$. Just as in the one-variable case, maximizing profits requires that firms hire capital and labor as long as the marginal benefit (marginal product) exceeds the marginal cost (factor price). Formally, the necessary first order conditions for profit maximization are

$$\alpha Ak_t^{\alpha-1} = r_t \text{ and } (1-\alpha)Ak_t^{\alpha} = w_t.$$

Constrained Maximization with Multiple Choice Variables

Let's extend the discussion from the previous section to the case where $f(x_1, x_2)$ is a strictly concave function of x_1 and x_2, but where the choice variables have to satisfy a resource constraints of the general form $F(x_1, x_2) = E$, where E is a positive constant. When resource constraints are present, there is a very important method that generates the first order conditions for the maximizing values of x_1 and x_2. It is called the *Lagrangian Method* , named after its inventor, the mathematician Joseph- Louis Lagrange. He showed that the first order conditions that must be satisfied by the maximizing values of x_1 and x_2 are

$$\frac{\partial f}{\partial x_1} = \lambda \frac{\partial F}{\partial x_1}, \quad \frac{\partial f}{\partial x_2} = \lambda \frac{\partial F}{\partial x_2}, \quad \text{and} \quad F(x_1, x_2) = E,$$

where λ is a variable called the *Lagrange multiplier.*

The first order conditions are easy to remember because they can reproduced by maximizing the *Lagrangian function*, $L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[E - F(x_1, x_2)]$ with respect to x_1, x_2, λ. In other words, treat L as any other function and find the maximizing values by setting the partial derivatives of L to zero,

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \text{and} \quad \frac{\partial L}{\partial \lambda} = 0.$$

These three equations, when written out and rearranged algebraically, are exactly the three first order conditions stated above.

EXAMPLES FROM THE TEXT

In Sect. 2.3 from Chap. 2, households maximize their lifetime utility by choosing the optimal consumption path over their two periods of life subject to their lifetime budget constraint. Matching the household's problem with the general set-up above we have

$$f(x_1, x_2) \equiv \ln(c_{1t}) + \beta \ln(c_{2t+1}), \quad F(x_1, x_2) \quad = \quad c_{1t} + \frac{c_{2t}}{R_t}, \quad \text{and} \quad E \equiv w_t$$

The Lagrangian function in our application is

$$L(c_{1t}, c_{2t+1}, \lambda_t) \quad = \quad \ln(c_{1t}) + \beta \ln(c_{2t+1}) \quad + \quad \lambda_t \left(w_t - c_{1t} - \frac{c_{2t+1}}{R_t} \right).$$

Differentiating and setting the partial derivatives equal to zero, gives us

$$\frac{1}{c_{1t}} = \lambda_t, \quad \frac{\beta}{c_{2t+1}} = \frac{\lambda_t}{R_t}, \quad \text{and} \quad c_{1t} + \frac{c_{2t+1}}{R_t} \quad = \quad w_t,$$

(see the rules for differentiating the natural log function given above). Solving these three equations for the three unknowns (c_{1t}, c_{2t+1}, λ_t), yields the optimal consumption demand functions and a value for the Lagrange multiplier, $c_{1t} = \frac{w_t}{1+\beta}, c_{2t+1} = \frac{\beta R_t w_t}{1+\beta}, \quad \lambda_t = \frac{1+\beta}{w_t}$.

A.3. Nonnegativity Constraints and Corner Solutions

The choice variables of economic agents are often restricted to be nonnegative values. The optimization approach taken in Sect. A.2 does not explicitly acknowledge this type of constraint on the choice variables. In many situations this is not a problem because, given the choice variables and the particular functions chosen, the optimal solutions naturally come out to be positive values. However, in some applications it is quite possible that some of the unconstrained optimal choice variables may take on negative values. This is not the proper solution if there are economic constraints preventing that possibility.

Fortunately, the Lagrangian method can be modified to account for nonnegativity constraints. The first order conditions with nonnegatvity constraints on x_1 and x_2 are

(i)

$$\frac{\partial L}{\partial x_1} \leq 0, x_1 \geq 0,$$

(ii)

$$\frac{\partial L}{\partial x_2} \leq 0, x_2 \geq 0,$$

and

(iii)

$$\frac{\partial L}{\partial \lambda} = 0.$$

where in (i) and (ii), *at least one* of the inequalities must be a strict equality. In the situation where the optimal values of both choices variables is strictly positive, then $x_1 > 0$ and $x_2 > 0$, so by the rule just stated $\frac{\partial L}{\partial x_1} = 0$ and $\frac{\partial L}{\partial x_2} = 0$, exactly as in the case where nonngegativity constraints are not accounted for. However, if an unconstrained choice of, say x_1, turns out to be negative, then the nonnegativity constraint binds and we have

$$\frac{\partial L}{\partial x_1} < 0, x_1 = 0.$$

This condition can be interpreted intuitively in the following way. Begin by thinking of $\frac{\partial L}{\partial x_1}$ as the marginal net benefit of increasing the value of x_1 (note that the Lagranian function incorporates both benefits and costs). If at $x_1 = 0, \frac{\partial L}{\partial x_1} > 0$, then the marginal benefit is positive and it is rational to increase x_1 above zero. However, if $\frac{\partial L}{\partial x_1} < 0$, then it is rational to *reduce* x_1 below zero in order to cause the total net benefit to rise. If this is not permitted, then the best the decision maker can do is set $x_1 = 0$. Because $x_1 = 0$ is at the end or at the "corner" of the permissible choices for x_1, this is referred as a *corner solution*.

EXAMPLES FROM THE TEXT

The approach used to handle choice variables that cannot be negative also works when a choice variable is constrained to exceed any particular value. In Chap. 4 we encounter a situation where schooling has a positive lower bound of $\bar{e}$, where $e_t \geq \bar{e}$. The rules for finding the optimal choice of e_t are

$$\frac{\partial L}{\partial e_t} \leq 0, e_t \geq \bar{e}.$$

A corner solution results unless the derivative (marginal net benefit) is positive when evaluated at the point $e_t = \bar{e}$. If $\frac{\partial L}{\partial e_t} > 0$ at $e_t = \bar{e}$, then the optimal choice is the interior solution $e_t = \frac{\theta\left(\eta\left(e_{t-1}/\bar{e}\right)^{\theta} - \gamma T\right)}{\gamma(1-\theta)} > \bar{e}$, the value for e_t that satisfies $\frac{\partial L}{\partial e_t} = 0$. If $\frac{\partial L}{\partial e_t} \leq 0$ at $e_t = \bar{e}$, then the best the household can do is choose the corner solution $e_t = \bar{e}$.

A.4. Total Differentials and Linear Approximations

If $y = f(x_1, x_2)$ is a differentiable function of x_1 and x_2, one can define the *total differential* of f as

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2,$$

where dy, dx_1, and dx_2 are real variables that are interpreted as "changes" in the original variables. The concept of the total differential extends naturally to the case where the function has many arguments or independent variables.

If one imagines that the total differential is taken at a particular point where $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$, then it can be related to the notion of a *linear approximation* of $f(x_1, x_2)$,

$$y = f\left(\bar{x}_1, \bar{x}_2\right) + \frac{\partial f}{\partial x_1}\left(\bar{x}_1, \bar{x}_2\right) dx_1 + \frac{\partial f}{\partial x_2}\left(\bar{x}_1, \bar{x}_2\right) dx_2,$$

where dx_1 and dx_2 are interpreted as deviations from the values $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$, and the partial derivatives are evaluated at the point $(\bar{x}_1, \bar{x}_2)$. Note that, analogous to the interpretations of dx_1 and dx_2, it is natural to think of dy as $y - f\left(\bar{x}_1, \bar{x}_2\right)$.

EXAMPLES FROM THE TEXT

In Sect. 6.3 from Chap. 6, we analyze a system of nonlinear difference equations where we cannot explicitly solve for future values of the state variables in terms of current values. In this situation one can conduct a qualitative analysis of the system in the neighborhood of a steady state by taking a linear approximation to the nonlinear system of equations.

To see how this works in a simpler setting, consider taking a linear approximation to the nonlinear difference equation from Chap. 2,

$$k_{t+1} = f(k_t) \equiv Bk_t^{\alpha}.$$

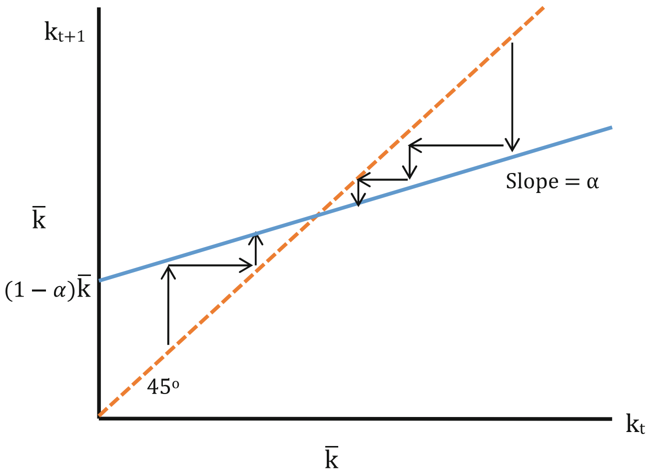
Taking a linear approximation of f in the neighborhood of the steady state gives us

$$k_{t+1} \approx f(\bar{k}) + f'(\bar{k})(k_t - \bar{k}) = \bar{k} + \alpha B\bar{k}^{\alpha-1}(k_t - \bar{k}).$$

We know the steady state k is $\bar{k} = B^{\frac{1}{1-\alpha}}$, implying that $\alpha B\bar{k}^{\alpha-1} = \alpha BB^{-1} = \alpha$. The linear difference equation that serves to approximate the behavior of the nonlinear difference equation near the steady state can then be written as

$$k_{t+1} = (1-\alpha)\bar{k} + \alpha k_t.$$

The difference equation is sketched below. Note that it exhibits the stability property possessed by the original nonlinear difference equation (near the steady state).



A.5. L'Hospital's Rule

On occasion one encounters a ratio of functions or expressions that take on an *indeterminate form* at a point of interest. An indeterminate form is one where the ratio becomes $\frac{0}{0}$ or $\frac{\infty}{\infty}$. In some cases indeterminate forms actually do have a determinate value that is simply not immediately obvious. *L'Hospital's Rule* indicates when this might be true.

The rule says that if you have two differentiable expressions, $f(x)$ and $h(x)$, and at a particular value of x, say $x = x_0$, the ratio $\frac{f(x)}{h(x)}$ takes an indeterminate form, then $\lim_{x \to x_0} \frac{f(x)}{h(x)} = \lim_{x \to x_0} \frac{f'(x)}{h'(x)}$. The result is useful because sometimes the ratio of derivatives has a determinate form.

EXAMPLES FROM THE TEXT

In Sect. 2.6 and *Problem* 7 of Chap. 2, we introduced a more general lifetime utility function with a single period utility flow from consumption of the form,

$$u_t = \frac{\left(c_t^{1-1/\sigma} - 1\right)}{(1 - 1/\sigma)}.$$

The motivation for needing a more general utility function is provided in the text, but part of the reason for its unusual form is to allow the logarithmic utility function, that we use in most of our models, to appear as a special case. Using L'Hospital Rule one can show that $u_t = \ln c_t$, when $\sigma = 1$.

To see this, first note that when $\sigma = 1$, the utility function has the indeterminate form $\frac{0}{0}$. Second, we need to use the result that the exponential function and the natural log functions are inverses of each other, i.e. $x^a = e^{a \ln x}$. This means we can write $c_t^{1-1/\sigma}$ as $e^{(1-1/\sigma)\ln c_t}$. Third, the rule for differentiating the exponential function $f(x) = e^{ax}$, is $f'(x) = ae^{ax}$. Finally, to apply the result, think of the expressions in the numerator and the denominator as functions of σ.

Now, we can write utility as

$$u_t = \frac{\left(e^{(1-1/\sigma)\ln c_t} - 1\right)}{(1 - 1/\sigma)}.$$

Differentiating the numerator and the denominator with respect to σ and then taking the ratio of the two derivatives gives

$$\frac{\frac{1}{\sigma^2}\ln c_t e^{(1-1/\sigma)\ln c_t}}{\frac{1}{\sigma^2}} = \ln c_t e^{(1-1/\sigma)\ln c_t}.$$

At $\sigma = 1$, the ratio is $u_t = \ln c_t$, because $e^0 = 1$.

A.6. Quadratic Equations

Some equations in the unknown variable x can be written in the following quadratic form

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. Mathematically, there are two solutions for x that satisfy the equation, although one or both may not make sense as solutions to an economic problem. The mathematical solution are given by the *quadratic formula*,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLES FROM THE TEXT

In *Problem* 19 of Chap. 2 we consider a special case of the CES utility function where $\sigma = \frac{2-\alpha}{1-\alpha}$. With $\delta = 1$ and $\sigma = \frac{2-\alpha}{1-\alpha}$, we can write the transition equation (9) from Chap. 2 as $k_t = \frac{(1-\alpha)Ak_{t-1}^{\alpha}}{n(1+d)} \frac{1}{1+\beta^{-\sigma}(\alpha A)^{1-\sigma}k_t}$. This expression can be written as $\beta^{-\sigma}(\alpha A)^{1-\sigma}k_t^2 + k_t + \frac{-(1-\alpha)Ak_{t-1}^{\alpha}}{n(1+d)} = 0$, which is a quadratic equation in the unknown k_t. Applying the quadratic formula reveals that there is only one positive solution, $k_t = \frac{\left(1 + 4\frac{\beta^{-\sigma}(\alpha A)^{1-\sigma}(1-\alpha)Ak_{t-1}^{\alpha}}{n(1+d)}\right)^{1/2} - 1}{2\beta^{-\sigma}(\alpha A)^{1-\sigma}}$.

A.7. Infinite Series

A *sequence* is an ordered list of terms, $a_0, a_1, a_2, \cdot, \cdot, \cdot, a_n$. A special case of a sequence is one where consecutive terms have the same ratio, known as a *geometric sequence*. This is possible when the terms of the sequence have a common base value that is raised to an increasing power as follows:

$a_0 = a^0 = 1$, $a_1 = a^1 = a$, $a_2 = a^2$, $a_3 = a^3, \cdot, \cdot, \cdot, a_n = a^n$. So the ratio of consecutive terms is always a.

Of more direct interest to us is the *sum* of a geometric sequence known as a *geometric series*, defined as

$$S_n = \sum_{i=0}^{n} a^i = 1 + a + \cdots + a^n.$$

Note that $S_n - aS_n = 1 - a^{n+1}$, so

$$S_n = \frac{1 - a^{n+1}}{1 - a}.$$

Finally, note when $0 \leq a < 1$, then if $n \to \infty$, the *infinite geometric series* is

$$S_\infty = \frac{1}{1-a}.$$

EXAMPLES FROM THE TEXT

In Eq. (2.31) from Sect. 2.6 of Chap. 2, we encounter an infinite series of the form $\left\{1+\left(\frac{n}{R}\right)\left[\frac{\beta R}{n}\right]^{\sigma}+\left(\frac{n}{R}\right)^{2}\left[\frac{\beta R}{n}\right]^{2\sigma}+\cdots\right\}$in an equation that can be used to solve for the consumption of the first generation of a dynastic chain linked by intergenerational altruism,

$$\Psi_1^{-1}c_{1t}\left\{1+\left(\frac{n}{R}\right)\left[\frac{\beta R}{n}\right]^{\sigma}+\left(\frac{n}{R}\right)^{2}\left[\frac{\beta R}{n}\right]^{2\sigma}+\cdots\right\}=W_\infty. \tag{2.31}$$

The geometric sum, in the curly brackets of (2.31), is finite provided $\beta^{\sigma}\left(\frac{n}{R}\right)^{1-\sigma} < 1$ or $\beta^{\sigma/(1-\sigma)} \le \frac{R}{n}$. If $R > n$, then this condition holds when $\sigma \le 1$ and $\beta \le 1$ Under these conditions, the value of the infinite series is $\frac{1}{1-\beta^{\sigma}\left(\frac{n}{R}\right)^{1-\sigma}}$ and the solution for consumption is $c_{1t} = \Psi_1\left(1-\beta^{\sigma}\left(\frac{n}{R}\right)^{1-\sigma}\right)W_\infty$.

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S. Das et al., *Economic Growth and Development*, Springer Texts in Business and Economics, https://doi.org/10.1007/978-3-319-89755-4