Appendix A
Lie Derivatives

In this appendix we obtain the Lie derivatives of: (a) a 0-form f (function), (b) a one form $\alpha = \mathbf{A} \cdot d\mathbf{x}$ and (c) a vector field $\mathbf{w}$ with respect to a vector field $\mathbf{u}$ from first principles.

The Lie derivative is directional derivative along a curve $\mathbf{x} = \mathbf{x}(\epsilon)$. The Lie derivative of a function f (0-form f) is:

$$\mathcal{L}_{\mathbf{u}} f = \frac{df}{d\epsilon} = \frac{d\mathbf{x}}{d\epsilon} \cdot \nabla f = \mathbf{u} \cdot \nabla f \quad \text{where} \quad \mathbf{u} = \frac{d\mathbf{x}}{d\epsilon}. \tag{A.1}$$

The Lie derivative of a 1-form $\alpha = A_i dx^i \equiv \mathbf{A} \cdot d\mathbf{x}$ is given by:

$$\begin{aligned}
\mathcal{L}_{\mathbf{u}} \alpha &= \frac{d\alpha}{d\epsilon} = \lim_{\epsilon \to 0} \left(\mathbf{A}' \cdot d\mathbf{x}' - \mathbf{A} \cdot d\mathbf{x}\right) / \epsilon \\
&= \frac{dA_i}{d\epsilon} dx^i + A_i \frac{d}{d\epsilon} \left(dx^i\right) \\
&= \frac{dx^j}{d\epsilon} \frac{\partial A_i}{\partial x^j} dx^i + A_i d\left(\frac{dx^i}{d\epsilon}\right) = \mathbf{u} \cdot \nabla \mathbf{A} \cdot d\mathbf{x} + A_i du^i \\
&= \mathbf{u} \cdot \nabla \mathbf{A} \cdot d\mathbf{x} + A_i \frac{\partial u^i}{\partial x^j} dx^j = [\mathbf{u} \cdot \nabla \mathbf{A} + \nabla \mathbf{u} \cdot \mathbf{A}] \cdot d\mathbf{x} \\
&\equiv [-\mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla(\mathbf{u} \cdot \mathbf{A})] \cdot d\mathbf{x}.
\end{aligned} \tag{A.2}$$

To obtain the Lie derivative of a vector field $\mathbf{w}$ with respect to a vector field $\mathbf{u}$, we use the infinitesimal transformations:

$$\mathbf{x}' = \mathbf{x} + \epsilon \mathbf{u}, \quad \text{i.e.} \ \frac{d\mathbf{x}}{d\epsilon} = \mathbf{u}, \quad \mathbf{x} = \mathbf{x}' - \epsilon \mathbf{u}. \tag{A.3}$$

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Thus, to $O(\epsilon)$:

$$\begin{aligned}
\frac{\partial}{\partial x'^i} &= \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} = \frac{\partial(x'^j - \epsilon u'^j)}{\partial x'^i}\frac{\partial}{\partial x^j} \equiv \left(\delta_{ij} - \epsilon\frac{\partial u^j}{\partial x^i}\right)\frac{\partial}{\partial x^j},\\
w'^i &= w^i + \epsilon u^j \frac{\partial w^i}{\partial x^j} \equiv w^i + \epsilon \mathbf{u}\cdot\nabla w^i,\\
\mathbf{w}'\cdot\nabla' &= (\mathbf{w} + \epsilon\mathbf{u}\cdot\nabla\mathbf{w})\cdot(\mathsf{I} - \epsilon\nabla\mathbf{u})\cdot\nabla\\
&\equiv \mathbf{w}\cdot\nabla + \epsilon\left[\mathbf{u}\cdot\nabla\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{u}\right]\cdot\nabla \equiv \mathbf{w}\cdot\nabla + \epsilon[\mathbf{u},\mathbf{w}],
\end{aligned} \tag{A.4}$$

Thus we obtain:

$$\mathcal{L}_{\mathbf{u}}(\mathbf{w}) = \lim_{\epsilon\to 0}\left(\mathbf{w}'\cdot\nabla' - \mathbf{w}\cdot\nabla\right)/\epsilon = [\mathbf{u},\mathbf{w}]^i\nabla_i \equiv [\mathbf{u},\mathbf{w}], \tag{A.5}$$

where

$$[\mathbf{u},\mathbf{w}] = [\mathbf{u}\cdot\nabla\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{u}]\cdot\nabla \tag{A.6}$$

is the Lie bracket.

Appendix B
Weber Transformations

The classical Weber transformation uses the Lagrangian map: $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ to integrate the Eulerian momentum equation to get the Clebsch representation for $\mathbf{u}$. The Eulerian momentum conservation equation can be written as:

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot\left[\rho\mathbf{u}\otimes\mathbf{u} + p\mathsf{I} + \left(\frac{B^2}{2\mu_0}\mathsf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0}\right)\right] = 0, \tag{B.1}$$

or as:

$$\frac{d\mathbf{u}}{dt} = T\nabla S - \nabla h + \frac{\mathbf{J}\times\mathbf{B}}{\rho} + \mathbf{B}\frac{\nabla\cdot\mathbf{B}}{\mu_0\rho}. \tag{B.2}$$

Use:

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= \frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) \quad \text{where} \quad \boldsymbol{\omega} = \nabla\times\mathbf{u}, \\ \frac{d}{dt}\left(\mathbf{u}\cdot d\mathbf{x}\right) &= \left[\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} + \nabla\left(|\mathbf{u}|^2\right)\right]\cdot d\mathbf{x}, \end{aligned} \tag{B.3}$$

to get:

$$\frac{d}{dt}\left(\mathbf{u}\cdot d\mathbf{x}\right) = \left[T\nabla S + \nabla\left(\frac{1}{2}|\mathbf{u}|^2 - h\right) + \frac{\mathbf{J}\times\mathbf{B}}{\rho} + \mathbf{B}\frac{\nabla\cdot\mathbf{B}}{\mu_0\rho}\right]\cdot d\mathbf{x}. \tag{B.4}$$

On the right-hand side (RHS) of (B.4) for the magnetic terms we use:

$$\frac{d}{dt}\left[\left(\frac{(\nabla\times\boldsymbol{\Gamma})\times\mathbf{B}}{\rho}\right)\cdot d\mathbf{x}\right] = -\left(\frac{\mathbf{J}\times\mathbf{B}}{\rho}\right)\cdot d\mathbf{x} \tag{B.5}$$

$$\frac{d}{dt}\left[\left(\frac{\nabla\cdot\mathbf{B}}{\rho}\right)\boldsymbol{\Gamma}\cdot d\mathbf{x}\right] = -\left(\frac{\nabla\cdot\mathbf{B}}{\rho}\right)\frac{\mathbf{B}}{\mu_0}\cdot d\mathbf{x}. \tag{B.6}$$

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On the right hand side of (B.4) for the gas bits we use:

$$\begin{aligned}&\frac{d}{dt}(\nabla\phi\cdot d\mathbf{x})=\nabla\left(\frac{1}{2}|\mathbf{u}|^2-h\right)\cdot d\mathbf{x},\\&\frac{d}{dt}(r\nabla S\cdot d\mathbf{x})=-T\nabla S\cdot d\mathbf{x},\quad \frac{d}{dt}\left(\tilde{\lambda}\nabla\mu\cdot d\mathbf{x}\right)=0,\\&\tilde{\lambda}=\frac{\lambda}{\rho},\quad r=\frac{\beta}{\rho}.\end{aligned}\tag{B.7}$$

to obtain the Clebsch representation $\mathbf{u}=\mathbf{u}_h+\mathbf{u}_M$ in (8.3)–(8.4).

Using (B.5)–(B.7) in (B.4) gives:

$$\frac{d}{dt}(\mathbf{w}\cdot d\mathbf{x})=0,\tag{B.8}$$

where

$$\mathbf{w}=\mathbf{u}-\left(\nabla\phi-r\nabla S-\frac{\nabla\times\mathbf{\Gamma}}{\rho}\times\mathbf{B}-\left(\frac{\nabla\cdot\mathbf{B}}{\rho}\right)\mathbf{\Gamma}\right).\tag{B.9}$$

Integration of (B.8) gives

$$\mathbf{w}\cdot d\mathbf{x}=f_0(\mathbf{x}_0)^k dx_0^k \quad \text{or} \quad w^j=f_0(\mathbf{x}_0)^k\partial x_0^k/\partial x^j.\tag{B.10}$$

Using the initial data: $w^j=f_0(\mathbf{x}_0)^j=f_{00}(\mathbf{x}_0)\partial g_{00}/\partial x_0^j$ at $t=0$ gives

$$\mathbf{w}=-\mu\nabla\nu,\tag{B.11}$$

where $\mu=-f_{00}$ and $\nu=g_{00}$. Equations (B.9)–(B.10) then give:

$$\mathbf{u}=\nabla\phi-\mu\nabla\nu-r\nabla S-\frac{\nabla\times\mathbf{\Gamma}}{\rho}\times\mathbf{B}-\left(\frac{\nabla\cdot\mathbf{B}}{\rho}\right)\mathbf{\Gamma},\tag{B.12}$$

which is the Clebsch representation (8.3)–(8.4) for $\mathbf{u}$.

The proof of (B.5) is sketched below. Note that $\mathbf{b}=\mathbf{B}/\rho$ is an advected vector field, satisfying (5.23) with $\mathbf{J}\to\mathbf{b}$. The one form on the LHS of (B.5) can be written as

$$\alpha=\mathbf{b}\lrcorner\left(\tilde{\mathbf{\Gamma}}\cdot d\mathbf{S}\right)=(\tilde{\mathbf{\Gamma}}\times\mathbf{b})\cdot d\mathbf{x}\equiv[(\nabla\times\mathbf{\Gamma})\times\mathbf{B}/\rho]\cdot d\mathbf{x}.\tag{B.13}$$

where $\tilde{\mathbf{\Gamma}} = \nabla \times \mathbf{\Gamma}$. The RHS of (B.5) is:

$$\begin{aligned}\frac{d\alpha}{dt} &= \frac{d\mathbf{b}}{dt}\lrcorner\tilde{\mathbf{\Gamma}}\cdot d\mathbf{S} + \mathbf{b}\lrcorner\frac{d}{dt}(\tilde{\mathbf{\Gamma}}\cdot d\mathbf{S})\\ &= 0 - \mathbf{b}\lrcorner(\mathbf{J}\cdot d\mathbf{S}) \equiv -\frac{\mathbf{J}\times\mathbf{B}}{\rho}\cdot d\mathbf{x}.\end{aligned} \tag{B.14}$$

This establishes (B.5). There are similar proofs for (B.6) and (B.7).

Appendix C
Cauchy Invariant $\mathbf{b} = \mathbf{B}/\rho$

In this appendix, we discuss the results (10.3)–(10.5) for the density ρ and magnetic field induction $\mathbf{B}$ in Lagrangian MHD derived in Newcomb (1962) (see also Parker 1979). The solutions for ρ and $\mathbf{B}$ are expressed in terms of the Lagrangian map $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$. One method, to derive (10.5) is to write Faraday's equation (2.4) in terms of the quantity:

$$\mathbf{b} = \frac{\mathbf{B}}{\rho}, \tag{C.1}$$

giving the equation:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\mathbf{b} = \left(\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{u}\right) \equiv \left(\frac{\partial \mathbf{b}}{\partial t} + [\mathbf{u}, \mathbf{b}]\right) = 0, \tag{C.2}$$

where $[\mathbf{u}, \mathbf{b}]$ is the Lie bracket of $\mathbf{u}$ and $\mathbf{b}$. The condition (C.2) states that the vector field $\mathbf{b}$ is Lie dragged by the vector field $\mathbf{u}$. Thus, $\mathbf{b}$ is a Lie dragged invariant, which implies:

$$\mathbf{b}\cdot\nabla = \mathbf{b}_0\cdot\nabla_0, \tag{C.3}$$

where $\mathbf{b}_0 = \mathbf{b}_0(\mathbf{x}_0)$ depends only on the Lagrange labels $\mathbf{x}_0$.

From (C.3), we obtain:

$$b^i \frac{\partial}{\partial x^i} = b_0^j \frac{\partial}{\partial x_0^j} \quad \text{so that} \quad b_0^j = b^i \frac{\partial x_0^j}{\partial x^i}. \tag{C.4}$$

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The quantity b_0^j is a Cauchy invariant, i.e. $db_0^j/dt = 0$ where $b_0^j = b_0^j(\mathbf{x}_0)$ depends only on the Lagrangian labels $\mathbf{x}_0$. To verify this result, we note:

$$\frac{db_0^j}{dt} = \frac{d}{dt}\left(b^i y_{ji}\right) = \frac{db^i}{dt} y_{ji} + b^i \frac{dy_{ji}}{dt}, \tag{C.5}$$

where $y_{ji} = \partial x_0^j/\partial x^i$. By noting that:

$$\frac{d}{dt} y_{ji} = -\frac{\partial u^s}{\partial x^i}\frac{\partial x_0^j}{\partial x^s} = -\frac{\partial u^s}{\partial x^i} y_{js}, \tag{C.6}$$

we obtain:

$$\frac{db_0^j}{dt} = \frac{\partial x_0^j}{\partial x^i}\left[\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\mathbf{u}\right]^i = 0, \tag{C.7}$$

because $\mathbf{b}$ satisfies (C.2). This proves that b_0^j is a Lie dragged invariant (i.e. a Cauchy invariant). From (C.2) we obtain:

$$\frac{B_0^j}{\rho_0} = \frac{B^i}{\rho} y_{ji} \quad \text{and hence} \quad \frac{B^i}{\rho} = x_{ij}\frac{B_0^j}{\rho_0}. \tag{C.8}$$

Using the Lagrangian mass continuity equation: $\rho d^3x = \rho_0 d^3x_0$ We obtain:

$$\rho = \frac{\rho_0}{J} \quad \text{and} \quad B^i = \frac{x_{ij}B_0^j}{J}, \tag{C.9}$$

where $J = \det(x_{ij})$ is the Jacobian of the Lagrangian map. This establishes (10.3)–(10.5).

Appendix D
Magnetosonic N-Waves

In this appendix, we discuss the phase and group velocity for the magneto-acoustic and Alfvén waves described by (12.54) and (12.53), namely:

$$F_{MS} = \omega^4 - \left(a^2 + b^2\right)\omega^2 k^2 + a^2 k^2 (\mathbf{b}\cdot\mathbf{k})^2 = 0, \tag{D.1}$$

$$F_A = \omega^2 - (\mathbf{b}\cdot\mathbf{k})^2 = 0. \tag{D.2}$$

We use cylindrical coordinates $\mathbf{r} = (x_1, x_2\cos\theta, x_2\sin\theta)^T$ where x_1 is distance along the field $\mathbf{B}$ and x_2 is cylindrical radius about $\mathbf{B}$. The corresponding coordinates in $\mathbf{k}$-space are $\mathbf{k} = (k_1, k_2\cos\Theta, k_2\sin\Theta)^T$. The wave frequency ω and wave number $\mathbf{k}$ are defined by the equations:

$$\mathbf{k} = \nabla S \quad \text{and} \quad \omega = -S_t, \tag{D.3}$$

where S is the wave phase or wave eikonal function (In this appendix S is the wave eikonal function, and not the entropy of the gas). From (D.1)–(D.3) the fast magneto-acoustic wave dispersion equation and the Alfvén dispersion equations may be written as:

$$\begin{aligned} F_{MS} &= \omega^4 - (a^2 + b^2)\omega^2\left(k_1^2 + k_2^2\right) + a^2 b^2 k_1^2\left(k_1^2 + k_2^2\right) = 0,\\ F_A &= \omega^2 - b^2 k_1^2 = 0. \end{aligned} \tag{D.4}$$

In terms of S and its derivatives, the dispersion equations are:

$$\begin{aligned} F_{MS} &= S_t^4 - (a^2 + b^2)\left(S_{x_1}^2 + S_{x_2}^2\right) S_t^2 + a^2 b^2 S_{x_1}^2\left(S_{x_1}^2 + S_{x_2}^2\right) = 0,\\ F_A &= S_t^2 - b^2 S_{x_1}^2 = (S_t - bS_{x_1})(S_t + bS_{x_1}) = 0. \end{aligned} \tag{D.5}$$

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Equations $F_{MS} = 0$ and $F_A = 0$ in (D.5) are first order, nonlinear partial differential equations for S, which may be integrated by using the method of characteristics (Sneddon 1957). In general, there are both multi-parameter, complete integral solutions, and envelope type solutions. In the MHD case, the envelope solutions corresponds to the group velocity surface for the waves. Equations (D.5) are the magnetosonic and Alfvén wave eikonal equations. One can also write the magneto-acoustic and Alfvén dispersion equations in the form:

$$\omega_m - \Omega_m(\mathbf{k}, \mathbf{x}) = 0, \tag{D.6}$$

where the subscript m identifies the wave mode of interest.

Writing $k^0 = S_t$ and $(t, x, y, z) = x^\alpha$ ($\alpha = 0, 1, 2, 3$), the characteristics of the wave eikonal equation $F = 0$ (here $F = F_{MS}$ or $F = F_A$) are given by:

$$\frac{dx^\alpha}{d\tau} = \frac{\partial F}{\partial k^\alpha}, \quad \frac{dS}{d\tau} = k^\alpha \frac{\partial F}{\partial x^\alpha}, \quad \frac{dk^\alpha}{d\tau} = -\left(\frac{\partial F}{\partial x^\alpha} + k^\alpha \frac{\partial F}{\partial S}\right), \quad \alpha = 0, 1, 2, \tag{D.7}$$

(Sneddon 1957), where τ is the affine parameter along the characteristics. Because $\partial F/\partial S = 0$, the characteristics (D.6) for the magnetoacoustic modes satisfy Hamilton's equations:

$$\frac{dx^\alpha}{d\tau} = \frac{\partial F}{\partial k^\alpha}, \quad \frac{dk^\alpha}{d\tau} = -\frac{\partial F}{\partial x^\alpha}, \quad \frac{dS}{d\tau} = 4F = 0, \tag{D.8}$$

where $F \equiv F_{MS}$. The equation $dS/d\tau = 4F$ follows by noting that $F(\lambda\mathbf{k}, \mathbf{x}) = \lambda^4 F(\mathbf{k}, \mathbf{x})$ for $F = F_{MS}$ where $k^\alpha = \partial S/\partial x^\alpha$ for $\alpha = 0, 1, 2$. Thus, F_{MS} is a homogeneous function of degree 4 in k^α. Thus the wave eikonal function S does not change moving along the characteristics.

Alternatively, if we separate off the individual wave modes in the form $F = \omega - \Omega(\mathbf{k}, \mathbf{x}) = 0$ where $\mathbf{k}$ is the spatial $\mathbf{k}$-vector then, the characteristics or ray equations take the form:

$$\begin{aligned} \frac{dx^i}{d\tau} &= \frac{\partial \Omega}{\partial k^i}, \quad \frac{dk^i}{d\tau} = -\frac{\partial \Omega}{\partial x^i}, \\ \frac{dt}{d\tau} &= 1, \quad \frac{dS}{d\tau} = \omega \frac{\partial F}{\partial \omega} - \mathbf{k} \cdot \frac{\partial \Omega}{\partial \mathbf{k}} = 0, \quad \frac{d\omega}{d\tau} = \frac{\partial \Omega}{\partial t}. \end{aligned} \tag{D.9}$$

The evolution equations for x^i, k^i and ω are Hamilton's equations of classical mechanics, with Hamiltonian Ω. To prove $dS/d\tau = 0$ in (D.8) we note:

$$\begin{aligned} \omega &= \Omega = kV_p(\mathbf{n}, \mathbf{x}, t), \quad \mathbf{n} = \frac{\mathbf{k}}{k}, \\ \mathbf{V}_g &= \frac{\partial \omega}{\partial \mathbf{k}} = V_p \mathbf{n} + (\mathsf{I} - \mathbf{nn}) \cdot \nabla_{\mathbf{n}} V_p, \end{aligned} \tag{D.10}$$

which implies:

$$\frac{dS}{d\tau} = \omega - \mathbf{k} \cdot \mathbf{V}_g = \omega - kV_p = \omega - \Omega = 0. \tag{D.11}$$

For the magneto-acoustic modes the wave dispersion equation (D.1) can be expressed in the form:

$$c^4 - \left(a^2 + b^2\right) c^2 + a^2 \left(\mathbf{b} \cdot \mathbf{n}\right)^2 = 0 \quad \text{where} \quad \mathbf{b} \cdot \mathbf{n} = b \cos \vartheta. \tag{D.12}$$

The group velocity

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = c\mathbf{n} - \frac{a^2(\mathbf{b} \cdot \mathbf{n}) \left[\mathbf{b} - \mathbf{b} \cdot \mathbf{nn}\right]}{c \left[2c^2 - (a^2 + b^2)\right]}. \tag{D.13}$$

Alternatively using $\omega = kc(\vartheta)$ where $\cos \vartheta = \mathbf{n} \cdot \mathbf{e}_1$ and $\mathbf{e}_1 = \mathbf{B}/B$ is the unit vector along $\mathbf{B}$ we obtain:

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = c(\vartheta)\mathbf{n} + c'(\vartheta)\mathbf{e}_\vartheta, \tag{D.14}$$

where $c'(\vartheta) = \partial c/\partial \vartheta$ and

$$\mathbf{n} = \cos \vartheta \mathbf{e}_1 + \sin \vartheta \mathbf{e}_2, \quad \mathbf{e}_\vartheta = -\sin \vartheta \mathbf{e}_1 + \cos \vartheta \mathbf{e}_2. \tag{D.15}$$

From (D.13)–(D.15) the group velocity surface $\mathbf{r} = \mathbf{V}_g t$ can be written in the form:

$$\begin{aligned} x_1 &= V_{g1} t = \left[c(\vartheta) \cos \vartheta - c'(\vartheta) \sin \vartheta\right] t, \\ x_2 &= V_{g2} t = \left[c'(\vartheta) \cos \vartheta + c(\vartheta) \sin \vartheta\right] t, \end{aligned} \tag{D.16}$$

where

$$c'(\vartheta) = \frac{a^2 b^2 \cos \vartheta \sin \vartheta}{c \left[2c^2 - (a^2 + b^2)\right]}. \tag{D.17}$$

the derivation of (D.17) follows by implicit differentiation of the dispersion equation (D.12).

The group velocity surface (D.16) also follows by determining the envelope of the family of plane waves described by the eikonal solution:

$$S = k \left(x_1 \cos \vartheta + x_2 \sin \vartheta\right) - kc(\vartheta)t, \tag{D.18}$$

obtained by requiring $S = 0$ and $S_\vartheta = 0$ simultaneously. These two conditions give the solutions (D.16) for $x_1(\vartheta, t)$ and $x_2(\vartheta, t)$ for the group velocity surface.

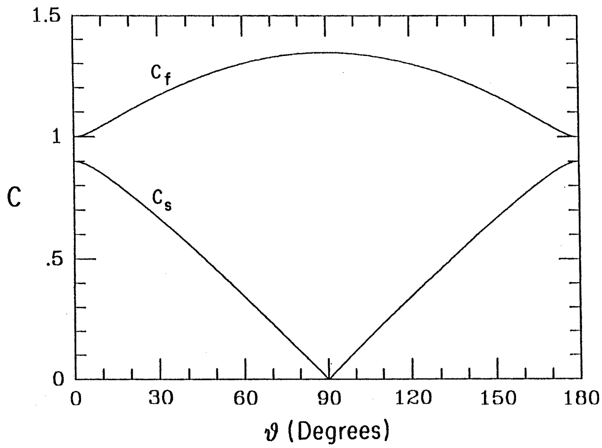

Fig. D.1 The phase velocities c_f and c_s from (D.19) for the fast and slow magnetosonic waves versus the angle ϑ between **k** and **B** ($\mathbf{k} \cdot \mathbf{B} = kB \cos\vartheta$) for the case $a = 1$ and $b = 0.9$

Figure D.1 shows plots of the fast and slow magnetosonic speeds, $c_f(\vartheta)$ and $c_s(\vartheta)$ versus ϑ for the case $a = 1$ and $b = 0.9$. The fast and slow magnetosonic modes are given by:

$$c_{f,s}^2 = \frac{1}{2}\left[\left(a^2 + b^2\right) \pm \left[\left(a^2 + b^2\right)^2 - 4a^2b^2\cos^2\vartheta\right]^{1/2}\right]. \qquad \text{(D.19)}$$

The main points to note are that $c_f > c_s$ for all ϑ ($0 < \vartheta < \pi$). The slow speed $c_s = 0$ at $\vartheta = \pi/2$ and the fast mode speed is maximal at $\vartheta = \pi/2$ where $c_f = (a^2 + b^2)^{1/2}$.

Figure D.2 shows the group velocity surface (D.16) for the fast and slow magnetosonic waves for the case $a = 1$ and $b = 0.9$. The group velocity $\mathbf{r} = \mathbf{V}_g t$ ($t = 1$) is the vector **OP** from the origin to a point $P(x_1, x_2)$ on the surface. The wave vector $\mathbf{k} = \nabla S$ is normal to the surface. The outer ellipsoidal-like surface is the fast magnetosonic group velocity surface, and the two cusped triangular-shaped surfaces are the slow mode surfaces (see e.g. Webb et al. 1993, 1994, 2001 for more detail). The formulation (D.16) of the MHD group velocity surfaces is similar to that of Whitham (1974) (equations (7.92)–(7.96)).

The dispersion equation and phase speed for the forward Alfvén wave from (D.4) are:

$$\omega = bk\cos\vartheta \quad \text{and} \quad V_A = \frac{\omega}{k} = b\cos\vartheta. \qquad \text{(D.20)}$$

The group velocity for the forward Alfvén wave from (D.14) is:

$$\mathbf{V}_{gA} = \frac{\partial\omega}{\partial\mathbf{k}} = bk\cos\vartheta\,\mathbf{n} - bk\sin\vartheta\,\mathbf{e}_\vartheta = b\mathbf{e}_1 \equiv \frac{\mathbf{B}}{\sqrt{\mu\rho}}. \qquad \text{(D.21)}$$

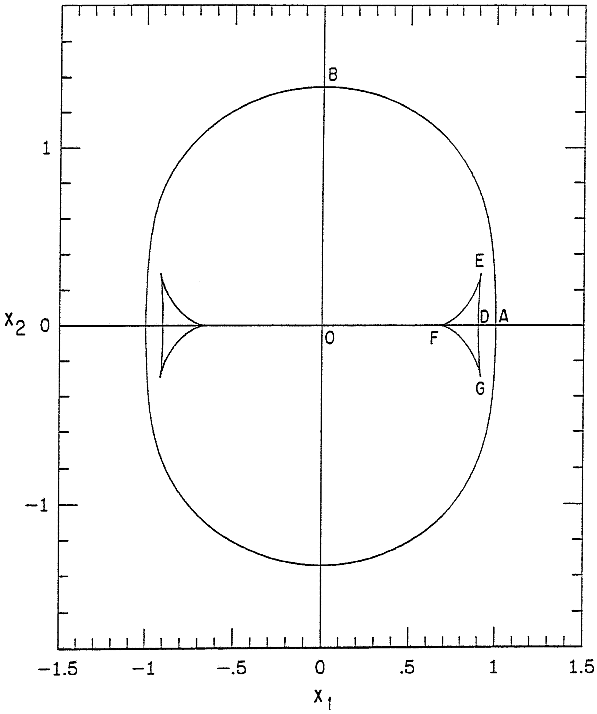

Fig. D.2 The group velocity surfaces for (a) the fast magnetosonic wave (outer ellipsoidal shaped curve) and (b) the slow magnetosonic group velocity surfaces (triangular shaped curves) for the case $a = 1$ and $b = 0.9$. x_1 is distance along the magnetic field and x_2 is distance perpendicular to **B**. The group velocity surfaces are calculated using the formulae (D.16) with $t = 1$

Thus, the Alfvén wave group velocity is directed along the magnetic field **B**, but the phase velocity $V_{pA} = V_A \cos\vartheta$ is parallel to the wave normal **n**.

An alternative approach to obtaining the group velocity surface developed by Lighthill (1960) is to plot the dispersion equation $F_{MS} = 0$ in the form:

$$\left(\frac{k_\perp}{\omega}\right)^2 = -\frac{\left[(k_\parallel/\omega)^2 - 1/a^2\right]\left[(k_\parallel/\omega)^2 - 1/b^2\right]}{\left[(k_\parallel/\omega)^2 - (1/a^2 + 1/b^2)\right]}. \tag{D.22}$$

Thus the wave number surface, or the slowness surface is the plot of $k_\perp/\omega$ versus $k_\parallel/\omega$ for a fixed ω (the slowness is defined as $\mathbf{k}/\omega$). From Whitham (1974), Section 11.4, one can identify the condition for stationary phase:

$$\delta S = \delta(\mathbf{k}\cdot\mathbf{r} - \omega t) = \delta\mathbf{k}\cdot(\mathbf{r} - \omega_{\mathbf{k}} t) = 0 \tag{D.23}$$

with the group velocity surface:

$$\mathbf{r} - \mathbf{V}_g t = 0 \quad \text{where} \quad \mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}}. \tag{D.24}$$

Differentiation of the dispersion equation $D(\omega, \mathbf{k}) = 0$ gives:

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = -\frac{D_{\mathbf{k}}(\mathbf{k}, \omega)}{D_\omega(\mathbf{k}, \omega)}. \tag{D.25}$$

In Lighthill's method of stationary phase (Lighthill 1960) the wave number surface $D(\mathbf{k}, \omega) = 0$ with ω fixed, has normal: $\mathbf{n} = -D_{\mathbf{k}}(\mathbf{k}, \omega)/|D_{\mathbf{k}}(\mathbf{k}, \omega)|$. Because $\mathbf{V}_g = V_g \mathbf{n}$, the group velocity surface can be written in the form:

$$\mathbf{r} = \frac{(\mathbf{k} \cdot \mathbf{r})\mathbf{n}}{\mathbf{k} \cdot \mathbf{n}} = \frac{\phi \mathbf{n}}{\mathbf{k} \cdot \mathbf{n}}, \tag{D.26}$$

where

$$\phi = \mathbf{r} \cdot \mathbf{k} \equiv S + \omega t. \tag{D.27}$$

Because S is stationary and ω is fixed, the phase $\phi = S + \omega t$ is a constant at a fixed t. This allows one to geometrically construct the group velocity surface from the wave-number surface (in our case the wave number surface is the slowness surface (D.22)). At a given point P on the wavenumber surface ($\mathbf{k} = \mathbf{OP}$ where $\mathbf{O}$ is the origin in $\mathbf{k}$-space), determine the wave normal $\mathbf{n} = -D_{\mathbf{k}}/|D_{\mathbf{k}}|$. Draw the tangent to the wave number surface through P, and find the perpendicular distance $OT = \mathbf{k} \cdot \mathbf{n}$ from the tangent plane to the origin. The group velocity surface from (D.26) is then given by

$$\mathbf{r} = \frac{\phi \mathbf{n}}{OT}. \tag{D.28}$$

The shape of the group velocity surface is given by $\mathbf{r} = \mathbf{n}/OT$ which is the reciprocal polar, or pedal curve of the wave number surface (because ϕ is constant we can set $\phi = 1$ in (D.28)).

To obtain the group velocity surface (D.28) from the wave number surface (D.22) note that the slowness:

$$\bar{k} = \frac{k}{\omega} = \frac{1}{c(\vartheta)}, \tag{D.29}$$

where $c(\vartheta)$ is the wave phase speed. The wave phase is:

$$\phi = \bar{\mathbf{k}} \cdot \mathbf{x} = \bar{k} r \cos(\vartheta - \chi), \tag{D.30}$$

where

$$\begin{aligned}\mathbf{x} =&(x_1, x_2) = r(\cos\chi, \sin\chi),\\ \bar{\mathbf{k}} =&(\bar{k}_1, \bar{k}_2) = \bar{k}\,(\cos\vartheta, \sin\vartheta)\,,\end{aligned} \tag{D.31}$$

For stationary phase variations:

$$\delta\phi = \frac{\partial\phi}{\partial\vartheta}\delta\vartheta = \delta\vartheta\left[-\bar{k}r\sin(\vartheta-\chi) + \frac{\partial\bar{k}}{\partial\vartheta}r\cos(\vartheta-\chi)\right] = 0. \tag{D.32}$$

From (D.32)

$$\frac{\bar{k}'(\vartheta)}{\bar{k}(\vartheta)} = \tan(\vartheta-\chi) = \tan(\zeta), \tag{D.33}$$

which implies $\zeta = \vartheta - \chi$ and

$$\begin{aligned}\tan\chi =& \tan(\vartheta-\zeta) = \frac{\tan\vartheta - k'/k}{1+\tan\vartheta\,(k'/k)}\\ =&\frac{\tan\vartheta + c'/c}{1-\tan\vartheta\,(c'/c)} = \frac{c\sin\vartheta + c'\cos\vartheta}{c\cos\vartheta - c'\sin\vartheta} = \frac{V_{g2}}{V_{g1}}.\end{aligned} \tag{D.34}$$

From (D.34):

$$\cos\chi = \frac{[c\cos\vartheta - c'(\vartheta)\sin\vartheta]}{\left[c^2+c'^2\right]^{1/2}} \qquad \sin\chi = \frac{[c\sin\vartheta + c'(\vartheta)\cos\vartheta]}{\left[c^2+c'^2\right]^{1/2}} \tag{D.35}$$

and (D.26) gives:

$$\mathbf{r} = \phi\left[c\cos\vartheta - c'(\vartheta)\sin\vartheta, c\sin\vartheta + c'(\vartheta)\cos\vartheta\right] = \mathbf{V}_g\phi, \tag{D.36}$$

which is the group velocity surface (D.16) for the case $\phi = t$.

Figures D.3 and D.4 show the magnetic field lines for the linear magnetosonic N wave arising from an initial delta function pressure distribution $\delta p = A\delta(\mathbf{x})$ at time $t = 0$, which is the analog of the acoustic N-wave described by Whitham (1974). In the acoustic N-wave, the solution consists of an N-wave in which there is a compression followed by a rarefaction located in the vicinity the sonic group velocity surface at the leading edge of the wave. The detailed form of the wave depends on whether the geometry is planar, cylindrical or spherical. The MHD analog of the acoustic $N-$ wave was obtained by Webb et al. (1993) in which the initial uniform magnetic field $\mathbf{B} = B_0\mathbf{e}_{x1}$ lies along the x_1-axis. The main point of interest in Figs. D.3 and D.4 is that there are singular N-wave type disturbances located on both the slow mode and fast mode group velocity surfaces. The magnetic

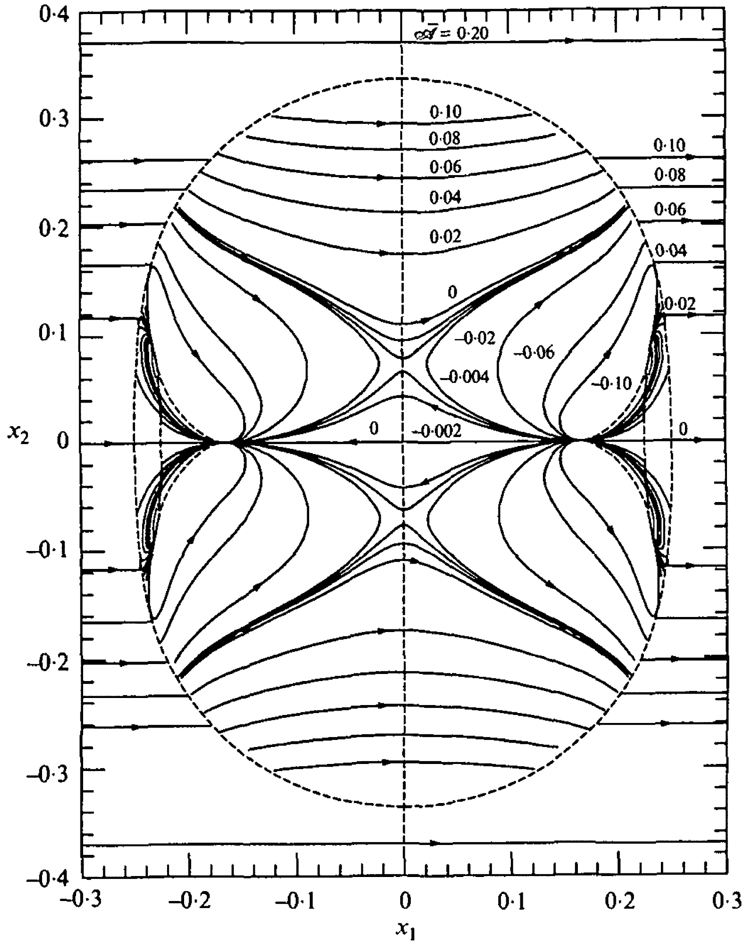

Fig. D.3 Magnetic field lines (contours of wave potential $\mathcal{A}$) for the magnetosonic N-wave for $a = 1$ and $b = 0.9$ at time $t = 0.25$. The perturbation parameter $\epsilon = A/p_0 = 0.2$. The fast and slow mode magnetosonic eikonals are given by the dashed curves (from Webb et al. 1993, Fig. 11)

field structure evolves with time. The fields (according to linear theory) reconnect at time $t = 0.31376$ after which times the slow mode cusps pull apart (right panel of Fig. D.4). The deltoid shaped slow magnetoacoustic surfaces act as sources and sinks for the magnetic field. The forward slow magnetoacoustic cusp point acts as a source and the backward slow magnetoacoustic cusp point acts as a sink. These points together, at early times, behave like a dipole, in which the forward point is a north pole and the backward slow cusp is the south pole. The polarity of the dipole is opposite to the uniform background magnetic field. In the left panel of Fig. D.4, the dipole collapses to a single field line connecting the north and south poles, and the

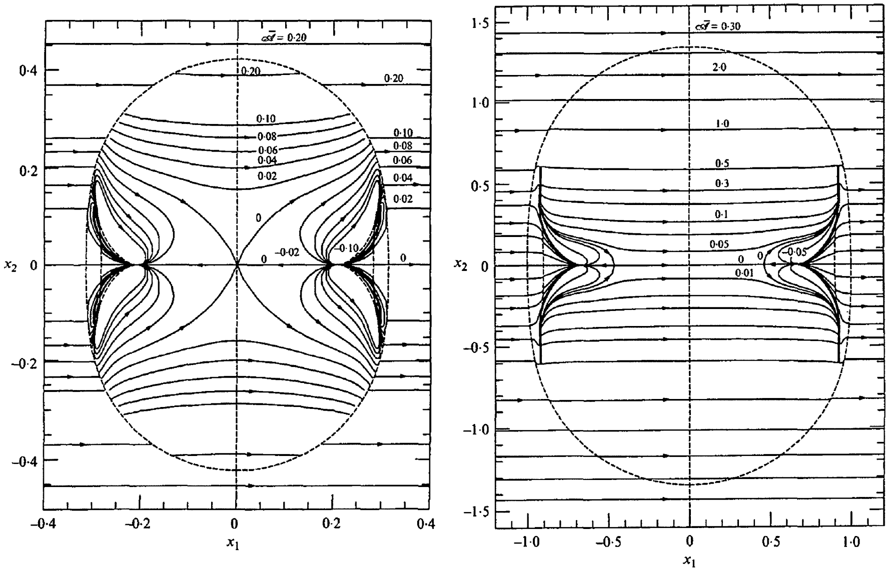

Fig. D.4 Magnetic field lines for the magnetosonic N-wave for the same parameters as in Fig. D.3 but at later times $t = 0.31376$ (left) and $t = 1$ (right) (from Webb et al. 1993, Figures 12 and 13)

external uniform magnetic field begins to dominate the solution. In the right panel of Fig. D.4, a straight line uniform magnetic field segment connecting the backward and forward slow mode disturbances develops, involving two neutral points at the ends of the segment.

We are not aware of numerical MHD simulations of the magnetoacoustic N-wave, but we expect that for small initial disturbances of δp confined near the origin should give rise to a magnetic field structure similar to that in Figs. D.3 and D.4. there is no disturbance outside the fast mode group velocity surface.

In Fig. D.2 the group velocity for the slow mode at the cusp F:

$$\mathbf{V}_c = \frac{ab}{\sqrt{a^2 + b^2}} \mathbf{e}_B, \tag{D.37}$$

is along the field. The cusp speed V_c describes surface waves on magnetic flux tubes (e.g. Roberts and Mangeney 1982; Roberts 1985). For slab magnetic field geometry the weakly nonlinear tube wave or surface wave is described by the Benjamin-Ono equation (Roberts and Mangeney 1982). For cylindrical flux tubes, weakly nonlinear long wavelength tube waves are governed by the Leibovich-Roberts equation (e.g. Roberts 1985; Weishaar 1989; Bogdan and Lerche 1988; Ruderman 2006).

Appendix E
Aharonov Bohm Effects in MHD

This appendix discusses the formulation of Yahalom (2013, 2016a, 2017a,b) of magnetic helicity H_M and non-barotropic cross helicity H_{CNB}. Yahalom developed a Clebsch variable variational principle for MHD that uses the action:

$$\begin{aligned}\mathcal{A} = \int \Bigg\{ & \left(\frac{1}{2}\rho u^2 - \rho e(\rho, S) + \frac{B^2}{2\mu_0}\right) \\ & + \phi \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right] - \rho\alpha \frac{d\chi}{dt} - \rho\beta \frac{d\eta}{dt} - \rho\sigma \frac{dS}{dt} \\ & - \frac{\mathbf{B}}{\mu_0} \cdot \nabla\chi \times \nabla\eta \Bigg\} d^3x\, dt. \end{aligned} \tag{E.1}$$

The stationary point requirements $\delta\mathcal{A}/\delta\mathbf{B} = 0$ and $\delta\mathcal{A}/\delta\mathbf{u} = 0$ gives the Clebsch expansions:

$$\begin{aligned}\mathbf{B} =& \nabla\chi \times \nabla\eta, \\ \mathbf{u} =& \nabla\phi + \alpha\nabla\chi + \beta\nabla\eta + \sigma\nabla S, \end{aligned} \tag{E.2}$$

for $\mathbf{B}$ and $\mathbf{u}$ ($r \equiv -\sigma$ in our formulation). One can write down the other variational equations by varying ρ, S and the Clebsch variables in the variational principle (see e.g. Yahalom 2017a,b).

The $\mathbf{B}$ field Clebsch variable expansion (E.2) is also used by Sakurai (1979). $\mathbf{A}$ and $\mathbf{B}$ have the forms:

$$\mathbf{A} = \chi\nabla\eta + \nabla\zeta, \quad \mathbf{B} = \nabla\chi \times \nabla\eta. \tag{E.3}$$

For a non-trivial $\mathbf{B}$ field topology there does not exist a global $\mathbf{A}$ (i.e. χ, η, ζ are not global single valued functions of $\mathbf{x}$). From (E.2) the magnetic helicity density

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$h_m = \mathbf{A} \cdot \mathbf{B}$ is given by:

$$h_m = \mathbf{A} \cdot \mathbf{B} = \nabla\zeta \cdot \nabla\chi \times \nabla\eta = \frac{\partial(\zeta, \chi, \eta)}{\partial(x, y, z)}. \tag{E.4}$$

Thus $h_m \neq 0$ only if χ, η and ζ are independent functions of $\mathbf{x}$. Semenov et al. (2002) show that the field topology changes due to jumps in ζ in magnetic fields with non-trivial topology for generalized versions of the MHD topological soliton (c.f. Kamchatnov 1982). A jump in ζ also occurs in the non-global $\mathbf{A}$ for the magnetic monopole (Urbantke 2003).

Yahalom (2013, 2017a,b) introduces an independent magnetic field potential, $\zeta(\mu)$, (μ is called the metage). The metage μ represents distance or affine parameter along the magnetic field line formed by the intersection of the $\eta = const.$ and $\chi = const.$ Euler potential surfaces. We obtain:

$$\nabla\zeta = \frac{\partial\zeta}{\partial\chi}\nabla\chi + \frac{\partial\zeta}{\partial\eta}\nabla\eta + \frac{\partial\zeta}{\partial\mu}\nabla\mu. \tag{E.5}$$

Using (E.5) in (E.4) gives:

$$h_m = \mathbf{A} \cdot \mathbf{B} = \frac{\partial\zeta}{\partial\mu}\nabla\mu \cdot \nabla\chi \times \nabla\eta = \frac{\partial\zeta}{\partial\mu}\left|\frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}\right|. \tag{E.6}$$

The magnetic helicity is given by:

$$H_M = \int_V \mathbf{A} \cdot \mathbf{B}\, d^3x = \int_V \frac{\partial\zeta}{\partial\mu}\, d\chi \wedge d\eta \wedge d\mu. \tag{E.7}$$

However, the differential of the magnetic flux:

$$d\Phi_B = \mathbf{B} \cdot d\mathbf{S} = (\nabla\chi \times \nabla\eta) \cdot d\mathbf{S} = d\chi d\eta. \tag{E.8}$$

Equation (E.6) follows by noting that:

$$\begin{aligned} d\mathbf{S} =& \mathbf{r}_\chi \times \mathbf{r}_\eta\, d\chi d\eta \quad \text{and} \quad \mathbf{B} = \nabla\chi \times \nabla\eta, \\ \mathbf{B} \cdot d\mathbf{S} =& (\nabla\chi \times \nabla\eta) \cdot (\mathbf{r}_\chi \times \mathbf{r}_\eta)\, d\chi d\eta = d\chi d\eta. \end{aligned} \tag{E.9}$$

Equation (E.9) follows by setting $(q^1, q^2, q^3) = (\chi, \eta, \mu)$ and noting:

$$\frac{\partial q^a}{\partial x^k}\frac{\partial x^k}{\partial q^b} = \delta^a_b \quad \text{which implies} \quad \mathbf{e}^a \cdot \mathbf{e}_b = \delta^a_b, \tag{E.10}$$

where $\mathbf{e}^a = \nabla q^a$ and $\mathbf{e}_b = \partial\mathbf{r}/\partial x^b$.

Using (E.6) in (E.5) and integrating over μ, we obtain:

$$H_M = \int [\zeta]\ d\chi d\eta \equiv \int [\zeta]\, d\Phi_B, \tag{E.11}$$

where $[\zeta]$ is the jump in ζ between the two ends of the field line (in the Aharonov Bohm problem, the path is a closed path). Equation (E.11) lead to the invariant:

$$[\zeta] = \frac{dH_M}{d\Phi_B}, \tag{E.12}$$

which is the magnetic helicity per unit magnetic flux. Thus, for a closed field line, the jump in $[\zeta]$ is non-zero for a non-trivial magnetic helicity. Yahalom (2013, 2016a, 2017a,b) refers to (E.12) as the MHD 'magnetic Aharonov-Bohm effect', in analogy with the Aharonov-Bohm effect in quantum mechanics.

Yahalom (2013, 2016a, 2017a,b) and Webb et al. (2014a,b) developed conservation laws for generalized cross helicity for both barotropic and non-barotropic MHD. The cross helicity H_C defined as:

$$H_C = \int_V \mathbf{u} \cdot \mathbf{B}\, d^3x, \tag{E.13}$$

is conserved for non-barotropic flows. The differential form of the cross helicity evolution equation from (3.62) is:

$$\frac{\partial}{\partial t}(\mathbf{u} \cdot \mathbf{B}) + \nabla \cdot \left[(\mathbf{u} \cdot \mathbf{B})\mathbf{u} + \mathbf{B}\left(h + \Phi - \frac{1}{2}u^2\right)\right] = T(\mathbf{B} \cdot \nabla S). \tag{E.14}$$

Integration of (E.14) over the volume V_m co-moving with the fluid, and assuming $\mathbf{B} \cdot \mathbf{n} = 0$ on ∂V, where $\mathbf{n}$ is the outward normal to ∂V, gives the helicity evolution equation:

$$\frac{dH_C}{dt} = \int_V T(\mathbf{B} \cdot \nabla S)\, d^3x. \tag{E.15}$$

Thus, $dH_C/dt = 0$ for barotropic flows where $\nabla S = 0$.

For non-barotropic flows, we define the generalized cross helicity as:

$$H_{CNB} = \int_V (\mathbf{u} - \sigma \nabla S) \cdot \mathbf{B}\, d^3x, \tag{E.16}$$

(note $\sigma = -r$ in Webb et al. (2014a,b) and Webb and Anco (2017); and in the definition of H_{CNB} in 3.68). Equation (E.14) gives:

$$\frac{dH_{CNB}}{dt} = 0, \quad \text{where} \quad \frac{d\sigma}{dt} = T(\mathbf{x}, t). \tag{E.17}$$

Using the Clebsch expansions (E.2) for **u** and **B**, we obtain:

$$H_C = \int \mathbf{B} \cdot \nabla \phi \, d^3x + \int \sigma \mathbf{B} \cdot \nabla S \, d^3x,$$
$$\equiv \int [\phi] d\Phi_B + \int \sigma \frac{\partial S}{\partial \mu} d\mu d\Phi_B. \tag{E.18}$$

Also

$$H_{CNB} = H_C - \int \sigma \mathbf{B} \cdot \nabla S \, d^3x = H_C - \int \sigma \frac{\partial S}{\partial \mu} d\mu d\Phi_B. \tag{E.19}$$

Here $[\phi]$ is the jump in ϕ across the discontinuity surface Σ. For simplicity we assume that there is one such surface, Σ, but there could be many such surfaces. From (E.18) and (E.19)

$$\frac{dH_C}{d\Phi_B} = [\phi] + \int \sigma \frac{\partial S}{\partial \mu} d\mu \equiv [\phi] + \oint \sigma dS, \quad \frac{dH_{CNB}}{d\Phi_B} = [\phi]. \tag{E.20}$$

Thus, $dH_{CNB}/d\Phi_B$ is an advected topological invariant (note $d[\phi]/dt = 0$ follows from the variational equation $\delta\mathcal{A}/\delta\rho = 0$). A more detailed analysis is given by Yahalom (2017a,b).

Appendix F
Equivalence Transformations

In this appendix we give a definition of equivalence transformations for a system of partial differential equations:

$$R^\sigma(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \tag{F.1}$$

(see also Bluman et al. 2010, p. 21). The differential equation system (F.1) is assumed to involve L constitutive parameters for functions $(K_1, K_2, \ldots K_N)$, which may depend on both the dependent and independent variables and on the derivatives of the dependent variables. A one parameter Lie group of equivalence transformations of a family $\mathcal{S}_k$ of differential equations of the type (F.1) of the form:

$$\tilde{x}^i = f^i(x, u, \epsilon), \quad \tilde{u}^p = g^p(x, u, \epsilon), \quad \tilde{K}_s = G_s(x, u, k, \epsilon), \tag{F.2}$$

maps members of the differential equation system (F.1) onto another member of the system $\mathcal{S}_k$ in the same family.

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Appendix G
Covariant, Non-relativistic MHD

In this appendix we discuss the use of generalized Eulerian coordinates q^k and Lagrangian labels a^k in the Lagrangian action principle described in Chap. 10. This approach has some similarities with the general relativistic MHD action principle developed by Achterberg (1983).

From Chap. 10, (Sect. 10.1), the Lagrangian map $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ is obtained by formally integrating the differential equation system:

$$\frac{\partial x^i(\mathbf{x}_0, t)}{\partial t} = u^i(\mathbf{x}, t) \quad \text{where} \quad x^i(\mathbf{x}_0, 0) = x_0^i, \tag{G.1}$$

and the Eulerian fluid velocity $u^i(\mathbf{x}, t)$ is assumed known. We also use generalized coordinates $q^i(\mathbf{a}, t)$ to describe the Lagrangian map, where

$$\frac{\partial q^i}{\partial t} = w^i(\mathbf{q}, t), \quad \text{where} \quad q^i(\mathbf{a}, 0) = a^i. \tag{G.2}$$

For example, we could use spherical polar coordinates $\mathbf{q} = (r, \theta, \phi)$ rather than Cartesian coordinates to describe the Eulerian position of the fluid element. w^i gives the generalized velocity corresponding to $\partial q^i/\partial t$.

We use holonomic coordinate bases vectors $\{\mathbf{e}_k\}$ and its dual basis $\{\mathbf{e}^k\}$, where

$$\mathbf{e}_k = \frac{\partial \mathbf{x}}{\partial q^k}, \quad \mathbf{e}^k = \frac{\partial q^k}{\partial \mathbf{x}}, \tag{G.3}$$

satisfy the orthogonality relations:

$$\langle \mathbf{e}_k, \mathbf{e}^m \rangle = \frac{\partial x^i}{\partial q^k} \frac{\partial q^m}{\partial x^i} = \delta^m{}_k, \tag{G.4}$$

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The metric tensors g_{mk} and g^{mk} are defined by the equations:

$$g_{mk} = \mathbf{e}_m \cdot \mathbf{e}_k \quad \text{and} \quad g^{mk} = \mathbf{e}^m \cdot \mathbf{e}^k. \tag{G.5}$$

Using (G.3)–(G.5) we obtain:

$$\mathbf{e}^s = g^{sp}\mathbf{e}_p, \quad \mathbf{e}_s = g_{sp}\mathbf{e}^p, \tag{G.6}$$

relating the bases $\{\mathbf{e}_s\}$ and $\{\mathbf{e}^s\}$. From the definition of the determinant $J = \det(\partial x^i/\partial q^j)$ it follows that

$$\mathbf{e}_a \cdot \mathbf{e}_b \times \mathbf{e}_c = J\varepsilon_{abc} \quad \text{where} \quad J = \det\left(\frac{\partial x^i}{\partial q^j}\right). \tag{G.7}$$

Note that

$$g = \det(g_{ab}) = \det\left(\frac{\partial x^i}{\partial q^a}\delta_{ij}\frac{\partial x^j}{\partial q^b}\right) = J^2 \quad \text{and} \quad J = \sqrt{g}. \tag{G.8}$$

The formulae:

$$\begin{aligned}\mathbf{e}^b \times \mathbf{e}^c &= \frac{\varepsilon_{bca}}{\sqrt{g}}\mathbf{e}_a \quad \text{or} \quad \mathbf{e}_a = \sqrt{g}\varepsilon_{abc}\mathbf{e}^b \times \mathbf{e}^c,\\ \mathbf{e}_b \times \mathbf{e}_c &= \sqrt{g}\varepsilon_{bca}\mathbf{e}^a \quad \text{or} \quad \mathbf{e}^a = \frac{\varepsilon_{abc}}{\sqrt{g}}\mathbf{e}_b \times \mathbf{e}_c.\end{aligned} \tag{G.9}$$

connect the two bases.

The mass continuity equation may be written in the form:

$$\rho d^3x = \rho_0 d^3x_0 \quad \text{or} \quad \rho J(\mathbf{x}, \mathbf{x}_0) = \rho_0, \tag{G.10}$$

where the determinant J is given by:

$$J(\mathbf{x}, \mathbf{x}_0) = \det\left(\frac{\partial x^i}{\partial x_0^j}\right). \tag{G.11}$$

By noting that

$$d^3x = J(\mathbf{x}, \mathbf{x}_0)d^3x_0, \quad d^3x_0 = \sqrt{g_0}d^3a, \quad d^3x = J(\mathbf{x}, \mathbf{x}_0)d^3x_0, \tag{G.12}$$

and the rule:

$$J(\mathbf{x}, \mathbf{x}_0) = J(\mathbf{x}, \mathbf{q})J(\mathbf{q}, \mathbf{a})J(\mathbf{a}, \mathbf{x}_0), \tag{G.13}$$

for the composition of determinants, we obtain:

$$J(\mathbf{x},\mathbf{x}_0) = \sqrt{g}J(\mathbf{q},\mathbf{a})/\sqrt{g_0}. \tag{G.14}$$

Thus the mass continuity equation may be written in the form:

$$\rho = \frac{\rho_0}{J(\mathbf{x},\mathbf{x}_0)} \equiv \frac{\rho_0\sqrt{g_0}}{\sqrt{g}J(\mathbf{q},\mathbf{a})}. \tag{G.15}$$

Following Newcomb (1962) we introduce the derivative transformation matrices:

$$q^i{}_j = \frac{\partial q^i}{\partial a^j}, \quad y^b{}_j = \frac{\partial a^b}{\partial q^j}, \tag{G.16}$$

associated with the transformations $\mathbf{q} \to \mathbf{a}$ and the inverse transformations $\mathbf{a} \to \mathbf{q}$. Note that

$$q^i{}_k y^k{}_j = \delta^i{}_j, \quad y^k{}_j = \frac{A_j{}^k}{J(\mathbf{q},\mathbf{a})}, \tag{G.17}$$

where

$$A_j{}^k = \mathrm{cofac}\left(\frac{\partial q^j}{\partial a^k}\right), \tag{G.18}$$

is the cofactor of the matrix $q^j{}_k$.

The conservation of magnetic flux moving with the flow, is equivalent to Faraday's equation (e.g. Parker 1979; Webb et al. 2014a), which can be written in the form:

$$B_0^\alpha dS_{0\alpha} = B^\beta dS_\beta. \tag{G.19}$$

The area elements $dS_{0\alpha}$ and dS_β are defined by the equations:

$$\begin{aligned} d^3x &= \sqrt{g}d^3q = \sqrt{g}d\sigma_\alpha dq^\alpha = dS_\alpha dq^\alpha, \\ d^3x_0 &= \sqrt{g_0}d^3a = \sqrt{g_0}d\sigma_{0\beta}dq^\beta = dS_{0\beta}da^\beta. \end{aligned} \tag{G.20}$$

Using $d^3x = J(\mathbf{x},\mathbf{x}_0)d^3x_0$, (G.20) gives:

$$dS_{0\beta} = \frac{q^\alpha{}_\beta}{J(\mathbf{x},\mathbf{x}_0)}dS_\alpha \equiv \frac{\sqrt{g_0}}{\sqrt{g}}\frac{q^\alpha{}_\beta}{J(\mathbf{q},\mathbf{a})}dS_\alpha. \tag{G.21}$$

Using (G.19) and (G.21) we obtain:

$$B^\alpha = \frac{\sqrt{g_0}}{\sqrt{g}} \frac{q^\alpha{}_\beta}{J(\mathbf{q}, \mathbf{a})} B_0^\beta \equiv \frac{q^\alpha{}_\beta B_0^\beta}{J(\mathbf{x}, \mathbf{x}_0)}, \tag{G.22}$$

as the relationship between the magnetic field components B^α and B_0^α. Equation (G.22) is equivalent to Faraday's equation.

A simpler derivation of (G.22) follows by noting that $\mathbf{b} = \mathbf{B}/\rho$ is a lie dragged vector field, i.e.

$$b^\alpha \frac{\partial}{\partial q^\alpha} = b_0^\beta \frac{\partial}{\partial a^\beta} \quad \text{or} \quad b^\alpha = b_0^\beta \frac{\partial q^\alpha}{\partial a^\beta}. \tag{G.23}$$

Then using $\mathbf{b} = \mathbf{B}/\rho$ and $\mathbf{b}_0 = \mathbf{B}_0/\rho_0$ the Cauchy invariant relation (G.23) implies the transformation (G.22).

Using (G.22) we obtain:

$$\nabla \cdot \mathbf{B} = \nabla_\alpha B^\alpha = \frac{1}{J(\mathbf{x}, \mathbf{x}_0)} \nabla_{0\alpha} B_0^\alpha, \tag{G.24}$$

where

$$\nabla_\alpha B^\alpha = \frac{\partial B^\alpha}{\partial q^\alpha} + \Gamma^\alpha_{s\alpha} B^s = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\alpha} \left(\sqrt{g} B^\alpha \right), \tag{G.25}$$

$$\frac{\partial \mathbf{e}_\alpha}{\partial q^\beta} = \Gamma^s_{\alpha\beta} \mathbf{e}_s, \quad V^\alpha_{;\beta} = \frac{\partial V^\alpha}{\partial q^\beta} + \Gamma^\alpha_{s\beta} V^s, \tag{G.26}$$

In (G.25) $\nabla \cdot \mathbf{B} = 0$ requires that $\nabla_0 \cdot \mathbf{B}_0 = 0$ for Gauss's law to hold.

From Chap. 10, the MHD action, taking into account the Lagrangian map, has the form:

$$\mathcal{A} = \int d^3x \int dt\, L, \tag{G.27}$$

where

$$L = \frac{1}{2} \rho u^2 - \varepsilon(\rho, S) - \rho \Phi(\mathbf{x}) - \frac{B^2}{2\mu_0}, \tag{G.28}$$

is the Lagrangian. Using generalized coordinates $q^\alpha = q^\alpha(\mathbf{a}, t)$, the action (G.27) can be written in the form:

$$\mathcal{A} = \int \sqrt{g} d^3q \int dt\, L = \int d^3a \int dt\, L_0, \tag{G.29}$$

where

$$L_0 = J(\mathbf{q}, \mathbf{a})\sqrt{g}L \equiv \sqrt{g_0}J(\mathbf{x}, \mathbf{x}_0)\, L. \tag{G.30}$$

Using the mass continuity equation (G.15) and the frozen in field theorem (G.22) we find:

$$L_0 = \sqrt{g_0}\left\{ \rho_0 \left[\frac{1}{2} g_{\alpha\beta} \frac{dq^\alpha}{dt}\frac{dq^\beta}{dt} - U\left(\frac{\rho_0}{J(\mathbf{x}, \mathbf{x}_0)}, S \right) - \Phi(\mathbf{x}) \right] \right.$$
$$\left. - \frac{g_{\mu\nu} q^\mu{}_\alpha q^\nu{}_\beta B_0^\alpha B_0^\beta}{2\mu_0 J(\mathbf{x}, \mathbf{x}_0)} \right\}, \tag{G.31}$$

as the form of the Lagrangian in generalized coordinates, where $U(\rho, S) = \varepsilon(\rho, S)/\rho$ is the internal energy density per unit mass.

The Euler-Lagrange equations for the action (G.29) with Lagrangian (G.31) are:

$$\frac{\delta \mathcal{A}}{\delta q^i} = \frac{\partial L_0}{\partial q^i} - \frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial q^i_t} \right) - \frac{\partial}{\partial a^j}\left(\frac{\partial L_0}{\partial q^i_j} \right) = 0. \tag{G.32}$$

Evaluating the partial derivatives of L_0 in (G.32) we obtain:

$$\frac{\delta \mathcal{A}}{\delta q^i} = -\sqrt{g}J(\mathbf{q}, \mathbf{a})\left\{ \rho g_{i\mu} \left(\frac{dw^\mu}{dt} + \Gamma^\mu_{\alpha\beta} w^\alpha w^\beta \right) \right.$$
$$\left. + \frac{\partial}{\partial q^i}\left(p + \frac{B^2}{2\mu_0} \right) - \frac{\nabla_\alpha \left(B^\alpha B_i \right)}{\mu_0} + \rho \frac{\partial \Phi}{\partial q^i} \right\} = 0, \tag{G.33}$$

where

$$w^\mu = \frac{\partial q^\mu(\mathbf{a}, t)}{\partial t} \equiv \frac{dq^\mu}{dt}, \tag{G.34}$$

is the generalized velocity corresponding to q^μ. The affine connection coefficients $\Gamma^\mu_{\alpha\beta}$ in (G.33) are given by the standard formulae:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right], \tag{G.35}$$

where $g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$ is the metric of (G.5). In (G.33),

$$A^\mu = \frac{dw^\mu}{dt} + \Gamma^\mu_{\alpha\beta} w^\alpha w^\beta, \tag{G.36}$$

is the acceleration vector of the fluid (i.e. $d/dt(w^\mu \mathbf{e}_\mu) = A^\mu \mathbf{e}_\mu$). From (G.33) the covariant form of the momentum equation is:

$$\rho A_i = -\nabla_i \left(p + \frac{B^2}{2\mu_0} \right) + \frac{\nabla_\alpha \left(B^\alpha B_i \right)}{\mu_0} - \rho \nabla_i \Phi, \tag{G.37}$$

where $A_i = g_{i\mu} A^\mu$ is the covariant form of the acceleration vector. It is straightforward to write down the contravariant form of the momentum equation (G.37). The frame independent form of (G.37) is given by:

$$\rho \mathbf{A} = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right) - \rho \nabla \Phi, \tag{G.38}$$

where $\mathbf{A} = A_i \mathbf{e}^i \equiv A^j \mathbf{e}_j$ is the acceleration vector of the fluid.

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