

# Appendix A: Solutions to Selected Exercises

## A.1 Solutions from Chapter 1

1. Using the conversion technique described in Section 1.3, or RPM described in Section 1.1:

$2015 = 2(1007) + 1$	$2015 = 1024$		
$1007 = 2(503) + 1$	$+ 512$	$1536$	
$503 = 2(251) + 1$	$+ 256$	$1792$	
$251 = 2(125) + 1$	$+ 128$	$1920$	
$125 = 2(62) + 1$	$+ 64$	$1984$	
$62 = 2(31) + 0$			
$31 = 2(15) + 1$	$+ 16$	$2000$	
$15 = 2(7) + 1$	$+ 8$	$2008$	
$7 = 2(3) + 1$	$+ 4$	$2012$	
$3 = 2(1) + 1$	$+ 2$	$2014$	
$1 = 2(0) + 1$	$+ 1$	$2015$	

$$2015_{10} = 11\ 111\ 011\ 111_2$$

2. Yes:  $83/2 = 41.5$   
 $83/3 = 27.666\ 666 \dots$   
 $83/5 = 16.6$   
 $83/7 = 11.857\ 142 \dots$

No further values need to be considered because  $11 > \sqrt{83}$  or  $11^2 = 121 > 83$ .

3.  $801 = 3 \cdot 267$   
 $802 = 2 \cdot 401$   
 $803 = 11 \cdot 73$   
 $805 = 5 \cdot 161 = 5 \cdot 7 \cdot 23$   
 $807 = 3 \cdot 269$   
 $809$  is prime (Try 2, 3, 5, 7, 11, 13, 17, 19 and  $23 \dots 29^2 = 841$ )

7. If  $t \leq \lfloor \sqrt{Q} \rfloor$  then  $t \leq \sqrt{Q}$  so  $t \times t \leq Q$  // squaring both sides  
 If  $t$  is an integer and  $t \times t \leq Q$  then  $t \leq \sqrt{Q}$   
 // taking the positive square root of both sides  
 Because  $t$  is some integer  $\leq \sqrt{Q}$ ,  $t$  is less than the largest integer that is  
 $\leq \sqrt{Q}$ ; that is,  $t \leq \lfloor \sqrt{Q} \rfloor$ .  $\square$

8.

$n$	$n(n+1)$	$+ 17$	
0	0(1)	$+ 17 = 17$	which is prime
1	1(2)	$+ 17 = 19$	which is prime
2	2(3)	$+ 17 = 23$	which is prime
3	3(4)	$+ 17 = 29$	which is prime
4	4(5)	$+ 17 = 37$	which is prime
5	5(6)	$+ 17 = 47$	which is prime
6	6(7)	$+ 17 = 59$	which is prime
7	7(8)	$+ 17 = 73$	which is prime
8	8(9)	$+ 17 = 89$	which is prime
9	9(10)	$+ 17 = 107$	which is prime
10	10(11)	$+ 17 = 127$	which is prime
11	11(12)	$+ 17 = 139$	which is prime
12	12(13)	$+ 17 = 163$	which is prime
13	13(14)	$+ 17 = 189$	which is prime
14	14(15)	$+ 17 = 217$	which is prime
15	15(16)	$+ 17 = 247$	which is prime
16	16(17)	$+ 17 = 279$	$= 17 \cdot 17$
17	17(18)	$+ 17 = 313$	$= 17 \cdot 19$
18	18(19)	$+ 17 = 359$	which is prime

14. (a)  $\frac{|A1 - A|}{|A|} = \frac{2.35 - 2.3456}{2.3456} = \frac{0.0044}{2.3456} = 0.0018758\dots \sim 0.19\%$   
 (b)  $\frac{|B1 - B|}{|B|} = \frac{2.3541 - 2.3}{2.3541} = \frac{0.0541}{2.3541} = 0.0229811\dots \sim 2.30\%$   
 (c)  $\frac{|(A1 - B1) - (A - B)|}{|A - B|} = \frac{|0.05 + 0.0085|}{0.0085} = \frac{0.0585}{0.0085} = 6.882352\dots \sim 688.24\%$   
 (d) Because  $A1 > A$  and  $B1 < B$  and  $A$  and  $B$  are similar in value, the two small errors, which are in opposite directions, are large relative to the difference between  $A$  and  $B$ .

17.  $\text{GCD}(2N+1, 3N+1) = \text{GCD}(N, 2N+1) = \text{GCD}(1, N) = 1$ .

## A.2 Solutions from Chapter 2

11. (a)  $9^4 = 6,561$   
 (b)  $(9)(8)(7)(6) = 3,024$   
 (c)  $(1)(8)(7)(6) = 336$   
 (d)  $\binom{9}{4} = 126$   
 (e) Any increasing 4-sequence on  $X$  that begins with 3 is a 3 followed by an increasing 3-sequence on  $\{4..9\}$ , so the number of increasing 4-sequences on  $X$  that begin with 3 is the number of increasing 3-sequences on  $\{4..9\}$ , which is  $\binom{6}{3} = 20$ .
18. Select the 4 letters and then select the 3 digits.

- (a) Choose 2 positions for the 2 T's  
 and choose another letter to place in the left-most free position for a letter  
 and choose another letter to place in the right-most free position for a letter  
 and choose a digit to place in the left-most position for a digit  
 and choose a digit to place in the middle position for a digit.

Then the number of such license plates =  $\binom{4}{2}(25)(25)(10)(10) = 375,000$ .

- (b) The number of 4-letter sequences with at least one "T"  
 = The number of 4-letter sequences minus the number of 4-letter sequences with no T's  
 =  $26^4 - 25^4 = 456,976 - 390,625 = 66,351$   
 The number of 3-digit sequences with at least one "4"  
 = The number of 3-digit sequences minus the number of 3-digit sequences with no 4's  
 =  $10^3 - 9^3 = 1,000 - 729 = 271$   
 Then the number of such license plates =  $(66,351)(271) = 17,981,121$ .

## A.3 Solutions from Chapter 3

10. *Proof.*

$$\text{Since} \quad \lfloor f \times n \rfloor \leq f \times n < \lfloor f \times n \rfloor + 1,$$

$$n - \lfloor f \times n \rfloor \geq n - f \times n > n - \lfloor f \times n \rfloor - 1.$$

Hence  $(n - \lfloor f \times n \rfloor) - 1 < n - f \times n \leq (n - \lfloor f \times n \rfloor)$ . // 2 consecutive integers

so 
$$\lceil n - f \times n \rceil = n - \lfloor f \times n \rfloor$$

and 
$$\lfloor f \times n \rfloor + \lceil (1 - f) \times n \rceil = n. \quad \square$$

11. (a)  $n^2 + n + 41 = n(n+1) + 41$ .

If  $n = 41$  then  $n^2 + n + 41 = 41(41 + 1) + 41 = 41(43)$  which is not prime.

If  $n = 40$  then  $n^2 + n + 41 = 40(40 + 1) + 41 = 41(41)$  which is not prime.

If  $n = 0$  then  $n^2 + n + 41 = 0(1) + 41 = 41 + 0 = 41$  which is prime.

If  $n = 1$  then  $n^2 + n + 41 = 1(2) + 41 = 41 + 2 = 43$  which is prime.

If  $n = 2$  then  $n^2 + n + 41 = 2(3) + 41 = 41 + 6 = 47$  which is prime.

If  $n = 3$  then  $n^2 + n + 41 = 3(4) + 41 = 41 + 12 = 53$  which is prime.

If  $n = 4$  then  $n^2 + n + 41 = 4(5) + 41 = 41 + 20 = 61$  which is prime.

If  $n = 5$  then  $n^2 + n + 41 = 5(6) + 41 = 41 + 30 = 71$  which is prime.

If  $n = 6$  then  $n^2 + n + 41 = 6(7) + 41 = 41 + 42 = 83$  which is prime.

If  $n = 7$  then  $n^2 + n + 41 = 7(8) + 41 = 41 + 56 = 97$  which is prime.

If  $n = 8$  then  $n^2 + n + 41 = 8(9) + 41 = 41 + 72 = 113$  which is prime.

If  $n = 9$  then  $n^2 + n + 41 = 9(10) + 41 = 41 + 90 = 131$  which is prime.

If  $n = 10$  then  $n^2 + n + 41 = 10(11) + 41 = 41 + 110 = 151$  which is prime.

...

If  $n = 39$  then  $n^2 + n + 41 = 39(40) + 41 = 41 + 1560 = 1601$  which is prime.

- (b) Let  $a = \sqrt{2}$  and let  $b = \sqrt{2}$ . Then both  $a$  and  $b$  are irrational // Theorem 3.5.4 but  $a * b = 2$  which is rational.

Or let  $a = \sqrt{2}$  and let  $b = \sqrt{8}$ . Then both  $a$  and  $b$  are irrational

// Theorem 3.5.4 plus the fact that  $b = 2\sqrt{2}$ .

but  $a * b = \sqrt{16} = 4$  which is rational.

- (c) Let  $a = 0$  and let  $b = \sqrt{2}$ . Then  $a$  is rational and  $b$  is irrational but  $a * b = 0$  which is rational. // zero is the only exception

14. Let  $c = \sqrt{2}^{\sqrt{2}}$ .

If  $c$  is rational then take  $a = \sqrt{2}$  and  $b = \sqrt{2}$ . Both are irrational but  $a^b$  is rational.

If  $c$  is irrational then take  $a = c$  and  $b = \sqrt{2}$ . Both are irrational but  $a^b =$

$$\left\{ (\sqrt{2})^{\sqrt{2}} \right\}^{\sqrt{2}} = (\sqrt{2})^2 \text{ which is rational.}$$

15.

P	Q	R	$(P \wedge Q) \rightarrow R$		$P \rightarrow (Q \rightarrow R)$		$[P \wedge (\sim R)] \rightarrow (\sim Q)$			
T	T	T	T	T	T	T	F	F	T	F
T	T	F	T	F	F	F	T	T	F	F
T	F	T	F	T	T	T	F	F	T	T
T	F	F	F	T	T	T	T	T	T	T
F	T	T	F	T	T	T	F	F	T	F
F	T	F	F	T	T	F	F	T	T	F
F	F	T	F	T	T	T	F	T	T	T
F	F	F	F	T	T	T	F	T	T	T
			↑	↑	↑	↑	↑	↑	↑	↑
			1	2	4	3	6	5	8	7

Columns 2, 4 and 8 are identical, so the three Boolean expressions are equivalent.

26. The answer is “yes”.

*Theorem:*  $n^2 < 3^n$  for  $\forall$  integers  $n \in \mathbf{N}$ .

*Proof.* // Here  $a = 0$  and  $P(n)$  is an inequality with a LHS and a RHS

// **but** the induction step is *easy only* after  $k = 2$  (not 0 or 1)

Step 1. If  $n = 0$  then  $n^2 = 0 < 1 = 3^n$ . //  $P(0)$  is True

If  $n = 1$  then  $n^2 = 1 < 3 = 3^n$ . //  $P(1)$  is True

If  $n = 2$  then  $n^2 = 4 < 9 = 3^n$ . //  $P(2)$  is True

Step 2. Assume  $\exists k >= 2$  where  $k^2 < 3^k$ . //  $P(k)$  is True

Step 3. If  $n = k + 1$  then

$$\begin{aligned}
 (k + 1)^2 &= k^2 + 2k + 1 \\
 &< k^2 + k^2 + k^2 && // 1 < 2 <= k < k^2 \\
 &= 3k^2 \\
 &< 3(3^k) = 3^{k+1}. && // \text{by Step 2}
 \end{aligned}$$

□

32. Let  $r$  denote the common ratio of this geometric sequence.

If  $r = 1$  then  $\forall n \in \mathbf{N} S_a + S_{a+1} + S_{a+2} + \dots + S_{a+n} = (n + 1)S_a$ .

If  $r \neq 1$  then  $\forall n \in \mathbf{N} S_a + S_{a+1} + S_{a+2} + \dots + S_{a+n} = S_a \times (r^{n+1} - 1)/(r - 1)$ .

*Proof.*

If  $r = 1$  then each of the  $(n + 1)$  consecutive entries is equal to the first,

so

$$S_a + S_{a+1} + S_{a+2} + \dots + S_{a+n} = (n + 1)S_a.$$

If  $r \neq 1$  then from Exercise 31  $\forall n \in \{a, \dots\} S_n = r^n \times K = r^{n-a} \times S_a$ ,

so

$$\begin{aligned}
 S_a + S_{a+1} + S_{a+2} + \dots + S_{a+n} &= S_a + rS_a + r^2S_a + \dots + r^nS_a \\
 &= S_a(1 + r + r^2 + \dots + r^n) \\
 &= S_a \times (r^{n+1} - 1)/(r - 1). && // \text{see Theorem 3.6.8}
 \end{aligned}$$

□

35. *Proof.* // We will prove (c) by mathematical induction. This also  
// is also a proof of (a) where  $r = 2$  and (b) where  $r = 3$ .

// Here  $a = 0$  and  $P(n)$  is an equation with a LHS and a RHS.

Step 1: If  $n = 0$  then LHS =  $(0 + 1)r^0 = 1$ ,

$$\text{and RHS} = \frac{[(r-1)0 + (r-2)]r^{0+1} + 1}{(r-1)^2} = \frac{(r-2)r + 1}{(r-1)^2} = 1.$$

//  $P(1)$  is True

Step 2: Assume  $\exists k \in \mathbf{N}$  where  $P(k)$  is True.

Step 3: If  $n = k + 1$  then // in the predicate  $P$

$$\begin{aligned} \text{LHS} &= \sum_{j=0}^{k+1} (j+1)r^j = \sum_{j=0}^k (j+1)r^j + ([k+1]+1)r^{k+1} \\ &= \frac{[(r-1)k + (r-2)]r^{k+1} + 1}{(r-1)^2} + (k+2)r^{k+1} \times \frac{(r-1)^2}{(r-1)^2} \quad // \text{by Step 2} \\ &= \frac{\{[rk - k + r - 2] + (k+2)[r^2 - 2r + 1]\}r^{k+1} + 1}{(r-1)^2} \\ &= \frac{\{[r(k+1) - (k+2)] + (k+2)r^2 - (k+2)2r + (k+2)\}r^{k+1} + 1}{(r-1)^2} \\ &= \frac{\{(k+1) + (k+2)r - (2k+4)\}r \times r^{k+1} + 1}{(r-1)^2} \\ &= \frac{\{(k+1)r + r - k - 1 - 2\}r^{k+2} + 1}{(r-1)^2} \\ &= \frac{\{(r-1)(k+1) + (r-2)\}r^{(k+1)+1} + 1}{(r-1)^2} \\ &= \text{RHS} \quad // \text{in the predicate } P \end{aligned}$$

$$// \text{Therefore, } \forall n \in \mathbf{N}, \sum_{j=0}^n (j+1)r^j = \frac{[(r-1)n + (r-2)]r^{n+1} + 1}{(r-1)^2}.$$

$$\begin{aligned} // \text{Therefore, when } r = 2, \forall n \in \mathbf{N}, \sum_{j=0}^n (j+1)2^j &= \frac{[(2-1)n + (2-2)]2^{n+1} + 1}{(2-1)^2} \\ &= n2^{n+1} + 1 \end{aligned}$$

$$\begin{aligned} // \text{and when } r = 3, \forall n \in \mathbf{N}, \sum_{j=0}^n (j+1)3^j &= \frac{[(3-1)n + (3-2)]3^{n+1} + 1}{(3-1)^2} \\ &= \frac{[2n+1]3^{n+1} + 1}{4}. \end{aligned}$$

36. *Proof.*

Step 1: If  $n = q$  then  $LHS = \binom{q}{q} = 1$ , and  $RHS = \binom{q+1}{q+1} = 1 = LHS$ .

Step 2: Assume that  $\exists$  an integer  $k \geq q$  where

$$\binom{q}{q} + \binom{q+1}{q} + \binom{q+2}{q} + \dots + \binom{k}{q} = \binom{k+1}{q+1}$$

Step 3: If  $n = q + 1$  then

$$\begin{aligned} LHS &= \binom{q}{q} + \binom{q+1}{q} + \binom{q+2}{q} + \dots + \binom{k}{q} + \binom{k+1}{q} \\ &= \binom{k+1}{q+1} + \binom{k+1}{q} \quad // \text{from Step 2.} \\ &= \binom{(k+1)+1}{q+1} \quad // \text{the Bad Banana Thm} \\ &= RHS. \quad \square \end{aligned}$$

### A.4 Solutions from Chapter 4

7. The integer,  $k = 2q + r$  where  $q \in \mathbf{Z}$  and  $r \in \{0, 1\}$ .

So  $q \leq \frac{k}{2} = q + \frac{r}{2} \leq q + \frac{1}{2}$  and therefore,  $\lfloor k/2 \rfloor = q$ .

Also,  $q - \frac{1}{2} \leq \frac{k-1}{2} = \frac{(2q+r)-1}{2} = q + \frac{r-1}{2} \leq q$  and therefore,  $\lceil \frac{k-1}{2} \rceil = q$ .  $\square$

8. Let  $k^* = k + 1$ .

$$\begin{aligned} \text{Then } \lceil k/2 \rceil &= \lceil (k^* - 1)/2 \rceil = \lfloor k^*/2 \rfloor \quad // \text{from Q7} \\ &= \lfloor (k+1)/2 \rfloor. \quad \square \end{aligned}$$

### A.5 Solutions from Chapter 5

39. (a) *Proof:* (using the indirect method)

Assume that  $d_{\text{ave}}$ , the average degree in  $G$ , is  $\geq 6$ . Since  $d_{\text{ave}} = (\text{the sum of the degrees of all the vertices})/|V|$ , from Eqn. 5.1.1, we get that  $2|E| = |V| \times d_{\text{ave}}$ . Then  $2|E| \geq |V| \times 6$  and hence  $|V| \leq 2|E|/6 = |E|/3$ .

Let  $d(G)$  be any drawing of  $G$  in the plane. Because,  $G$  has no bridges, every face of  $d(G)$  is bounded by a circuit. Since  $G$  has no loops, every face in  $d(G)$  has more than 1 edge in its boundary; and, since  $G$  has no pair of parallel edges, every face has at least 3 edges. Because every edge is in exactly 2 faces  $2|E| \geq 3|F|$  and hence  $|F| \leq 2|E|/3$ . Then (from Thm. 5.5.2)

$$2 = |V| - |E| + |F| \leq |E|/3 - |E| + 2|E|/3 = 0.$$

Because 2 is not  $\leq 0$ , our assumption must be false, so  $d_{\text{ave}}$  must be  $< 6$ .  $\square$

(b) One example of such a graph is:



40. (a) *Proof:* (using the indirect method)

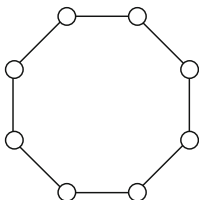
Assume that  $f_{\text{ave}}$ , the average face size in some drawing of  $G$ , is  $\geq 6$ . Since  $f_{\text{ave}} = (\text{the sum of the sizes of all the faces})/|F|$ , from Eqn. 5.5.1, we get that  $2|E| = |F| \times f_{\text{ave}}$ . Then  $2|E| \geq |F| \times 6$  and hence  $|F| \leq 2|E|/6 = |E|/3$ .

Let  $d(G)$  be any drawing of  $G$  in the plane. Because,  $G$  has no bridges, every face of  $d(G)$  is bounded by a circuit. Since  $G$  has no vertex of degree  $< 3$ , from Equation (5.1.1), we get that  $2|E| = |V| \times d_{\text{ave}} \geq |V| \times 3$  and hence  $|V| \leq 2|E|/3$ . Then (from Thm. 5.5.2)

$$2 = |V| - |E| + |F| \leq 2|E|/3 - |E| + |E|/3 = 0.$$

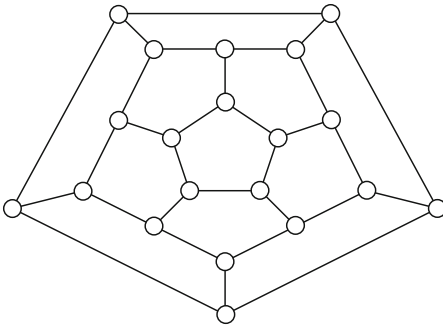
Because 2 is not  $\leq 0$ , our assumption must be false, so  $f_{\text{ave}}$  must be  $< 6$ .  $\square$

(b) One example of such a graph is:

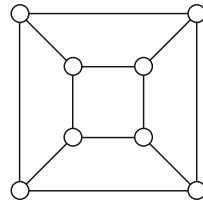




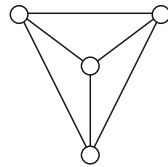
41. (a) Examples of planar drawings are:



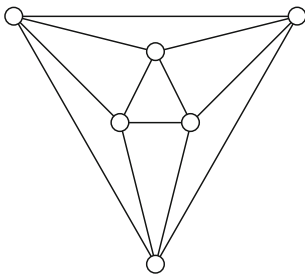
Dodecahedron



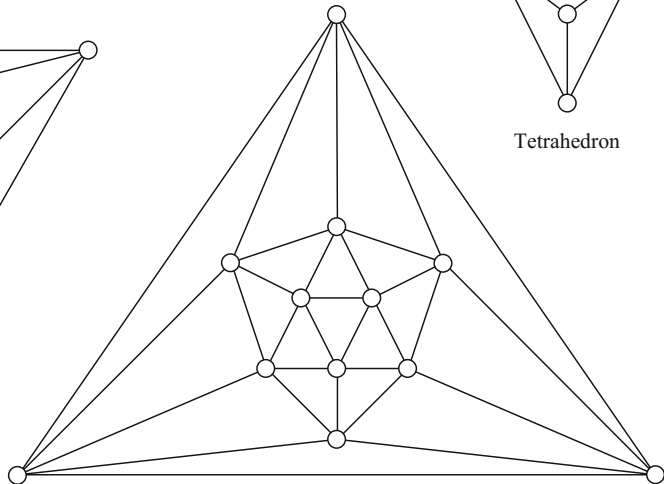
Cube



Tetrahedron



Octahedron



Icosahedron

- (b) The dual of the *tetrahedron* is the *tetrahedron* itself.
- The dual of the *cube* is the *octahedron*.
- The dual of the *octahedron* is the *cube*.
- The dual of the *dodecahedron* is the *icosahedron*.
- The dual of the *icosahedron* is the *dodecahedron*.

(c) *Proof.*

In a 3-dimensional polyhedron, a vertex is a point where 3 or more faces meet, so each vertex has degree  $\geq 3$ . Also, the planar graph of the surface of the polyhedron is bridgeless, connected and simple.

If  $s = 3$  then all of the faces are triangles. The result given in Q39 gives us that  $d$  must be  $< 6$ . The cases where  $s = 3$  and  $d = 3, 4$  and  $5$  are: the tetrahedron, the octahedron, and the icosahedron. No other cases are possible when  $s = 3$ .

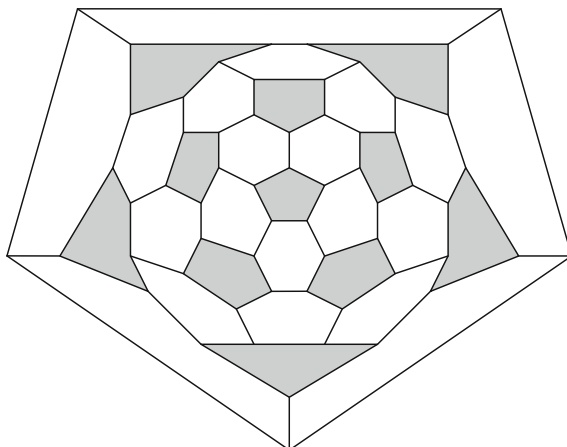
If  $d = 3$  then all vertices have degree 3. The result given in Q40 gives us that  $s$  must be  $< 6$ . The cases where  $d = 3$  and  $s = 3, 4$  and  $5$  are: the tetrahedron, the cube, and the dodecahedron. No other cases are possible when  $d = 3$ .

Assume now that both  $d \geq 4$  and  $s \geq 4$ . From Equation (5.1.1), we get that  $2|E| \geq |V| \times 4$  and hence  $|V| \leq |E|/2$ . From Equation (5.5.1), we get that  $2|E| \geq |F| \times 4$  and hence  $|F| \leq |E|/2$ . Then (from Thm. 5.5.2) we would get

$$2 = |V| - |E| + |F| \leq |E|/2 - |E| + |E|/2 = 0.$$

Because 2 is not  $\leq 0$ , our assumption must be false; the only possible Platonic Solids are the 5 listed. □

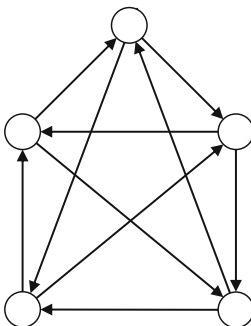
42. (a)



- (b) Each vertex lies on exactly one of the 12 pentagons, so  $|V| = (12)(5) = 60$ . Each vertex has degree 3, so  $2|E| = (60)(3)$ . Thus,  $|E| = 90$ .

### A.6 Solutions from Chapter 6

2. (a) No. See, for example, answer (b).  
 (b)



- (c) Let  $W(x)$  denote the number of wins by player  $x$ , and  $W_{ave}$  denote the average number of wins over all players. Then

$$6W_{ave} = \text{total \# of wins} = \text{total \# of games} = \binom{6}{2} = 15. \text{ So } W_{ave} = 15/6 = 2\frac{1}{2} \text{ and}$$

therefore not all players can have the same number of wins.

- (d) Let  $L(x)$  denote the number of losses by player  $x$ . Then  

$$\sum\{[W(x) - L(x)] : x \in V\} = \sum\{W(x) : x \in V\} - \sum\{L(x) : x \in V\}$$

$$= \text{total \# of games} - \text{total \# of games} = 0.$$
 So  $\sum\{[W(x) - L(x)] : x \in V \setminus \{a\}\} = 0 - [W(a) - L(a)] < 0$  and at least one of the terms in the sum must be negative; i.e. there must be a player  $b$  with more losses than wins.

- (e) Theorem: Every tournament contains a directed Hamilton Path.

*Proof.* Let  $T_n$  denote any particular tournament resulting from an orientation of  $K_n$ .

// We will prove the theorem by Mathematical Induction on  $n$ .

Step 1. If  $n = 1$  then  $V = \{a\}$  and the trivial path  $(a)$  contains all the vertices exactly once and is a directed Hamilton Path in  $T_n$ .

Step 2. Assume  $\exists k \geq 1$  where every tournament with  $k$  players has a directed Hamilton Path.

Step 3. If  $n = k + 1$  and  $T$  is a tournament with  $k + 1$  players, select any player  $w$  and remove  $w$ , and all arcs to and from  $w$ , from  $T$ . What remains is a tournament with  $k$  players,  $T_k$ . By step 2, there is a directed Hamilton Path in  $T_k$ ,

$$\pi = (v_0, e_1, v_1, e_2, v_2, e_3, \dots, e_k, v_k)$$

where each  $e_i$  is an arc oriented from  $v_{i-1}$  to  $v_i$ . Denote by  $\alpha_i$  the arc joining  $w$  and  $v_i$  if it is oriented from  $v_i$  to  $w$ , and by  $\beta_i$  if it is oriented from  $w$  to  $v_i$ .

If the edge joining  $w$  and  $v_k$  is oriented from  $v_k$  to  $w$ , then

$$\pi_1 = (v_0, e_1, v_1, e_2, v_2, e_3, \dots, e_k, v_k, \alpha_k, w)$$

is a directed Hamilton Path in  $T$ .

Otherwise, the edge joining  $w$  and  $v_k$  is oriented from  $w$  to  $v_k$ . Now, let  $q$  be the smallest index where the edge joining  $w$  and  $v_q$  is oriented from  $w$  to  $v_q$ . Then  $0 \leq q \leq k$ .

If  $q = 0$  then

$$\pi_2 = (w, \beta_0, v_0, e_1, v_1, e_2, v_2, e_3, \dots, e_k, v_k)$$

is a directed Hamilton Path in  $T$ .

If  $q > 0$  then the edge joining  $w$  and  $v_{q-1}$  is oriented from  $v_{q-1}$  to  $w$ , and

$$\pi_3 = (v_0, e_1, v_1, e_2, \dots, v_{q-1}, \alpha_{q-1}, w, \beta_q, v_q, \dots, e_k, v_k)$$

is a directed Hamilton Path in  $T$ . □

6. There is only one topological sorting of an acyclic digraph  $D$  if and only if  $D$  contains a directed Hamilton Path.
7. There are many acyclic orientations – first assign the integers from 1 to  $|V|$  to the vertices, then orient the edges so all arcs go from a lower integer to a higher one.
11. First topologically sort the vertices.  
Algorithm 6.3.2 yields that the number of dipaths from  $a$  to  $b$  is 46.  
Algorithm 6.3.3 yields that the length of a shortest dipath from  $a$  to  $b$  is 18.  
Algorithm 6.3.4 yields that the length of a longest dipath from  $a$  to  $b$  is 51.
17. The Ford-Fulkerson Algorithm yields a Max-Flow with value 24.
20. (a) In the construction of the flow network  $N(G)$  where the vertices of  $G$  are partitioned into two independent sets,  $L$  and  $R$ : give the arc  $(s, v)$  from the new source  $s$  to vertex  $v$  in  $L$  a capacity equal  $b(v)$ , give the arc  $(v, t)$  to the new sink  $t$  from a vertex  $v$  in  $R$  a capacity equal  $b(v)$ , and give all other arcs a capacity of one unit.  
Then any  $b$ -matching  $M$  in  $G$  of size  $k$ , corresponds to a feasible flow  $F_M$  in  $N(G)$  with value  $k$ ; and, any feasible flow  $F$  in  $N(G)$  with value  $k$  corresponds to a  $b$ -matching  $M_F$  in  $G$  of size  $k$ .
  - (b) // A maximum cardinality 1-matching in  $G$  has size 8.  
A maximum cardinality 2-matching in  $G$  has size 16.
  - (c) A maximum cardinality 3-matching in  $G$  has size 22.

## A.7 Solutions from Chapter 7

6. (a) 7 !R 7  
(b) 6 R 4 and 4 R 5 but 6 !R 5  
(c) 6 R 4 but 4 !R 6  
(d) 3 R 5 and 5 R 3 but 3  $\neq$  5
8. R is reflexive because the trivial path  $\pi = (v)$  joins any vertex  $v$  to itself.  
R is symmetric because if a path  $\pi$  joins any vertex  $v$  to another vertex  $w$ , then the path  $\pi^R$  joins  $w$  to  $v$ .  
R is transitive because if  $\pi_1$  joins vertex  $v$  to vertex  $w$ , and  $\pi_2$  joins  $w$  to another vertex  $x$ , then the concatenation of  $\pi_1$  and  $\pi_2$  joins vertex  $v$  to vertex  $x$ .  
R is an equivalence relation. The equivalence classes of vertices are the vertices in the connected components.
9. R is reflexive because the trivial dipath  $\pi = (v)$  joins any vertex  $v$  to itself.  
R is transitive because if  $\pi_1$  joins vertex  $v$  to vertex  $w$ , and  $\pi_2$  joins  $w$  to another vertex  $x$ , then the concatenation of  $\pi_1$  and  $\pi_2$  joins vertex  $v$  to vertex  $x$ .  
If  $D$  has no cycles, what properties does R have? Recall that a relation  $R$  on  $S$  is anti-symmetric means whenever  $a$  and  $b$  are distinct elements of  $S$  if  $a R b$  then  $b \not R a$ . If  $a$  and  $b$  are distinct vertices, and  $a R b$  and  $b R a$  then there is a dipath  $\pi_1$  from vertex  $a$  to vertex  $b$  and there is a dipath  $\pi_2$  from vertex  $a$  to vertex  $b$ .

Hence there is a closed dipath  $\pi$  equal the concatenation of  $\pi_1$  and  $\pi_2$ .

Then there must be a cycle in  $\mathbf{D}$ . // by Thm 6.1.1

Thus, if  $\mathbf{D}$  has no cycles,  $R$  is (also) anti-symmetric.

Because  $R$  is reflexive, transitive and anti-symmetric it is a Partial Order.

29. (a)  $f \sim g$  and  $A \in \mathbf{R}^+$   
 $\Rightarrow f \ll g$  and  $A \in \mathbf{R}^+ \Rightarrow Af \ll g$  // by #3 of Thm 7.5.3  
 $A \in \mathbf{R}^+ \Rightarrow f \ll Af$  // by #2 of Thm 7.5.3  
 $\Rightarrow g \ll f$  and  $f \ll Af \Rightarrow g \ll Af$  // by Thm 7.5.1 □  
 $\Rightarrow Af \sim g$
- (b)  $g \ll (g + f)$  // by #2 of Thm 7.5.3  
 $f \ll g \Rightarrow (f + g) \ll g$  // by #4 of Thm 7.5.3 □  
 $\Rightarrow (f + g) \sim g$
- (c)  $f_1 \ll g$  and  $f_2 \sim g$   
 $\Rightarrow f_1 \ll g$  and  $f_2 \ll g \Rightarrow (f_1 + f_2) \ll g$  // by #5 of Thm 7.5.3  
 $f_2 \ll (f_2 + f_1) = (f_1 + f_2)$  // by #1 of Thm 7.5.3  
 $g \ll f_2$  and  $f_2 \ll (f_2 + f_1) \Rightarrow g \ll (f_2 + f_1)$   
 $\Rightarrow (f_1 + f_2) \sim g$  □
- (d)  $f_1 \ll g_1$  and  $f_2 \ll g_2 \Rightarrow (f_1 + f_2) \ll (g_1 + g_2)$  // by #6 of Thm 7.5.3  
 $g_1 \ll f_1$  and  $g_2 \ll f_2 \Rightarrow (g_1 + g_2) \ll (f_1 + f_2)$  // by #6 of Thm 7.5.3  
 $\Rightarrow (f_1 + f_2) \sim (g_1 + g_2)$
- (e)  $f_1 \ll g_1$  and  $f_2 \ll g_2 \Rightarrow (f_1 \times f_2) \ll (g_1 \times g_2)$  // by #7 of Thm 7.5.3  
 $g_1 \ll f_1$  and  $g_2 \ll f_2 \Rightarrow (g_1 \times g_2) \ll (f_1 \times f_2)$  // by #7 of Thm 7.5.3  
 $\Rightarrow (f_1 \times f_2) \sim (g_1 \times g_2)$  □

30. Let  $K = 1$  and let  $M = 1$ . If  $n \geq M$  then  
 $(f_1 + f_2) = n + n^2 \leq n^3 + 2n^2 = (g_1 + g_2)$ ,  
 so  $(f_1 + f_2) = n + n^2 \ll n^3 + 2n^2 = (g_1 + g_2)$ .

Suppose that  $K \in \mathbf{R}^+$  and  $M \in \mathbf{P}$ . If  $n^* = M + \lceil K \rceil$ , then  $n^* \in \mathbf{P}$ ,  $n^* \geq M$ ,  $n^* \geq K$ , and when  $n$  takes the value  $n^*$

$$(g_1 + g_2) = n^3 + 2n^2 = n(n^2 + 2n) \geq K(n^2 + 2n) > K(n + n^2) = K(f_1 + f_2).$$

So  $(g_1 + g_2) = n^3 + 2n^2 \not\ll n + n^2 = (f_1 + f_2)$ .

Thus  $(f_1 + f_2) = n + n^2 \lll n^3 + 2n^2 = (g_1 + g_2)$   
 and  $(f_1 + f_2) = n + n^2 \not\sim n^3 + 2n^2 = (g_1 + g_2)$ . □

31. Proof.

Anti-symmetry: Suppose that  $f \neq g$ .

$$f \mathcal{SD} g \Rightarrow f \lll g \quad // \text{Thm. 7.5.7}$$

$$\Rightarrow g \lll f$$

$$\Rightarrow g \lll f \quad // \text{Thm. 7.5.5}$$

$$\Rightarrow g \mathcal{SD} f \quad // \text{contra-positive of Thm. 7.5.7}$$

Transitivity: Suppose that  $f \mathcal{SD} g$  and  $g \mathcal{SD} h$ . // Is  $f \mathcal{SD} h$ ?

$$\forall K_1 \in \mathbf{R}^+, \exists M_1(K_1) \in \mathbf{P} \text{ such that if } n > M_1(K_1) \text{ then } K_1 \times f(n) < g(n).$$

If  $K_1=1$ ,  $\exists M_1(1) \in \mathbf{P}$  such that if  $n > M_1(1)$  then  $1 \times f(n) < g(n)$ .

$$\forall K_2 \in \mathbf{R}^+, \exists M_2(K_2) \in \mathbf{P} \text{ such that if } n > M_2(K_2) \text{ then } K_2 \times g(n) < h(n).$$

Let  $K$  be any element of  $\mathbf{R}^+$ , and let  $M(K) = M_1(1) + M_2(K)$  // then  $M(K) \in \mathbf{P}$

If  $n > M(K)$  then

$$n > M_1(1) \text{ so } f(n) < g(n) \text{ and } K \times f(n) < K \times g(n),$$

$$\text{also } n > M_2(K) \text{ so } K \times g(n) < h(n),$$

$$\text{and hence } K \times f(n) < h(n).$$

Thus  $f \mathcal{SD} h$ . □

32. Tabulating the first few values of these sequences gives:

$n$	$f(n)$	$g(n)$	
1	1	1	$// n = 1 = 2(0) + 1$ so $r = 0$
2	2	4	
3	6	6	
4	24	96	
5	120	120	

For any  $r \in \mathbf{P}$ , we have

$$f(2r) = (2r)!$$

$$< g(2r) = (2r)! (2r)$$

$$< f(2r+1) = (2r)! (2r+1)$$

$$= g(2r+1) = (2r)! (2r+1) \quad // \text{both } f \text{ and } g \text{ are increasing integer sequences}$$

If  $n = 2r$  then  $g(n) = f(n) \times n$  and if  $n = 2r + 1$  then  $f(n) = g(n)$ .

Let  $K = 1$  and let  $M = 1$ . If  $n \geq M$  then  $f(n) \leq K \times g(n)$ . Therefore  $f \lll g$ .

Recall that  $f1 \lll f2$  means  $\forall K \in \mathbf{R}^+$  and  $\forall M \in \mathbf{P}$ ,  $\exists n^* \geq M$  where  $f1(n^*) > K \times f2(n^*)$ .

Let  $K$  be any given positive real number and let  $M$  be any given positive integer.

Let  $r^* = M + \lceil K \rceil$ . //  $r^* \in \mathbf{P}$  and  $r^* > M, K$   
 If  $n^* = 2r^*$  then  $n^* \geq M$  and  $g(n^*) = f(n^*) \times n^* > K \times f(n^*)$  so  $g \not\ll f$ .  
 Therefore  $f \ll\ll g$ .

Recall that  $f1 \mathcal{SD} f2$  means  $\forall K \in \mathbf{R}^+, \exists M(K) \in \mathbf{P}$  such that if  $n > M(K)$  then  $K \times f1(n) < f2(n)$ .

Therefore,  $f1 \mathcal{SD} f2$  means  $\exists K \in \mathbf{R}^+$  such that  $\forall M \in \mathbf{P}, \exists n^* > M$  where  $K \times f1(n^*) \geq f2(n)$ .

Let  $K = 1$  and let  $M$  be any given positive integer.  
 If  $n^* = 2M + 1$  then  $n^* \geq M$  and  $g(n^*) = f(n^*) \geq K \times f(n^*)$ .  
 Therefore  $f \not\mathcal{SD} g$ . □

### Chapter 8

- Suppose  $E_n$  is defined recursively on  $\mathbf{P}$  by

$$E_0 = 0, E_1 = 2, \text{ and } E_{n+1} = 2n\{E_n + E_{n-1}\} \text{ for all } n \geq 1.$$

Determine the value of  $E_{10}$ .

$E_2 = E_{1+1} = 2(1)\{E_1 + E_0\} =$	$2\{2 + 0\} =$	$4$
$E_3 = E_{2+1} = 2(2)\{E_2 + E_1\} =$	$4\{4 + 2\} =$	$24$
$E_4 = E_{3+1} = 2(3)\{E_3 + E_2\} =$	$6\{24 + 4\} =$	$168$
$E_5 = E_{4+1} = 2(4)\{E_4 + E_3\} =$	$8\{168 + 24\} =$	$1536$
$E_6 = E_{5+1} = 2(5)\{E_5 + E_4\} =$	$10\{1536 + 168\} =$	$17040$
$E_7 = E_{6+1} = 2(6)\{E_6 + E_5\} =$	$12\{17040 + 1536\} =$	$222912$
$E_8 = E_{7+1} = 2(7)\{E_7 + E_6\} =$	$14\{222912 + 17040\} =$	$3359328$
$E_9 = E_{8+1} = 2(8)\{E_8 + E_7\} =$	$16\{3359328 + 222912\} =$	$57315840$
$E_{10} = E_{9+1} = 2(9)\{E_9 + E_8\} =$	$18\{57315840 + 3359328\} =$	$1092153024$

- (a) The 15 possible pairings are:
  1.  $x_1$  with  $x_2$  and  $x_3$  with  $x_4$  and  $x_5$  with  $x_6$
  2.  $x_1$  with  $x_2$  and  $x_3$  with  $x_5$  and  $x_4$  with  $x_6$
  3.  $x_1$  with  $x_2$  and  $x_3$  with  $x_6$  and  $x_4$  with  $x_5$
  4.  $x_1$  with  $x_3$  and  $x_2$  with  $x_4$  and  $x_5$  with  $x_6$
  5.  $x_1$  with  $x_3$  and  $x_2$  with  $x_5$  and  $x_4$  with  $x_6$
  6.  $x_1$  with  $x_3$  and  $x_2$  with  $x_6$  and  $x_4$  with  $x_5$
  7.  $x_1$  with  $x_4$  and  $x_2$  with  $x_3$  and  $x_5$  with  $x_6$
  8.  $x_1$  with  $x_4$  and  $x_2$  with  $x_5$  and  $x_3$  with  $x_6$
  9.  $x_1$  with  $x_4$  and  $x_2$  with  $x_6$  and  $x_3$  with  $x_5$
  10.  $x_1$  with  $x_5$  and  $x_2$  with  $x_3$  and  $x_4$  with  $x_6$
  11.  $x_1$  with  $x_5$  and  $x_2$  with  $x_4$  and  $x_3$  with  $x_6$

12.  $x_1$  with  $x_5$  and  $x_2$  with  $x_6$  and  $x_3$  with  $x_4$   
 13.  $x_1$  with  $x_6$  and  $x_2$  with  $x_3$  and  $x_4$  with  $x_5$   
 14.  $x_1$  with  $x_6$  and  $x_2$  with  $x_4$  and  $x_3$  with  $x_5$   
 15.  $x_1$  with  $x_6$  and  $x_2$  with  $x_5$  and  $x_3$  with  $x_4$  // So  $P_3 = 15$
- (b) Suppose  $n \geq 2$ . Element  $x_1$  may be paired with any of the  $(2n - 1)$  other elements in  $A$ . This leaves  $(2n - 2) = 2(n - 1)$  elements still to be paired, and that can be done in  $P_{n-1}$  ways. Thus the number of pairings of  $2n$  elements,  $P_n = (2n - 1) \times P_{n-1}$ .
- (c) Theorem:  $P_n = (1)(3)(5) \dots (2n - 1)$   
 // product of the first  $n$  positive odd integers

*Proof.* // by mathematical induction

Step 1.  $P_1 = 1$  which is the first positive odd integer.

Step 2. Assume  $\exists k \geq 1$  where  $P_k = (1)(3)(5) \dots (2k - 1)$ .

Step 3. If  $n = k + 1$  then  $n \geq 2$  and

$$\begin{aligned} P_{k+1} &= (2[k+1]-1)P_k && // \text{using the RE} \\ &= (2k+1) \times (1)(3)(5) \dots (2k-1) && // \text{by Step 2} \\ &= (1)(3)(5) \dots (2k-1) \times (2[k+1]-1) && \square \end{aligned}$$

$$8. y_{n+1} = \frac{[n+1]([n+1]-1)}{2} + c = \frac{[n+1](n)}{2} + c = \frac{n(n-1)+2n}{2} + c = y_n + n.$$

9. (a)  $f(1) = 11$ ,  $f(2) = 23$ ,  $f(3) = 47$ ,  $f(4) = 95$ ,  $f(5) = 191$ ,  
 $f(6) = 383$ ,  $f(7) = 767$ ,  $f(8) = 1535$ ,  $f(9) = 3071$ ,  $f(10) = 6143$ .
- (b)  $f(1) - f(0) = 6$  but  $f(2) - f(1) = 12$  so  $f$  is not an arithmetic sequence.  
 $f(1) / f(0) = 11/5 = 121/55$  but  $f(2) / f(1) = 23/11 = 115/55$  so  $f$  is not an geometric sequence.

$$\begin{aligned} 11. (a) \quad s_1 &= (1/5)s_0 - 8 = (1/5)(60) & -8 &= 12 & -8 &= +4 \\ s_2 &= (1/5)s_1 - 8 = (1/5)(4) & -8 &= 0.8 & -8 &= -7.2 \\ s_3 &= (1/5)s_2 - 8 = (1/5)(-7.2) & -8 &= -1.44 & -8 &= -9.44 \end{aligned}$$

- (b) In this RE,  $a = 1/5$  and  $c = -8$  so  $\frac{c}{1-a} = \frac{-8}{4/5} = -10$  and we have  $s_0 = 60$ .

$$\begin{aligned} \text{The particular solution is } s_n &= a^n \left[ I - \frac{c}{1-a} \right] + \frac{c}{1-a} \\ &= (1/5)^n [60 - (-10)] + (-10) \\ &= (1/5)^n [70] - 10. \end{aligned}$$

- (c) Yes. The limit is  $-10$ .  
 (d) No. Because the sequence does not converge to zero.



21.	$n$	$F_n$	$T_n = 1 + F_0 + F_1 + \dots + F_n$
	0	1	2
	1	1	3
	2	2	5
	3	3	8
	4	5	13
	5	8	21

**Theorem** For  $\forall n \in \mathbf{P}$ ,  $T_n = 1 + F_0 + F_1 + \dots + F_n$  equals  $F_{n+2}$ .

*Proof.*

Step 1. If  $n = 0$  then  $T_n = 1 + F_0 = 1 + 1 = 2 = F_2 = F_{0+2}$ .

Step 2. Assume that  $\exists k \geq 0$  such that  $T_k$  equals  $F_{k+2}$ .

Step 3. If  $n = k + 1$  then  $n \geq 1$  so

$$\begin{aligned}
 T_{k+1} &= 1 + F_0 + F_1 + \dots + F_k + F_{k+1} \\
 &= T_k + F_{k+1} \\
 &= F_{k+2} + F_{k+1} \quad // \text{by Step 2} \\
 &= F_{k+3} \quad // \text{by the Fibonacci RE} \\
 &= F_{(k+1)+2}. \quad \square
 \end{aligned}$$

23. (a) The General Solution of this Recurrence Equation is

$$\begin{aligned}
 S_n &= A(r)^n + Bn(r)^n \quad // \text{since } r_1 = r_2 = r \\
 &= A(11)^n + Bn(11)^n \quad // \text{since } r = 11
 \end{aligned}$$

(b) Find the Particular Solution where  $S_0 = 1$  and  $S_1 = 5$ .

$$\begin{aligned}
 S_0 = 1 &= A(11)^0 + B(0)(11)^0 = A \\
 S_1 = 5 &= A(11)^1 + B(1)(11)^1 = 11A + 11B
 \end{aligned}$$

Then  $A = 1$   
 $B = [5 - 11(1)]/11 = -6/11$

Hence, the Particular Solution where  $S_0 = 1$  and  $S_1 = 5$  is

$$\begin{aligned}
 S_n &= (1)(11)^n + (-6/11)n(11)^n = 11^n - 6n(11)^{n-1} \\
 // \quad S_0 &= 11^0 - 6(0)(11)^{0-1} = 1 - 0 = 1 \\
 // \text{ and } \quad S_1 &= 11^1 - 6(1)(11)^0 = 11 - 6 = 5.
 \end{aligned}$$

29. (a)  $e^{1/3} = 1.395\ 612\ 425\dots$

(b)	$j$		$\text{term}_j = \frac{1}{j!} \left(\frac{1}{3}\right)^j$	partial sum
	0		1	$= 1.000\ 000\ 000\dots$
	1		$1/3$	$= 1.333\ 333\ 333\dots = 4/3$
	2	$1/(6 \times 3)$	$= 1/18$	$= 0.055\ 555\ 555\dots$
	3	$1/(9 \times 18)$	$= 1/162$	$= 0.006\ 172\ 839\dots$
	4	$1/(12 \times 162)$	$= 1/1944$	$= 0.000\ 514\ 403\dots$
	5	$1/(15 \times 1944)$	$= 1/29160$	$= 0.000\ 034\ 293\dots$
				$1.395\ 612\ 425\dots = 2713/1944$
				$1.395\ 610\ 425\dots = 40696/29160$

(c) Yes.

(d)  $e^{-1/3} = 0.716\ 531\ 310\dots$

	$j$		$\text{term}_j = \frac{1}{j!} \left(\frac{-1}{3}\right)^j$	partial sum
	0		1	$= +1.000\ 000\ 000\dots$
	1		$-1/3$	$= -0.333\ 333\ 333\dots$
	2	$+1/(6 \times 3)$	$= +1/18$	$= +0.055\ 555\ 555\dots$
	3	$-1/(9 \times 18)$	$= -1/162$	$= -0.006\ 172\ 839\dots$
	4	$+1/(12 \times 162)$	$= +1/1944$	$= +0.000\ 514\ 403\dots$
	5	$-1/(15 \times 1944)$	$= -1/29160$	$= -0.000\ 034\ 293\dots$
				$0.666\ 666\ 666\dots = 2/3$
				$0.722\ 222\ 222\dots = 13/18$
				$0.716\ 049\ 382\dots = 116/162$
				$0.716\ 563\ 786\dots = 1393/1944$
				$0.716\ 529\ 492\dots = 20894/29160$

Yes. This partial sum also gives 3 decimal places of accuracy.

## Chapter 9

4. (a)  $[(7)(6)]/[(2)(1)] = 21$

(b) (1, 2, 3, 4, 5)

(c) (3, 4, 5, 6, 7)

(d) (2, 3, 5, 6, 7)

7. The sequences are (read left to right, top to bottom):

321	421	431	432	521	531	532	541	542	543
621	631	632	641	642	643	651	652	653	654

8. (a) Let  $n = 3$ ,  $B = 8$ ,  $V_1 = 10$ ,  $W_1 = 5$ ,  $V_2 = 7$ ,  $W_2 = 4$ ,  $V_3 = 7$ , and  $W_3 = 4$ .

The Greedy Solution is  $\{O_1\}$  with total value = 10 and total weight = 5.

The Optimal Solution is  $\{O_2, O_3\}$  with total value = 14 and total weight = 8.

(b) Let  $n = 3$ ,  $B = 8$ ,  $V_1 = 10$ ,  $W_1 = 5$ ,  $\text{Ratio}_1 = 2$

$$V_2 = 7, W_2 = 4, \text{Ratio}_2 = 1.75$$

$$V_3 = 7, W_3 = 4, \text{Ratio}_3 = 1.75$$

The Greedy Solution is  $\{O_1\}$  with total value = 10 and total weight = 5.

The Optimal Solution is  $\{O_2, O_3\}$  with total value = 14 and total weight = 8.

$$\begin{aligned}
 9. \text{ (a)} \quad \sum_{j=a}^b (x_{j-1} - x_j) &= x_{a-1} - x_a \\
 &+ x_a - x_{a+1} \\
 &+ x_{a+1} - x_{a+2} \\
 &\quad \dots \\
 &+ x_{b-2} - x_{b-1} \\
 &+ x_{b-1} - x_b \\
 &= x_{a-1} - x_b. \quad // \text{ All other } x_j \text{'s "cancel out"}.
 \end{aligned}$$

$$\text{(b)} \quad \sum_{j=a}^b (x_j - x_{j-1}) = - \sum_{j=a}^b (x_{j-1} - x_j) = -(x_{a-1} - x_b) = x_b - x_{a-1}.$$

(c) Let  $y_j = x_{j+1}$  for each index  $j$ . Then

$$\sum_{j=a}^b (x_{j+1} - x_j) = \sum_{j=a}^b (y_j - y_{j-1}) = y_b - y_{a-1} = x_{b+1} - x_a.$$

(d) Let  $x_j = n^j$  for each index  $j$ . Then

$$\begin{aligned}
 \sum_{j=a}^b n^j(n-1) &= \sum_{j=a}^b (n^{j+1} - n^j) = \sum_{j=a}^b (x_{j+1} - x_j) = x_{b+1} - x_a \\
 &= n^{b+1} - n^a.
 \end{aligned}$$

(e) Let  $x_j = n^{k-j}$  for each index  $j$ . Then

$$\begin{aligned}
 \sum_{j=a}^b n^{k-j}(n-1) &= \sum_{j=a}^b (n^{k-j+1} - n^{k-j}) \\
 &= \sum_{j=a}^b (x_{j-1} - x_j) = x_{a-1} - x_b \quad // \text{ from part (a)} \\
 &= n^{k-(a-1)} - n^{k-b} = n^{k-a+1} - n^{k-b}.
 \end{aligned}$$

## Chapter 10

$$3. \text{ (a)} \quad \text{prob}(\mathbf{A}) = 27/80$$

$$\text{prob}(\mathbf{B}) = 11/80$$

$$\text{prob}(\mathbf{A \text{ and } B}) = 4/80$$

$$\text{prob}(\mathbf{A \text{ or } B}) = \text{prob}(\mathbf{A}) + \text{prob}(\mathbf{B}) - \text{prob}(\mathbf{A \text{ and } B})$$

$$= \frac{27}{80} + \frac{11}{80} - \frac{4}{80} = \frac{34}{80}$$

$$\text{prob}(\mathbf{A|B}) = \text{prob}(\mathbf{A \text{ and } B})/\text{prob}(\mathbf{B}) = \frac{4/80}{11/80} = \frac{4}{11}$$

$$\text{prob}(\mathbf{B|A}) = \text{prob}(\mathbf{B \text{ and } A})/\text{prob}(\mathbf{A}) = \frac{4/80}{27/80} = \frac{4}{27}$$

(b) **A** and **B** are NOT mutually exclusive because  $prob(\mathbf{A \text{ and } B}) = 4/80 \neq 0$ .

(c) **A** and **B** are NOT independent because

$$\begin{aligned} prob(\mathbf{A \text{ and } B}) &= 4/80 = 320/6400 \\ &\neq prob(\mathbf{A}) * prob(\mathbf{B}) = (27/80) * (11/80) = 297/6400. \end{aligned}$$

5. (a) # delegations that exclude Mike =  $\binom{n-1}{k}$

$$prob(\text{Mike is excluded}) = \binom{n-1}{k} / \binom{n}{k} = \frac{(n-1)!}{k!(n-k-1)!} \times \frac{k!(n-k)!}{n!} = \frac{n-k}{n}$$

$$prob(\text{Mike is included}) = 1 - \frac{n-k}{n} = \frac{k}{n}$$

Alternatively...

# delegations that include Mike =  $\binom{n-1}{k-1}$

$$prob(\text{Mike is included}) = \binom{n-1}{k-1} / \binom{n}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} \times \frac{k!(n-k)!}{n!} = \frac{k}{n}$$

(b)  $prob(\text{Flora is included}) = \frac{k}{n}$

(c) # delegations that include both Mike and Flora =  $\binom{n-2}{k-2}$

$$\begin{aligned} prob(\text{Mike and Flora are included}) &= \binom{n-2}{k-2} / \binom{n}{k} \\ &= \frac{(n-2)!}{(k-2)!(n-k)!} \times \frac{k!(n-k)!}{n!} \\ &= \frac{k(k-1)}{n(n-1)}. \end{aligned}$$

This equals  $prob(\text{Mike is included}) * prob(\text{Flora is included})$

$$\Leftrightarrow \frac{k(k-1)}{n(n-1)} = \frac{k(k)}{n(n)}$$

$$\Leftrightarrow \frac{k-1}{n-1} = \frac{k}{n}$$

$$\Leftrightarrow n(k-1) = k(n-1)$$

$$\Leftrightarrow nk - n = kn - k$$

$$\Leftrightarrow -n = -k$$

$$\Leftrightarrow k = n$$

Since  $1 < k < n$ , the events “Mike is in” and “Flora is in” are NOT independent.

6.  $prob(\{X \text{ gets H and you get H}\} \text{ or } \{X \text{ gets T and you get T}\})$

$$\begin{aligned}
 &= prob(X \text{ gets H and you get H}) + prob(X \text{ gets T and you get T}) \\
 &= prob(X \text{ gets H}) * prob(\text{you get H}) + prob(X \text{ gets T}) * prob(\text{you get T}) \\
 &= p * 1/2 + (1 - p) * (1 - 1/2) \\
 &= 1/2
 \end{aligned}$$

9. (a)

T =	Prob	# probes	(#probes) × (Prob)
$X_{10}$	0.40	1	0.40
$X_7$	0.22	2	0.44
$X_4$	0.10	3	0.30
$X_9$	0.07	4	0.28
$X_6$	0.06	5	0.30
$X_3$	0.05	6	0.30
$X_1$	0.04	7	0.28
$X_8$	0.03	8	0.24
$X_2$	0.02	9	0.18
$X_5$	<u>0.01</u>	10	<u>0.10</u>
	1.00		2.82 = Expected # probes

(b)

# probes	T =	Prob	(#probes) × (Prob)
1	$X_5$	0.01	0.01
2	$X_2, X_8$	0.05	0.10
3	$X_1, X_3, X_6, X_9$	0.22	0.66
4	$X_4, X_7, X_{10}$	<u>0.72</u>	<u>2.88</u>
		1.00	3.65 = Expected # probes

(c) Yes

10. (a)

	passed test	failed test	
passed course	3230	48	3278
failed course	170	552	722
	3400	600	4000

(b)  $P(\text{failed test}|\text{passed course}) = \frac{48}{3278} = 0.014643075\dots \sim 1.5\%$

(c)  $P(\text{passed test}|\text{failed course}) = \frac{170}{722} = 0.235457063\dots \sim 23.5\%$

13. (a)  $10^4 = 10\ 000$   
 (b)  $10 * 9 * 8 * 7 = 5\ 040$   
 (c) The number of passwords with the digit “5” repeated  $j$  times is  $\binom{4}{j} 9^{4-j}$ .

$j$	$\binom{4}{j} 9^{4-j}$
0	$(1)9^4 = 6\ 561$
1	$(4)9^3 = 2\ 916$
2	$(6)9^2 = 486$
3	$(4)9^1 = 36$
4	$(1)9^0 = 1$
Total:	$10\ 000$

(d) $j$	$j \times \text{prob}(X=j)$	
0	$0 \times 6561/10000 = 0.0000$	
1	$1 \times 2916/10000 = 0.2916$	
2	$2 \times 486/10000 = 0.0972$	
3	$3 \times 36/10000 = 0.0108$	
4	$4 \times 1/10000 = \underline{0.0004}$	
	$E(\# \text{ of } 5\text{'s}) = 0.4000$	// equals $4 \times (1/10)$ // $np$ in some binomial experiment

15. Using a Binomial experiment as a model for this process

where a trial is a laptop purchase  
 a “success” is making a claim on the guarantee  
 // we’re interested in the number of claims made,  $X$   
 so  $n = 43, p = 1 - 85\% = 15\% = 0.15$  and  $q = 0.85$   
 and  $X$  is the number of claims made.

We want  $P(X > 3) = \sum_{k=3}^{43} P(X = k)$  // a very long calculation

$= 1 - \sum_{k=0}^2 P(X = k)$  // using the hint

$P(X = 0) = \binom{43}{0} p^0 \times q^{43} = 1 \times 1 \times q^{43} = (1)(1)(0.85)^{43} = 0.000\ 922\ 600 \dots$

$P(X = 1) = \binom{43}{1} p^1 \times q^{42} = 43 \times p \times q^{42} = (43)(0.15)(0.85)^{42} = 0.007\ 000\ 911 \dots$

$P(X = 2) = \binom{43}{2} p^2 \times q^{41} = \frac{43 \times 42}{2 \times 1} (0.15)^2 (0.85)^{41}$   
 $= (903)(0.0225)(0.85)^{41} = 0.025\ 944\ 554 \dots$

Hence,  $P(X < 3) = 0.000\ 922\ 600 \dots + 0.007\ 000\ 911 \dots + 0.025\ 944\ 554 \dots$   
 $= 0.033\ 868\ 065 \dots$

and  $P(X \geq 3) = 1 - 0.033\ 868\ 065\dots = 0.966\ 131\ 934\dots$

// Also,  $\binom{43}{0}p^0 \times q^{43} + \binom{43}{1}p^1 \times q^{42} + \binom{43}{2}p^2 \times q^{41}$   
 //  $= \{1 \times 1 \times q^{43} + 43 \times p \times q^{42} + 903 \times p^2 \times q^{41}\}$   
 //  $= \{0.7225 + 5.4825 + 20.3175\} \times q^{41}$   
 //  $= \{26.5225\} \times (0.001276956\dots) = 0.033\ 868\ 067\dots$   
 // We should expect about  $np = 43 \times (0.15) = 6.45$  claims.

17. (a) 38%

- (b) Using a Binomial experiment as a model for this where the trials correspond to the top 10 marks a “success” is a woman obtaining a top ten mark // we’re interested in the number of women that obtain a top ten mark,  $X$  we assume that  $p = 38\%$ , so  $q = 62\%$ .

$$P(X = 6) = \binom{10}{6} p^6 \times q^4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} (0.38)^6 (0.62)^4$$

$$= (210)(0.003010936\dots)(0.14776336)$$

$$= 0.093\ 430\ 276\dots$$

(c)  $E(X) = np = (10)(0.38) = 3.8$

23. The possible outcomes from the game are summarized in the following table, with the amount won shown in brackets:

	Diamond	Spade, Heart or Club
Ace	1 (\$10.00)	3 (\$5.00)
Jack, Queen or King	3 (\$7.00)	9 (\$4.50)
2 through 10	9 (\$4.50)	27 (\$0.00)

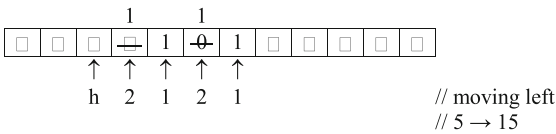
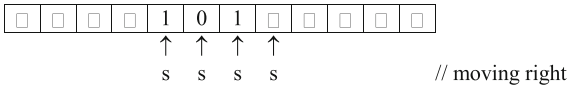
Probability Distribution for  $f$

$v$	$prob(f = v)$	$v \times prob(f = v)$
-\$2.50	27/52	-\$67.50/52
\$2.00	18/52	\$36.00/52
\$2.50	3/52	\$7.50/52
\$4.50	3/52	\$13.50/52
\$7.50	<u>1/52</u>	<u>\$7.50/52</u>
	52/52	$E(f) = -\$ 3.00/52$
		$= -\$ 0.057692307\dots$
		$= -5.7692307\dots\text{cents}$

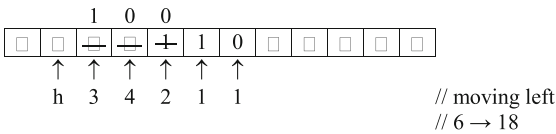
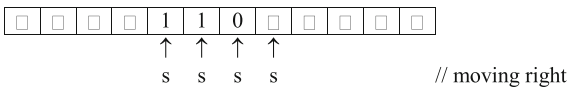
The average payoff per play is negative because the person operating the game (the “house”) needs to cover the cost of operating the game, and wants to make a profit from it. The money needed for these costs and profits comes from having a negative average payoff per play.

### Chapter 11

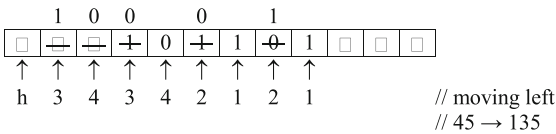
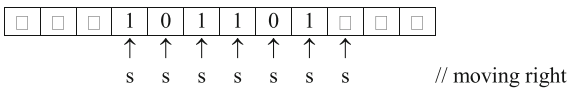
1. (a) (i)



(ii)



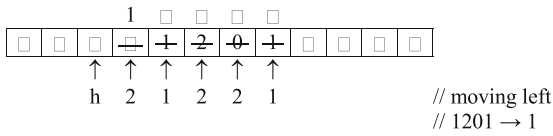
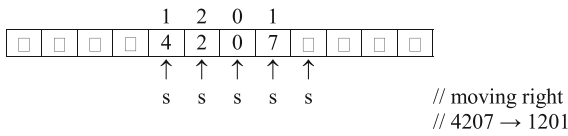
(iii)



(b)  $n \rightarrow 3n$



2. (a)



(b)  $n \rightarrow n \text{ MOD } 3$

4. Let  $L$  denote the “language” of strings of  $a$ ’s and  $b$ ’s described. Then  $L = \{a^n b^n : n \in \mathbf{N}\}$ . Furthermore,  $w \in L$  if and only if  $w$  is the empty string or  $w = axb$  where  $x \in L$ .

- |                   |                 |                  |                 |
|-------------------|-----------------|------------------|-----------------|
| (s, a: □, R, 1)   | (1, a: a, R, 1) | (2, a: □, L, No) | (3, a: a, L, 3) |
| (s, b: □, R, No)  | (1, b: b, R, 1) | (2, b: □, L, 3)  | (3, b: b, L, 3) |
| (s, □: □, R, Yes) | (1, □: □, L, 2) | (2, □: □, L, No) | (3, □: □, R, s) |

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