Appendix A: Solutions to Selected Exercises

A.1 Solutions from Chapter 1

1. Using the conversion technique described in Section 1.3, or RPM described in Section 1.1:

$2015 = 2\,(1007) + 1$	$2015 =$	1024	
$1007 = 2\,(503) + 1$		$+\,512$	1536
$503 = 2\,(251) + 1$		$+\,256$	1792
$251 = 2\,(125) + 1$		$+\,128$	1920
$125 = 2\,(62) + 1$		$+\,64$	1984
$62 = 2\,(31) + 0$			
$31 = 2\,(15) + 1$		$+\,16$	2000
$15 = 2\,(7) + 1$		$+\,8$	2008
$7 = 2\,(3) + 1$		$+\,4$	2012
$3 = 2\,(1) + 1$		$+\,2$	2014
$1 = 2\,(0) + 1$		$+\,1$	2015

   $2015\{10\} = 11\ 111\ 011\ 111\{2\}$

2. Yes: $83/2 = 41.5$
   $83/3 = 27.666\ 666\ \ldots$
   $83/5 = 16.6$
   $83/7 = 11.857\ 142\ \ldots$
   No further values need to be considered because $11 > \sqrt{83}$ or $11^2 = 121 > 83$.

3. $801 = 3*267$
   $802 = 2*401$
   $803 = 11*73$
   $805 = 5*161 = 5*7*23$
   $807 = 3*269$
   809 is prime (Try 2, 3, 5, 7, 11, 13, 17, 19 and 23… $29^2 = 841$)

T. Jenkyns and B. Stephenson, *Fundamentals of Discrete Math for Computer Science: A Problem-Solving Primer*, Undergraduate Topics in Computer Science,
https://doi.org/10.1007/978-3-319-70151-6

7. If $t <= \lfloor\sqrt{Q}\rfloor$ then $t <= \sqrt{Q}$ so $t \times t <= Q$ // squaring both sides
If t is an integer and $t \times t <= Q$ then $t <= \sqrt{Q}$
// taking the positive square root of both sides
Because t is some integer $<= \sqrt{Q}$, t is less than the largest integer that is $<= \sqrt{Q}$; that is, $t <= \lfloor\sqrt{Q}\rfloor$. ▯

8.

n	$n(n+1)$	$+\ 17$	
0	0(1)	+ 17 = 17	which is prime
1	1(2)	+ 17 = 19	which is prime
2	2(3)	+ 17 = 23	which is prime
3	3(4)	+ 17 = 29	which is prime
4	4(5)	+ 17 = 37	which is prime
5	5(6)	+ 17 = 47	which is prime
6	6(7)	+ 17 = 59	which is prime
7	7(8)	+ 17 = 73	which is prime
8	8(9)	+ 17 = 89	which is prime
9	9(10)	+ 17 = 107	which is prime
10	10(11)	+ 17 = 127	which is prime
11	11(12)	+ 17 = 139	which is prime
12	12(13)	+ 17 = 163	which is prime
13	13(14)	+ 17 = 189	which is prime
14	14(15)	+ 17 = 217	which is prime
15	15(16)	+ 17 = 247	which is prime
16	16(17)	+ 17 = 279	= 17*17
17	17(18)	+ 17 = 313	= 17*19
18	18(19)	+ 17 = 359	which is prime

14. (a) $\dfrac{|A1 - A|}{|A|} = \dfrac{2.35 - 2.3456}{2.3456} = \dfrac{0.0044}{2.3456} = 0.0018758\ldots \sim 0.19\%$

(b) $\dfrac{|B1 - B|}{|B|} = \dfrac{2.3541 - 2.3}{2.3541} = \dfrac{0.0541}{2.3541} = 0.0229811\ldots \sim 2.30\%$

(c) $\dfrac{|(A1 - B1) - (A - B)|}{|A - B|} = \dfrac{|0.05 + 0.0085|}{0.0085} = \dfrac{0.0585}{0.0085} = 6.882352\ldots \sim 688.24\%$

(d) Because A1 > A and B1 < B and A and B are similar in value, the two small errors, which are in opposite directions, are large relative to the difference between A and B.

17. GCD(2N+1, 3N+1) = GCD(N, 2N+1) = GCD(1, N) = 1.

A.2 Solutions from Chapter 2

11. (a) $9^4 = 6{,}561$
    (b) $(9)(8)(7)(6) = 3{,}024$
    (c) $(1)(8)(7)(6) = 336$
    (d) $\binom{9}{4} = 126$
    (e) Any increasing 4-sequence on X that begins with 3 is a 3 followed by an increasing 3-sequence on {4..9}, so the number of increasing 4-sequences on X that begin with 3 is the number of increasing 3-sequences on {4..9}, which is $\binom{6}{3} = 20$.

18. Select the 4 letters and then select the 3 digits.
    (a) Choose 2 positions for the 2 T's
    and choose another letter to place in the left-most free position for a letter
    and choose another letter to place in the right-most free position for a letter
    and choose a digit to place in the left-most position for a digit
    and choose a digit to place in the middle position for a digit.
    Then the number of such license plates $= \binom{4}{2}(25)(25)(10)(10) = 375{,}000$.
    (b) The number of 4-letter sequences with at least one "T"
    = The number of 4-letter sequences minus the number of 4-letter sequences with no T's
    $= 26^4 - 25^4 = 456{,}976 - 390{,}625 = 66{,}351$
    The number of 3-digit sequences with at least one "4"
    = The number of 3-digit sequences minus the number of 3-digit sequences with no 4's
    $= 10^3 - 9^3 = 1{,}000 - 729 = 271$
    Then the number of such license plates $= (66{,}351)(271) = 17{,}981{,}121$.

A.3 Solutions from Chapter 3

10. *Proof.*

Since $\lfloor f \times n \rfloor <= f \times n < \lfloor f \times n \rfloor + 1$,

$$n - \lfloor f \times n \rfloor >= n - f \times n > n - \lfloor f \times n \rfloor - 1.$$

Hence $(n - \lfloor f \times n \rfloor) - 1 < n - f \times n <= (n - \lfloor f \times n \rfloor)$. // 2 consecutive integers

so $\lceil n - f \times n \rceil = n - \lfloor f \times n \rfloor$

and $\lfloor f \times n \rfloor + \lceil (1 - f) \times n \rceil = n.$ ∎

11. (a) $n^2 + n + 41 = n(n+1) + 41.$

If $n = 41$ then $n^2 + n + 41 = 41(41 + 1) + 41 = 41(43)$ which is not prime.
If $n = 40$ then $n^2 + n + 41 = 40(40 + 1) + 41 = 41(41)$ which is not prime.

If $n = 0$ then $n^2 + n + 41 = 0(1) + 41 = 41 + 0 = 41$ which is prime.
If $n = 1$ then $n^2 + n + 41 = 1(2) + 41 = 41 + 2 = 43$ which is prime.
If $n = 2$ then $n^2 + n + 41 = 2(3) + 41 = 41 + 6 = 47$ which is prime.
If $n = 3$ then $n^2 + n + 41 = 3(4) + 41 = 41 + 12 = 53$ which is prime.
If $n = 4$ then $n^2 + n + 41 = 4(5) + 41 = 41 + 20 = 61$ which is prime.
If $n = 5$ then $n^2 + n + 41 = 5(6) + 41 = 41 + 30 = 71$ which is prime.
If $n = 6$ then $n^2 + n + 41 = 6(7) + 41 = 41 + 42 = 83$ which is prime.
If $n = 7$ then $n^2 + n + 41 = 7(8) + 41 = 41 + 56 = 97$ which is prime.
If $n = 8$ then $n^2 + n + 41 = 8(9) + 41 = 41 + 72 = 113$ which is prime.
If $n = 9$ then $n^2 + n + 41 = 9(10) + 41 = 41 + 90 = 131$ which is prime.
If $n = 10$ then $n^2 + n + 41 = 10(11) + 41 = 41 + 110 = 151$ which is prime.
…
If $n = 39$ then $n^2 + n + 41 = 39(40) + 41 = 41 + 1560 = 1601$ which is prime.

(b) Let $\boldsymbol{a} = \sqrt{2}$ and let $\boldsymbol{b} = \sqrt{2}$. Then both $\boldsymbol{a}$ and $\boldsymbol{b}$ are irrational // Theorem 3.5.4 but $\boldsymbol{a} * \boldsymbol{b} = 2$ which is rational.
Or let $\boldsymbol{a} = \sqrt{2}$ and let $\boldsymbol{b} = \sqrt{8}$. Then both $\boldsymbol{a}$ and $\boldsymbol{b}$ are irrational
// Theorem 3.5.4 plus the fact that $\boldsymbol{b} = 2\sqrt{2}$.
but $\boldsymbol{a} * \boldsymbol{b} = \sqrt{16} = 4$ which is rational.

(c) Let $\boldsymbol{a} = 0$ and let $\boldsymbol{b} = \sqrt{2}$. Then $\boldsymbol{a}$ is rational and $\boldsymbol{b}$ is irrational but $\boldsymbol{a} * \boldsymbol{b} = 0$ which is rational. // zero is the only exception

14. Let $\boldsymbol{c} = \sqrt{2}^{\sqrt{2}}$.
If $\boldsymbol{c}$ is rational then take $\boldsymbol{a} = \sqrt{2}$ and $\boldsymbol{b} = \sqrt{2}$. Both are irrational but $\boldsymbol{a}^{\boldsymbol{b}}$ is rational.
If $\boldsymbol{c}$ is irrational then take $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \sqrt{2}$. Both are irrational but $\boldsymbol{a}^{\boldsymbol{b}} = \left\{ \left(\sqrt{2}\right)^{\sqrt{2}} \right\}^{\sqrt{2}} = \left(\sqrt{2}\right)^2$ which is rational.

15.

P	Q	R	$(P \wedge Q)$	$\rightarrow R$	$P \rightarrow$	$(Q \rightarrow R)$	$[P$	$\wedge (\sim R)]$	$\rightarrow$	$(\sim Q)$
T	T	T	T	T	T	T	F	F	T	F
T	T	F	T	F	F	F	T	T	F	F
T	F	T	F	T	T	T	F	F	T	T
T	F	F	F	T	T	T	T	T	T	T
F	T	T	F	T	T	T	F	F	T	F
F	T	F	F	T	T	F	F	T	T	F
F	F	T	F	T	T	T	F	F	T	T
F	F	F	F	T	T	T	F	T	T	T
			↑ 1	↑ 2	↑ 4	↑ 3	↑ 6	↑ 5	↑ 8	↑ 7

Columns 2, 4 and 8 are identical, so the three Boolean expressions are equivalent.

26. The answer is "yes".

Theorem: $n^2 < 3^n$ for $\forall$ integers $n \in \mathbf{N}$.
Proof. // Here $a = 0$ and $P(n)$ is an inequality with a LHS and a RHS
// ***but*** the induction step is easy only after $k = 2$ (not 0 or 1)
Step 1. If $n = 0$ then $n^2 = 0 < 1 = 3^n$. // $P(0)$ is True
If $n = 1$ then $n^2 = 1 < 3 = 3^n$. // $P(1)$ is True
If $n = 2$ then $n^2 = 4 < 9 = 3^n$. // $P(2)$ is True
Step 2. Assume $\exists\ k >= 2$ where $k^2 < 3^k$. // $P(k)$ is True
Step 3. If $n = k+1$ then

$$
\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&< k^2 + k^2 + k^2 && // \ 1 < 2 <= k < k^2 \\
&= 3k^2 \\
&< 3(3^k) = 3^{k+1}. && // \text{ by Step2}
\end{aligned}
$$

∎

32. Let r denote the common ratio of this geometric sequence.
If $r = 1$ then $\forall n \in \mathbf{N}\ S_a + S_{a+1} + S_{a+2} + \ldots + S_{a+n} = (n+1)S_a$.
If $r \neq 1$ then $\forall n \in \mathbf{N}\ S_a + S_{a+1} + S_{a+2} + \ldots + S_{a+n} = S_a \times (r^{n+1} - 1)/(r-1)$.

Proof.
If $r = 1$ then each of the $(n+1)$ consecutive entries is equal to the first,
so $S_a + S_{a+1} + S_{a+2} + \ldots + S_{a+n} = (n+1)S_a$.
If $r \neq 1$ then from Exercise 31 $\forall n \in \{a\ldots\}\ S_n = r^n \times K = r^{n-a} \times S_a$,
so

$$
\begin{aligned}
S_a + S_{a+1} + S_{a+2} + \ldots + S_{a+n} &= S_a + rS_a + r^2S_a + \ldots + r^nS_a \\
&= S_a(1 + r + r^2 + \ldots + r^n) \\
&= S_a \times (r^{n+1} - 1)/(r-1). && // \text{ see Theorem 3.6.8}
\end{aligned}
$$

∎

35. *Proof.* // We will prove (c) by mathematical induction. This also // is also a proof of (a) where $r = 2$ and (b) where $r = 3$.

// Here $a = 0$ and $P(n)$ is an equation with a LHS and a RHS.

Step 1: If $n = 0$ then LHS $= (0+1)r^0 = 1$,

$$\text{and} \quad \text{RHS} = \frac{[(r-1)0 + (r-2)]r^{0+1} + 1}{(r-1)^2} = \frac{(r-2)r+1}{(r-1)^2} = 1.$$

// $P(1)$ is True

Step 2: Assume $\exists\ k \in \mathrm{N}$ where $P(k)$ is True.

Step 3: If $n = k+1$ then // in the predicate P

$$\begin{aligned}
\text{LHS} &= \sum_{j=0}^{k+1} (j+1)r^j = \sum_{j=0}^{k} (j+1)r^j + ([k+1]+1)r^{k+1} \\
&= \frac{[(r-1)k + (r-2)]r^{k+1} + 1}{(r-1)^2} + (k+2)r^{k+1} \times \frac{(r-1)^2}{(r-1)^2} && \text{// by Step 2} \\
&= \frac{\{[rk - k + r - 2] + (k+2)[r^2 - 2r + 1]\}r^{k+1} + 1}{(r-1)^2} \\
&= \frac{\{[r(k+1) - (k+2)] + (k+2)r^2 - (k+2)2r + (k+2)\}r^{k+1} + 1}{(r-1)^2} \\
&= \frac{\{(k+1) + (k+2)r - (2k+4)\}r \times r^{k+1} + 1}{(r-1)^2} \\
&= \frac{\{(k+1)r + r - k - 1 - 2\}r^{k+2} + 1}{(r-1)^2} \\
&= \frac{\{(r-1)(k+1) + (r-2)\}r^{(k+1)+1} + 1}{(r-1)^2} \\
&= \text{RHS} && \text{// in the predicate } P
\end{aligned}$$

// Therefore, $\forall\, n \in \mathbf{N},\ \sum_{j=0}^{n} (j+1)r^j = \dfrac{[(r-1)n + (r-2)]r^{n+1} + 1}{(r-1)^2}$.

// Therefore, when r = 2, $\forall n \in \mathbf{N}, \sum_{j=0}^{n} (j+1)2^j = \dfrac{[(2-1)n + (2-2)]2^{n+1} + 1}{(2-1)^2}$

$$= n2^{n+1} + 1$$

// and when r = 3, $\forall\, n \in \mathbf{N}, \sum_{j=0}^{n} (j+1)3^j = \dfrac{[(3-1)n + (3-2)]3^{n+1} + 1}{(3-1)^2}$

$$= \frac{[2n+1]3^{n+1} + 1}{4}.$$

36. *Proof.*

Step 1: If $n = q$ then LHS $= \binom{q}{q} = 1$, and RHS $= \binom{q+1}{q+1} = 1 =$ LHS.

Step 2: Assume that $\exists$ an integer $k >= q$ where

$$\binom{q}{q} + \binom{q+1}{q} + \binom{q+2}{q} + \ldots + \binom{k}{q} = \binom{k+1}{q+1}$$

Step 3: If $n = q + 1$ then

$$\begin{aligned} \text{LHS} &= \binom{q}{q} + \binom{q+1}{q} + \binom{q+2}{q} + \ldots + \binom{k}{q} + \binom{k+1}{q} \\ &= \binom{k+1}{q+1} + \binom{k+1}{q} && \text{// from Step 2.} \\ &= \binom{(k+1)+1}{q+1} && \text{// the Bad Banana Thm} \\ &= \text{RHS.} \end{aligned}$$

∎

A.4 Solutions from Chapter 4

7. The integer, $k = 2q + r$ where $q \in \mathbf{Z}$ and $r \in \{0, 1\}$.
   So $q <= \frac{k}{2} = q + \frac{r}{2} <= q + \frac{1}{2}$ and therefore, $\lfloor k/2 \rfloor = q$.
   Also, $q - \frac{1}{2} <= \frac{k-1}{2} = \frac{(2q+r)-1}{2} = q + \frac{r-1}{2} <= q$ and therefore, $\lceil \frac{k-1}{2} \rceil = q$. ∎

8. Let $k^* = k + 1$.
   Then $\lceil k/2 \rceil = \lceil (k^* - 1)/2 \rceil = \lfloor k^*/2 \rfloor$ // from Q7
   $= \lfloor (k+1)/2 \rfloor$. ∎

A.5 Solutions from Chapter 5

39. (a) *Proof*: (using the indirect method)

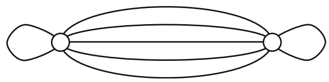
Assume that $\boldsymbol{d}_{\mathbf{ave}}$, the average degree in $\boldsymbol{G}$, is $>= 6$. Since $\boldsymbol{d}_{\mathbf{ave}} =$ (the sum of the degrees of all the vertices)/$|\boldsymbol{V}|$, from Eqn. 5.1.1, we get that $2|\boldsymbol{E}| = |\boldsymbol{V}| \times \boldsymbol{d}_{\mathbf{ave}}$. Then $2|\boldsymbol{E}| >= |\boldsymbol{V}| \times 6$ and hence $|\boldsymbol{V}| <= 2|\boldsymbol{E}|/6 = |\boldsymbol{E}|/3$.

Let $d(\boldsymbol{G})$ be any drawing of $\boldsymbol{G}$ in the plane. Because, $\boldsymbol{G}$ has no bridges, every face of $d(\boldsymbol{G})$ is bounded by a circuit. Since $\boldsymbol{G}$ has no loops, every face in $d(\boldsymbol{G})$ has more than 1 edge in its boundary; and, since $\boldsymbol{G}$ has no pair of parallel edges, every face has at least 3 edges. Because every edge is in exactly 2 faces $2|\boldsymbol{E}| >= 3|\boldsymbol{F}|$ and hence $|\boldsymbol{F}| <= 2|\boldsymbol{E}|/3$. Then (from Thm. 5.5.2)

$$2 = |V| - |E| + |F| <= |E|/3 - |E| + 2|E|/3 = 0.$$

Because 2 is not $<= 0$, our assumption must be false, so d_{ave} must be < 6. ∎

(b) One example of such a graph is:



40. (a) *Proof*: (using the indirect method)

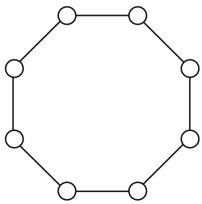
Assume that f_{ave}, the average face size in some drawing of G, is $>= 6$. Since f_{ave} = (the sum of the sizes of all the faces)/$|F|$, from Eqn. 5.5.1, we get that $2|E| = |F| \times f_{ave}$. Then $2|E| >= |F| \times 6$ and hence $|F| <= 2|E|/6 = |E|/3$.

Let $d(G)$ be any drawing of G in the plane. Because, G has no bridges, every face of $d(G)$ is bounded by a circuit. Since G has no vertex of degree < 3, from Equation (5.1.1), we get that $2|E| = |V| \times d_{ave} >= |V| \times 3$ and hence $|V| <= 2|E|/3$. Then (from Thm. 5.5.2)

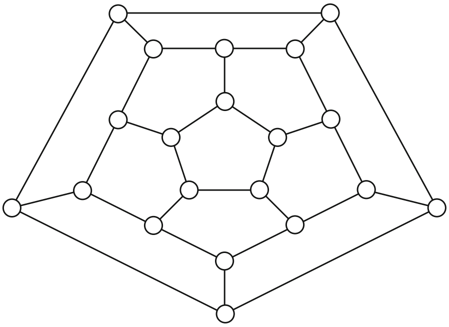
$$2 = |V| - |E| + |F| <= 2|E|/3 - |E| + |E|/3 = 0.$$

Because 2 is not $<= 0$, our assumption must be false, so f_{ave} must be < 6. ∎

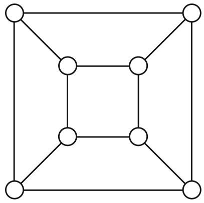
(b) One example of such a graph is:



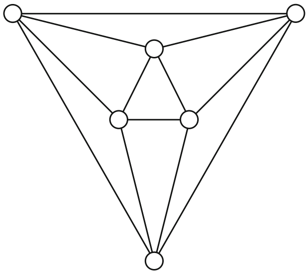
41. (a) Examples of planar drawings are:



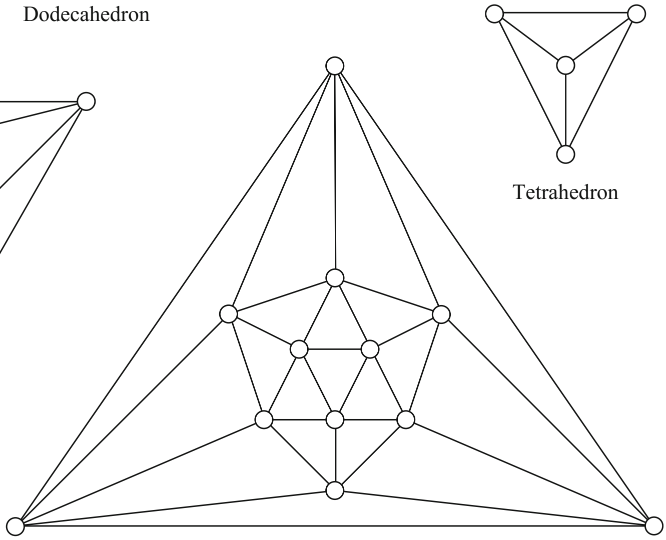
Dodecahedron



Cube



Octahedron



Tetrahedron

Icosahedron

(b) The dual of the *tetrahedron* is the *tetrahedron* itself.
The dual of the *cube* is the *octahedron*.
The dual of the *octahedron* is the *cube*.
The dual of the *dodecahedron* is the *icosahedron*.
The dual of the *icosahedron* is the *dodecahedron*.

(c) *Proof*.

In a 3-dimensional polyhedron, a vertex is a point where 3 or more faces meet, so each vertex has degree >=3. Also, the planar graph of the surface of the polyhedron is bridgeless, connected and simple.

If $s = 3$ then all of the faces are triangles. The result given in Q39 gives us that d must be < 6. The cases where $s = 3$ and $d = 3$, 4 and 5 are: the tetrahedron, the octahedron, and the icosahedron. No other cases are possible when $s = 3$.

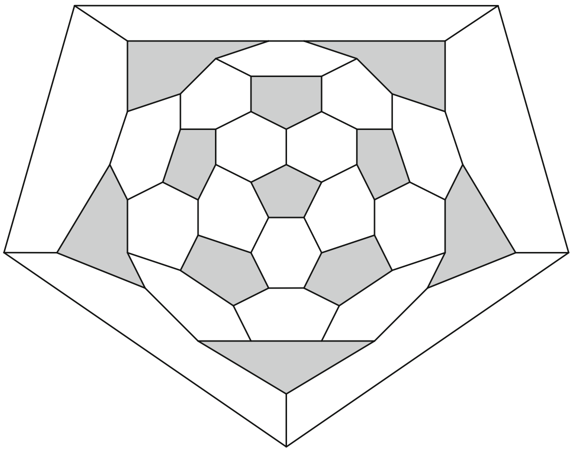
If $d = 3$ then all vertices have degree 3. The result given in Q40 gives us that s must be < 6. The cases where $d = 3$ and $s = 3$, 4 and 5 are: the tetrahedron, the cube, and the dodecahedron. No other cases are possible when $d = 3$.

Assume now that both $d >= 4$ and $s >= 4$. From Equation (5.1.1), we get that $2|E| >= |V| \times 4$ and hence $|V| <= |E|/2$. From Equation (5.5.1), we get that $2|E| >= |F| \times 4$ and hence $|F| <= |E|/2$. Then (from Thm. 5.5.2) we would get

$$2 = |V| - |E| + |F| <= |E|/2 - |E| + |E|/2 = 0.$$

Because 2 is not $<= 0$, our assumption must be false; the only possible Platonic Solids are the 5 listed. ▯

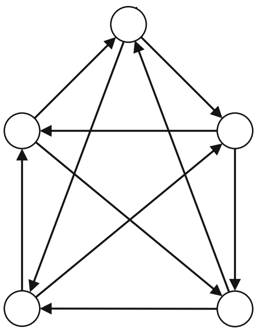
42. (a)



(b) Each vertex lies on exactly one of the 12 pentagons, so $|V| = (12)(5) = 60$. Each vertex has degree 3, so $2|E| = (60)(3)$. Thus, $|E| = 90$.

A.6 Solutions from Chapter 6

2. (a) No. See, for example, answer (b).
   (b)



(c) Let $\boldsymbol{W}(\boldsymbol{x})$ denote the number of wins by player $\boldsymbol{x}$, and $\boldsymbol{W}_{\text{ave}}$ denote the average number of wins over all players. Then

$$6\boldsymbol{W}_{\text{ave}} = \text{total \# of wins} = \text{total \# of games} = \binom{6}{2} = 15. \text{ So } \boldsymbol{W}_{\text{ave}}$$

$$= 15/6 = 2\tfrac{1}{2} \text{ and}$$

therefore not all players can have the same number of wins.

(d) Let $\boldsymbol{L}(\boldsymbol{x})$ denote the number of losses by player $\boldsymbol{x}$. Then

$$\sum\{[\boldsymbol{W}(\boldsymbol{x}) - \boldsymbol{L}(\boldsymbol{x})] : x \in \boldsymbol{V}\} = \sum\{\boldsymbol{W}(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{V}\} - \sum\{\boldsymbol{L}(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{V}\}$$
$$= \text{total \# of games} \quad - \text{ total \# of games} = 0.$$

So $\sum\{[\boldsymbol{W}(\boldsymbol{x}) - \boldsymbol{L}(\boldsymbol{x})] : \boldsymbol{x} \in \boldsymbol{V}\backslash\{\boldsymbol{a}\}\} = 0 - [\boldsymbol{W}(\boldsymbol{a}) - \boldsymbol{L}(\boldsymbol{a})] < 0$ and at least one of the terms in the sum must be negative; i.e. there must be a player $\boldsymbol{b}$ with more losses than wins.

(e) Theorem: Every tournament contains a directed Hamilton Path.

Proof. Let $\mathbf{T}_n$ denote any particular tournament resulting from an orientation of $\mathbf{K}_n$.

// We will prove the theorem by Mathematical Induction on n.

Step 1. If $n = 1$ then $\boldsymbol{V} = \{\boldsymbol{a}\}$ and the trivial path $(\boldsymbol{a})$ contains all the vertices exactly once and is a directed Hamilton Path in $\mathbf{T}_n$.

Step 2. Assume $\exists\ k >= 1$ where every tournament with k players has a directed Hamilton Path.

Step 3. If $n = k + 1$ and $\mathbf{T}$ is a tournament with $k + 1$ players, select any player $\boldsymbol{w}$ and remove $\boldsymbol{w}$, and all arcs to and from $\boldsymbol{w}$, from $\mathbf{T}$. What remains is a tournament with k players, $\mathbf{T}_k$. By step 2, there is a directed Hamilton Path in $\mathbf{T}_k$,

$$\pi = (v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_k, v_k)$$

where each e_i is an arc oriented from v_{i-1} to v_i. Denote by $\boldsymbol{\alpha}_i$ the arc joining $\boldsymbol{w}$ and v_i if it is oriented from v_i to $\boldsymbol{w}$, and by $\boldsymbol{\beta}_i$ if it is oriented from $\boldsymbol{w}$ to v_i.

If the edge joining $\boldsymbol{w}$ and v_k is oriented from v_k to $\boldsymbol{w}$, then

$$\pi 1 = (v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_k, v_k, \boldsymbol{\alpha}_k, \boldsymbol{w})$$

is a directed Hamilton Path in $\mathbf{T}$.

Otherwise, the edge joining $\boldsymbol{w}$ and v_k is oriented from $\boldsymbol{w}$ to v_k. Now, let q be the smallest index where the edge joining $\boldsymbol{w}$ and v_q is oriented from $\boldsymbol{w}$ to v_q. Then $0 <= q <= k$.

If $q = 0$ then

$$\pi 2 = (\boldsymbol{w}, \boldsymbol{\beta}_0, v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_k, v_k)$$

is a directed Hamilton Path in $\mathbf{T}$.

If $q > 0$ then the edge joining $\boldsymbol{w}$ and v_{q-1} is oriented from v_{q-1} to $\boldsymbol{w}$, and

$$\pi 3 = (v_0, e_1, v_1, e_2, \ldots, v_{q-1}, \boldsymbol{\alpha}_{q-1}, \boldsymbol{w}, \boldsymbol{\beta}_q, v_q, \ldots, e_k, v_k)$$

is a directed Hamilton Path in $\mathbf{T}$. ∎

6. There is only one topological sorting of an acyclic digraph ***D*** if and only if ***D*** contains a directed Hamilton Path.
7. There are many acyclic orientations – first assign the integers from 1 to $|V|$ to the vertices, then orient the edges so all arcs go from a lower integer to a higher one.
11. First topologically sort the vertices.
    Algorithm 6.3.2 yields that the number of dipaths from ***a*** to ***b*** is 46.
    Algorithm 6.3.3 yields that the length of a shortest dipath from ***a*** to ***b*** is 18.
    Algorithm 6.3.4 yields that the length of a longest dipath from ***a*** to ***b*** is 51.
17. The Ford-Fulkerson Algorithm yields a Max-Flow with value 24.
20. (a) In the construction of the flow network ***N***(***G***) where the vertices of ***G*** are partitioned into two independent sets, ***L*** and ***R***: give the arc (***s***, ***v***) from the new source ***s*** to vertex ***v*** in ***L*** a capacity equal ***b***(***v***), give the arc (***v***, ***t***) to the new sink ***t*** from a vertex ***v*** in ***R*** a capacity equal ***b***(***v***), and give all other arcs a capacity of one unit.
    Then any ***b-matching M*** in ***G*** of size *k*, corresponds to a feasible flow $\boldsymbol{F_M}$ in ***N***(***G***) with value *k*; and, any feasible flow ***F*** in ***N***(***G***) with value *k* corresponds to a ***b-matching*** $\boldsymbol{M_F}$ in ***G*** of size *k*.
    (b) // A maximum cardinality 1-matching in ***G*** has size 8.
    A maximum cardinality 2-matching in ***G*** has size 16.
    (c) A maximum cardinality 3-matching in ***G*** has size 22.

A.7 Solutions from Chapter 7

6. (a) 7 !R 7
   (b) 6 R 4 and 4 R 5 but 6 !R 5
   (c) 6 R 4 but 4 !R 6
   (d) 3 R 5 and 5 R 3 but $3 \neq 5$

8. R is reflexive because the trivial path $\pi = (v)$ joins any vertex v to itself.
   R is symmetric because if a path π joins any vertex v to another vertex w, then the path π^R joins w to v.
   R is transitive because if $\pi 1$ joins vertex v to vertex w, and $\pi 2$ joins w to another vertex x, then the concatenation of $\pi 1$ and $\pi 2$ joins vertex v to vertex x.
   R is an equivalence relation. The equivalence classes of vertices are the vertices in the connected components.
9. R is reflexive because the trivial dipath $\pi = (v)$ joins any vertex v to itself.
   R is transitive because if $\pi 1$ joins vertex v to vertex w, and $\pi 2$ joins w to another vertex x, then the concatenation of $\pi 1$ and $\pi 2$ joins vertex v to vertex x.
   If ***D*** has no cycles, what properties does R have? Recall that a relation *R* on *S* is anti-symmetric means whenever *a* and *b* are distinct elements of *S* if *a R b* then *b !R a*. If *a* and *b* are distinct vertices, and *a R b* and *b R a* then
   there is a dipath $\pi 1$ from vertex *a* to vertex *b* and
   there is a dipath $\pi 2$ from vertex *a* to vertex *b*.

Hence there is a closed dipath π equal the concatenation of $\pi 1$ and $\pi 2$.
Then there must be a cycle in $\boldsymbol{D}$. // by Thm 6.1.1

Thus, if $\boldsymbol{D}$ has no cycles, R is (also) anti-symmetric.
Because R is reflexive, transitive and anti-symmetric it is a Partial Order.

29. (a) $f \sim g$ and $A \in \mathbf{R}^+$
$\Rightarrow f << g$ and $A \in \mathbf{R}^+ \Rightarrow Af << g$ // by #3 of Thm 7.5.3
$A \in \mathbf{R}^+ \Rightarrow f << Af$ // by #2 of Thm 7.5.3
$\Rightarrow g << f$ and $f << Af \Rightarrow g << Af$ // by Thm 7.5.1 ∎
$\Rightarrow Af \sim g$

(b) $g << (g+f)$ // by #2 of Thm 7.5.3
$f << g \Rightarrow (f+g) << g$ // by #4 of Thm 7.5.3 ∎
$\Rightarrow (f+g) \sim g$

(c) $f1 << g$ and $f2 \sim g$
$\Rightarrow f1 << g$ and $f2 << g \Rightarrow (f1+f2) << g$ // by #5 of Thm 7.5.3
$f2 << (f2+f1) = (f1+f2)$ // by #1 of Thm 7.5.3
$g << f2$ and $f2 << (f2+f1) \Rightarrow g << (f2+f1)$
$\Rightarrow (f1+f2) \sim g$ ∎

(d) $f1 << g1$ and $f2 << g2 \Rightarrow (f1+f2) << (g1+g2)$ // by #6 of Thm 7.5.3
$g1 << f1$ and $g2 << f2 \Rightarrow (g1+g2) << (f1+f2)$ // by #6 of Thm 7.5.3
$\Rightarrow (f1+f2) \sim (g1+g2)$

(e) $f1 << g1$ and $f2 << g2 \Rightarrow (f1 \times f2) << (g1 \times g2)$ // by #7 of Thm 7.5.3
$g1 << f1$ and $g2 << f2 \Rightarrow (g1 \times g2) << (f1 \times f2)$ // by #7 of Thm 7.5.3
$\Rightarrow (f1 \times f2) \sim (g1 \times g2)$ ∎

30. Let $K = 1$ and let $M = 1$. If $n >= M$ then
$(f1+f2) = n+n^2 <= n^3+2n^2 = (g1+g2)$,
so $(f1+f2) = n+n^2 << n^3+2n^2 = (g1+g2)$.

Suppose that $K \in \mathbf{R}^+$ and $M \in \mathbf{P}$. If $n^* = M + \lceil K \rceil$, then $n^* \in \mathbf{P}$, $n^* >= M$, $n^* >= K$, and when n takes the value n^*
$(g1+g2) = n^3+2n^2 = n(n^2+2n) >= K(n^2+2n)$
$> \quad K(n+n^2) = K(f1+f2)$.

So $(g1+g2) = n^3+2n^2 \; !<< n+n^2 = (f1+f2)$.

Thus $(f1+f2) = n+n^2 \; <<< \; n^3+2n^2 = (g1+g2)$
and $(f1+f2) = n+n^2 \; !\sim \; n^3+2n^2 = (g1+g2)$. ∎

31. Proof.

Anti-symmetry: Suppose that $f \neq g$.

$$
\begin{aligned}
f\ \mathcal{SD}\ g &\Rightarrow f <<< g && \text{// Thm. 7.5.7}\\
&\Rightarrow g\ !<< f &&\\
&\Rightarrow g\ !<<< f && \text{// Thm. 7.5.5}\\
&\Rightarrow g\ !\mathcal{SD}\ f && \text{// contra-positive of Thm. 7.5.7}
\end{aligned}
$$

Transitivity: Suppose that $f\ \mathcal{SD}\ g$ and $g\ \mathcal{SD}\ h$. // Is $f\ \mathcal{SD}\ h$?

$\forall K_1 \in \mathbf{R}^+,\ \exists M_1(K_1) \in \mathbf{P}$ such that if $n > M_1(K_1)$ then $K_1 \times f(n) < g(n)$.

If $K_1 = 1$, $\exists M_1(1) \in \mathbf{P}$ such that if $n > M_1(1)$ then $1 \times f(n) < g(n)$.

$\forall K_2 \in \mathbf{R}^+,\ \exists M_2(K_2) \in \mathbf{P}$ such that if $n > M_2(K_2)$ then $K_2 \times g(n) < h(n)$.

Let K be any element of $\mathbf{R}^+$, and let $M(K) = M_1(1) + M_2(K)$ // then $M(K) \in \mathbf{P}$

If $n > M(K)$ then

$$
\begin{aligned}
& n > M_1(1) \text{ so } f(n) < g(n) \text{ and } K \times f(n) < K \times g(n),\\
\text{also}\quad & n > M_2(K) \text{ so } K \times g(n) < h(n),\\
\text{and hence}\quad & K \times f(n) < h(n).
\end{aligned}
$$

Thus $f\ \mathcal{SD}\ h$. ▯

32. Tabulating the first few values of these sequences gives:

n	$f(n)$	$g(n)$	
1	1	1	// $n = 1 = 2(0) + 1$ so $r = 0$
2	2	4	
3	6	6	
4	24	96	
5	120	120	

For any $r \in \mathbf{P}$, we have

$$
\begin{aligned}
f(2r) &= (2r)!\\
< g(2r) &= (2r)!\,(2r)\\
< f(2r+1) &= (2r)!\,(2r+1)\\
= g(2r+1) &= (2r)!\,(2r+1) \quad \text{// both } f \text{ and } g \text{ are increasing integer sequences}
\end{aligned}
$$

If $n = 2r$ then $g(n) = f(n) \times n$ and if $n = 2r + 1$ then $f(n) = g(n)$.
Let $K = 1$ and let $M = 1$. If $n >= M$ then $f(n) <= K \times g(n)$. Therefore $f << g$.

Recall that $f1\ !<<\ f2$ means $\forall K \in \mathbf{R}^+$ and $\forall M \in \mathbf{P}$, $\exists n^* >= M$ where $f1(n^*) > K \times f2(n^*)$.

Let K be any given positive real number and let M be any given positive integer.

Let $r* = M + \lceil K \rceil$. // $r* \in \mathbf{P}$ and $r* > M, K$
If $n* = 2r*$ then $n* >= M$ and $g(n*) = f(n*) \times n* > K \times f(n*)$ so $g \,!<< f$.
Therefore $f <<< g$.

Recall that $f1$ $\mathcal{SD}$ $f2$ means $\forall K \in \mathbf{R}^+$, $\exists M(K) \in \mathbf{P}$ such that if $n > M(K)$ then $K \times f1(n) < f2(n)$.

Therefore, $f1$ $\mathcal{SD}$ $f2$ means $\exists K \in \mathbf{R}^+$ such that $\forall M \in \mathbf{P}$, $\exists n* > M$ where $K \times f1(n*) >= f2(n)$.

Let $K = 1$ and let M be any given positive integer.
If $n* = 2M + 1$ then $n* >= M$ and $g(n*) = f(n*) >= K \times f(n*)$.
Therefore f $!\mathcal{SD}$ g. ▯

Chapter 8

1. Suppose E_n is defined recursively on $\mathbf{P}$ by

$$E_0 = 0,\ E_1 = 2,\ \text{and}\ E_{n+1} = 2n\{E_n + E_{n-1}\}\ \text{for all}\ n >= 1.$$

Determine the value of E_{10}.

$$\begin{array}{llr}
E_2 = E_{1+1} = 2(1)\{E_1 + E_0\} = & 2\{2+0\} = & 4 \\
E_3 = E_{2+1} = 2(2)\{E_2 + E_1\} = & 4\{4+2\} = & 24 \\
E_4 = E_{3+1} = 2(3)\{E_3 + E_2\} = & 6\{24+4\} = & 168 \\
E_5 = E_{4+1} = 2(4)\{E_4 + E_3\} = & 8\{168+24\} = & 1\,536 \\
E_6 = E_{5+1} = 2(5)\{E_5 + E_4\} = & 10\{1536+168\} = & 17\,040 \\
E_7 = E_{6+1} = 2(6)\{E_6 + E_5\} = & 12\{17\,040+1\,536\} = & 222\,912 \\
E_8 = E_{7+1} = 2(7)\{E_7 + E_6\} = & 14\{222\,912+17\,040\} = & 3\,359\,328 \\
E_9 = E_{8+1} = 2(8)\{E_8 + E_7\} = & 16\{3\,359\,328+222\,912\} = & 57\,315\,840 \\
E_{10} = E_{9+1} = 2(9)\{E_9 + E_8\} = & 18\{57\,315\,840+3\,359\,328\} = & 1\,092\,153\,024
\end{array}$$

7. (a) The 15 possible pairings are:
   1. x_1 with x_2 and x_3 with x_4 and x_5 with x_6
   2. x_1 with x_2 and x_3 with x_5 and x_4 with x_6
   3. x_1 with x_2 and x_3 with x_6 and x_4 with x_5
   4. x_1 with x_3 and x_2 with x_4 and x_5 with x_6
   5. x_1 with x_3 and x_2 with x_5 and x_4 with x_6
   6. x_1 with x_3 and x_2 with x_6 and x_4 with x_5
   7. x_1 with x_4 and x_2 with x_3 and x_5 with x_6
   8. x_1 with x_4 and x_2 with x_5 and x_3 with x_6
   9. x_1 with x_4 and x_2 with x_6 and x_3 with x_5
   10. x_1 with x_5 and x_2 with x_3 and x_4 with x_6
   11. x_1 with x_5 and x_2 with x_4 and x_3 with x_6

12. x_1 with x_5 and x_2 with x_6 and x_3 with x_4
13. x_1 with x_6 and x_2 with x_3 and x_4 with x_5
14. x_1 with x_6 and x_2 with x_4 and x_3 with x_5
15. x_1 with x_6 and x_2 with x_5 and x_3 with x_4 // So $\boldsymbol{P_3} = 15$

(b) Suppose $n >= 2$. Element x_1 may be paired with any of the ($2n$ – 1) other elements in $\boldsymbol{A}$. This leaves ($2n$ – 2) = 2(n – 1) elements still to be paired, and that can be done in $\boldsymbol{P_{n-1}}$ ways. Thus the number of pairings of $2n$ elements, $\boldsymbol{P_n} = (2n - 1) \times \boldsymbol{P_{n-1}}$.

(c) Theorem: $\boldsymbol{P_n} = (1)(3)(5) \ldots (2n - 1)$

// product of the first n positive odd integers

Proof. // by mathematical induction

Step 1. $\boldsymbol{P_1} = 1$ which is the first positive odd integer.
Step 2. Assume $\exists\ k >= 1$ where $\boldsymbol{P_k} = (1)(3)(5) \ldots (2k - 1)$.
Step 3. If $n = k + 1$ then $n >= 2$ and

$$\begin{aligned} \boldsymbol{P_{k+1}} &= (2[k+1]-1)\boldsymbol{P_k} && \text{// using the RE} \\ &= (2k+1) \times (1)(3)(5)\ldots(2k-1) && \text{// by Step 2} \\ &= (1)(3)(5)\ldots(2k-1) \times (2[k+1]-1) && \square \end{aligned}$$

8. $$y_{n+1} = \frac{[n+1]([n+1]-1)}{2} + \mathrm{c} = \frac{[n+1](n)}{2} + \mathrm{c} = \frac{n(n-1)+2n}{2} + \mathrm{c} = y_n + n.$$

9. (a) $f(1) = 11$, $f(2) = 23$, $f(3) = 47$, $f(4) = 95$, $f(5) = 191$, $f(6) = 383$, $f(7) = 767$, $f(8) = 1535$, $f(9) = 3071$, $f(10) = 6143$.

(b) $f(1) - f(0) = 6$ but $f(2) - f(1) = 12$ so f is not an arithmetic sequence. $f(1) / f(0) = 11/5 = 121/55$ but $f(2) / f(1) = 23/11 = 115/55$ so f is not an geometric sequence.

11. (a)
$$\begin{aligned} s_1 &= (1/5)s_0 - 8 = (1/5)(60) - 8 = 12 - 8 = +4 \\ s_2 &= (1/5)s_1 - 8 = (1/5)(4) - 8 = 0.8 - 8 = -7.2 \\ s_3 &= (1/5)s_2 - 8 = (1/5)(-7.2) - 8 = -1.44 - 8 = -9.44 \end{aligned}$$

(b) In this RE, $a = 1/5$ and $c = -8$ so $\dfrac{c}{1-a} = \dfrac{-8}{4/5} = -10$ and we have $s_0 = 60$.

The particular solution is
$$\begin{aligned} s_n &= a^n\left[I - \frac{c}{1-a}\right] + \frac{c}{1-a} \\ &= (1/5)^n[60 - (-10)] + (-10) \\ &= (1/5)^n[70] - 10. \end{aligned}$$

(c) Yes. The limit is –10.

(d) No. Because the sequence does not converge to zero.

21.

n	F_n	$T_n = 1 + F_0 + F_1 + \ldots + F_n$
0	1	2
1	1	3
2	2	5
3	3	8
4	5	13
5	8	21

Theorem For $\forall n \in \mathbf{P}$, $T_n = 1 + F_0 + F_1 + \ldots + F_n$ equals $\boldsymbol{F}_{n+2}$.
Proof.

Step 1. If $n = 0$ then $\boldsymbol{T}_n = 1 + \boldsymbol{F}_0 = 1 + 1 = 2 = \boldsymbol{F}_2 = \boldsymbol{F}_{0+2}$.
Step 2. Assume that $\exists\ k >= 0$ such that $\boldsymbol{T}_k$ equals $\boldsymbol{F}_{k+2}$.
Step 3. If $n = k + 1$ then $n >= 1$ so

$$\begin{aligned} \boldsymbol{T}_{k+1} &= 1 + \boldsymbol{F}_0 + \boldsymbol{F}_1 + \ldots + \boldsymbol{F}_k \quad + \boldsymbol{F}_{k+1} \\ &= \boldsymbol{T}_k \quad + \boldsymbol{F}_{k+1} \\ &= \boldsymbol{F}_{k+2} \quad + \boldsymbol{F}_{k+1} && \text{// by Step 2} \\ &= \boldsymbol{F}_{k+3} && \text{// by the Fibonacci RE} \\ &= \boldsymbol{F}_{(k+1)+2}. \end{aligned}$$

□

23. (a) The General Solution of this Recurrence Equation is

$$\begin{aligned} S_n &= \boldsymbol{A}(r)^n \quad + \boldsymbol{B}n(r)^n && \text{// since } r_1 = r_2 = r \\ &= \boldsymbol{A}(11)^n + \boldsymbol{B}n(11)^n && \text{// since } r = 11 \end{aligned}$$

(b) Find the Particular Solution where $S_0 = 1$ and $S_1 = 5$.

$$S_0 = 1 = \boldsymbol{A}(11)^0 + \boldsymbol{B}(0)(11)^0 = \boldsymbol{A}$$

$$S_1 = 5 = \boldsymbol{A}(11)^1 + \boldsymbol{B}(1)(11)^1 = 11\boldsymbol{A} + 11\boldsymbol{B}$$

Then $\boldsymbol{A} = 1$
$\boldsymbol{B} = [5 - 11(1)]/11 = -6/11$

Hence, the Particular Solution where $S_0 = 1$ and $S_1 = 5$ is

$$S_n = (1)(11)^n + (-6/11)n(11)^n = 11^n - 6n(11)^{n-1}$$

// $S_0 = 11^0 - 6(0)(11)^{n-1} = 1 - 0 = 1$
// and $S_1 = 11^1 - 6(1)(11)^0 = 11 - 6 = 5.$

29. (a) $e^{1/3} = 1.395\ 612\ 425\ldots$

(b)

j			$\text{term}_j = \frac{1}{j!}\left(\frac{1}{3}\right)^j$	partial sum
0		1	= 1.000 000 000...	1.000 000 000... = 1
1		1/3	= 0.333 333 333...	1.333 333 333... = 4/3
2	1/(6 × 3)	= 1/18	= 0.055 555 555...	1.388 888 888... = 25/18
3	1/(9 × 18)	= 1/162	= 0.006 172 839...	1.395 061 728... = 226/162
4	1/(12 × 162)	= 1/1944	= 0.000 514 403...	1.395 576 132... = 2713/1944
5	1/(15 × 1944)	= 1/29160	= 0.000 034 293...	1.395 610 425... = 40696/29160

(c) Yes.

(d) $e^{-1/3} = 0.716\ 531\ 310\ldots$

j			$\text{term}_j = \frac{1}{j!}\left(\frac{-1}{3}\right)^j$	partial sum
0		1	= +1.000 000 000...	1.000 000 000... = 1
1		−1/3	= −0.333 333 333...	0.666 666 666... = 2/3
2	+ 1/(6 × 3)	= +1/18	= +0.055 555 555...	0.722 222 222... = 13/18
3	− 1/(9 × 18)	= −1/162	= −0.006 172 839...	0.716 049 382... = 116/162
4	+ 1/(12 × 162)	= +1/1944	= +0.000 514 403...	0.716 563 786... = 1393/1944
5	− 1/(15 × 1944)	= −1/29160	= −0.000 034 293...	0.716 529 492... = 20894/29160

Yes. This partial sum also gives 3 decimal places of accuracy.

Chapter 9

4. (a) [(7)(6)]/[(2)(1)] = 21
   (b) (1, 2, 3, 4, 5)
   (c) (3, 4, 5, 6, 7)
   (d) (2, 3, **5**, 6, 7)

7. The sequences are (read left to right, top to bottom):

321	421	431	432	521	531	532	541	542	543
621	631	632	641	642	643	651	652	653	654

8. (a) Let $n = 3$, B = 8, $V_1 = 10$, $W_1 = 5$, $V_2 = 7$, $W_2 = 4$, $V_3 = 7$, and $W_3 = 4$.
   The Greedy Solution is $\{O_1\}$ with total value = 10 and total weight = 5.
   The Optimal Solution is $\{O_2, O_3\}$ with total value = 14 and total weight = 8.
   (b) Let $n = 3$, B = 8, $V_1 = 10$, $W_1 = 5$, $\text{Ratio}_1 = 2$
   $V_2 = 7$, $W_2 = 4$, $\text{Ratio}_2 = 1.75$
   $V_3 = 7$, $W_3 = 4$, $\text{Ratio}_3 = 1.75$
   The Greedy Solution is $\{O_1\}$ with total value = 10 and total weight = 5.
   The Optimal Solution is $\{O_2, O_3\}$ with total value = 14 and total weight = 8.

9. (a)
$$\begin{aligned}\sum_{j=a}^{b}(x_{j-1}-x_j) &= \quad x_{a-1} - x_a\\ &+ \quad x_a - x_{a+1}\\ &+ \quad x_{a+1} - x_{a+2}\\ &\quad \dots\\ &+ \quad x_{b-2} - x_{b-1}\\ &+ \quad x_{b-1} - x_b\\ &= \quad x_{a-1} - x_b. \qquad \text{// All other } x_j\text{'s "cancel out".}\end{aligned}$$

(b) $$\sum_{j=a}^{b}(x_j-x_{j-1}) = -\sum_{j=a}^{b}(x_{j-1}-x_j) = -(x_{a-1}-x_b) = x_b - x_{a-1}.$$

(c) Let $y_j = x_{j+1}$ for each index j. Then

$$\sum_{j=a}^{b}(x_{j+1}-x_j) = \sum_{j=a}^{b}(y_j-y_{j-1}) = y_b - y_{a-1} = x_{b+1} - x_a.$$

(d) Let $x_j = n^j$ for each index j. Then

$$\begin{aligned}\sum_{j=a}^{b} n^j(n-1) &= \sum_{j=a}^{b}(n^{j+1}-n^j) = \sum_{j=a}^{b}(x_{j+1}-x_j) = x_{b+1}-x_a\\ &= n^{b+1} - n^a.\end{aligned}$$

(e) Let $x_j = n^{k-j}$ for each index j. Then

$$\begin{aligned}\sum_{j=a}^{b} n^{k-j}(n-1) &= \sum_{j=a}^{b}(n^{k-j+1}-n^{k-j})\\ &= \sum_{j=a}^{b}(x_{j-1}-x_j) = x_{a-1}-x_b \qquad \text{// from part (a)}\\ &= n^{k-(a-1)} - n^{k-b} = n^{k-a+1} - n^{k-b}.\end{aligned}$$

Chapter 10

3. (a) $prob(\mathbf{A}) = 27/80$

$prob(\mathbf{B}) = 11/80$

$prob(\mathbf{A\ and\ B}) = 4/80$

$$\begin{aligned}prob(\mathbf{A\ or\ B}) &= prob(\mathbf{A}) + prob(\mathbf{B}) - prob(\mathbf{A\ and\ B})\\ &= \frac{27}{80} + \frac{11}{80} - \frac{4}{80} = \frac{34}{80}\end{aligned}$$

$$prob(\mathbf{A}|\mathbf{B}) = prob(\mathbf{A\ and\ B})/prob(\mathbf{B}) = \frac{4/80}{11/80} = \frac{4}{11}$$

$$prob(\mathbf{B}|\mathbf{A}) = prob(\mathbf{B\ and\ A})/prob(\mathbf{A}) = \frac{4/80}{27/80} = \frac{4}{27}$$

(b) **A** and **B** are NOT mutually exclusive because *prob*(**A and B**) = 4/80 ≠ 0.
(c) **A** and **B** are NOT independent because

$$prob(\mathbf{A\ and\ B}) = 4/80 = 320/6400$$
$$\neq prob(\mathbf{A}) * prob(\mathbf{B}) = (27/80) * (11/80) = 297/6400.$$

5. (a) # delegations that exclude Mike $= \binom{n-1}{k}$

$$prob\,(\text{Mike is excluded}) = \binom{n-1}{k} \Big/ \binom{n}{k} = \frac{(n-1)!}{k!(n-k-1)!} \times \frac{k!(n-k)!}{n!} = \frac{n-k}{n}$$

$$prob\,(\text{Mike is included}) = 1 - \frac{n-k}{n} = \frac{k}{n}$$

Alternatively…

delegations that include Mike $= \binom{n-1}{k-1}$

$$prob\,(\text{Mike is included}) = \binom{n-1}{k-1} \Big/ \binom{n}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} \times \frac{k!(n-k)!}{n!} = \frac{k}{n}$$

(b) *prob*(Flora is included) $= \frac{k}{n}$

(c) # delegations that include both Mike and Flora $= \binom{n-2}{k-2}$

$$prob\,(\text{Mike and Flora are included}) = \binom{n-2}{k-2} \Big/ \binom{n}{k}$$
$$= \frac{(n-2)!}{(k-2)!(n-k)!} \times \frac{k!(n-k)!}{n!}$$
$$= \frac{k(k-1)}{n(n-1)}.$$

This equals *prob* (Mike is included) ∗ *prob* (Flora is included)

$$\Leftrightarrow \quad \frac{k(k-1)}{n(n-1)} = \frac{k(k)}{n(n)}$$
$$\Leftrightarrow \quad \frac{k-1}{n-1} = \frac{k}{n}$$
$$\Leftrightarrow \quad n(k-1) = k(n-1)$$
$$\Leftrightarrow \quad nk - n = kn - k$$
$$\Leftrightarrow \quad -n = -k$$
$$\Leftrightarrow \quad k = n$$

Since $1 < k < n$, the events "Mike is in" and "Flora is in" are NOT independent.

6. $prob(\{\text{X gets H and you get H}\} \text{ or } \{\text{X gets T and you get T}\})$

$$= prob(\text{X gets H and you get H}) + prob\,(\text{X gets T and you get T})$$
$$= prob(\text{X gets H}) * prob\,(\text{you get H}) + prob\,(\text{X gets T}) * prob\,(\text{you get T})$$
$$= p * 1/2 + (1 - p) * (1 - 1/2)$$
$$= 1/2$$

9. (a)

T =	Prob	# probes	(#probes) × (Prob)
X_{10}	0.40	1	0.40
X_7	0.22	2	0.44
X_4	0.10	3	0.30
X_9	0.07	4	0.28
X_6	0.06	5	0.30
X_3	0.05	6	0.30
X_1	0.04	7	0.28
X_8	0.03	8	0.24
X_2	0.02	9	0.18
X_5	0.01	10	0.10
	1.00		2.82 = Expected # probes

(b)

# probes	T =	Prob	(#probes) × (Prob)
1	X_5	0.01	0.01
2	X_2, X_8	0.05	0.10
3	X_1, X_3, X_6, X_9	0.22	0.66
4	X_4, X_7, X_{10}	0.72	2.88
		1.00	3.65 = Expected # probes

(c) Yes

10. (a)

	passed test	failed test	
passed course	3230	48	3278
failed course	170	552	722
	3400	600	4000

(b) $P(\text{failed test}|\text{passed course}) = \dfrac{48}{3278} = 0.014643075\ldots \sim 1.5\%$

(c) $P(\text{passed test}|\text{failed course}) = \dfrac{170}{722} = 0.235457063\ldots \sim 23.5\%$

13. (a) $10^4 = 10\ 000$
(b) $10 * 9 * 8 * 7 = 5\ 040$
(c) The number of passwords with the digit "5" repeated j times is $\binom{4}{j} 9^{4-j}$.

j	$\binom{4}{j} 9^{4-j}$	
0	$(1)9^4 =$	6 561
1	$(4)9^3 =$	2 916
2	$(6)9^2 =$	486
3	$(4)9^1 =$	36
4	$(1)9^0 =$	1
Total:		10 000

(d)

j	$j \times prob(\boldsymbol{X}=j)$
0	$0 \times 6561/10000 = 0.0000$
1	$1 \times 2916/10000 = 0.2916$
2	$2 \times 486/10000 = 0.0972$
3	$3 \times 36/10000 = 0.0108$
4	$4 \times 1/10000 = 0.0004$
	E(# of 5's) = 0.4000

// equals 4× (1/10)
// *np* in some binomial experiment

15. Using a Binomial experiment as a model for this process

where a trial is a laptop purchase
a "success" is *making a claim* on the guarantee
// we're interested in the number of claims made, X
so $n = 43$, $p = 1 - 85\% = 15\% = 0.15$ and $q = 0.85$
and X is the *number of claims* made.

$$\text{We want } P(X >= 3) = \sum_{k=3}^{43} P(X = k) \qquad \text{// a very long calculation}$$

$$= 1 - \sum_{k=0}^{2} P(X = k) \qquad \text{// using the hint}$$

$$P(X = 0) = \binom{43}{0} p^0 \times q^{43} = 1 \times 1 \times q^{43} = (1)(1)(0.85)^{43} = 0.000\,922\,600\ldots$$

$$P(X = 1) = \binom{43}{1} p^1 \times q^{42} = 43 \times p \times q^{42} = (43)(0.15)(0.85)^{42} = 0.007\,000\,911\ldots$$

$$P(X = 2) = \binom{43}{2} p^2 \times q^{41} = \frac{43 \times 42}{2 \times 1}(0.15)^2(0.85)^{41}$$

$$= (903)(0.0225)(0.85)^{41} = 0.025\,944\,554\ldots$$

Hence, $P(X < 3) = 0.000\,922\,600\ldots + 0.007\,000\,911\ldots + 0.025\,944\,554\ldots$
$= 0.033\,868\,065\ldots$

and $P(X >= 3) = 1 - 0.033\,868\,065\ldots = 0.966\,131\,934\ldots$

// Also, $\binom{43}{0}p^0 \times q^{43} + \binom{43}{1}p^1 \times q^{42} + \binom{43}{2}p^2 \times q^{41}$

// $= \{1 \times 1 \times q^2 \quad + 43 \times p \times q^1 \quad + 903 \times p^2\} \times q^{41}$

// $= \{0.7225 \quad + 5.4825 \quad + 20.3175\} \times q^{41}$

// $= \{26.5225\} \times (0.001276956\ldots) = 0.033\,868\,067\ldots$

// We should expect about $np = 43 \times (0.15) = 6.45$ claims.

17. (a) 38%
    (b) Using a Binomial experiment as a model for this
    where the trials correspond to the top 10 marks
    a "success" is a <u>*woman*</u> obtaining a top ten mark
    // we're interested in the number of women that obtain a top ten mark, X
    we assume that $p = 38\%$, so $q = 62\%$.

$$P(X = 6) = \binom{10}{6} p^6 \times q^4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}(0.38)^6(0.62)^4$$
$$= (210)(0.003010936\ldots)(0.14776336)$$
$$= 0.093\,430\,276\ldots$$

    (c) $E(X) = np = (10)(0.38) = 3.8$

23. The possible outcomes from the game are summarized in the following table, with the amount won shown in brackets:

	Diamond	Spade, Heart or Club
Ace	1 ($10.00)	3 ($5.00)
Jack, Queen or King	3 ($7.00)	9 ($4.50)
2 through 10	9 ($4.50)	27 ($0.00)

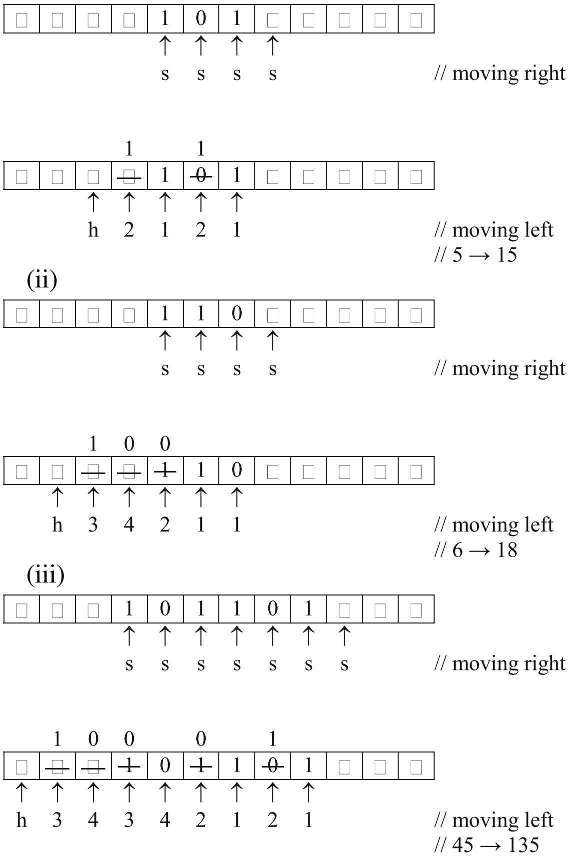
Probability Distribution for ***f***

v	$prob(f = v)$	$v \times prob(f = v)$
–$2.50	27/52	–$67.50/52
$2.00	18/52	$36.00/52
$2.50	3/52	$7.50/52
$4.50	3/52	$13.50/52
$7.50	1/52	$7.50/52
	52/52	E(f) = –$ 3.00/52
		= –$ 0.057692307…
		= –5.7692307…cents

The average payoff per play is negative because the person operating the game (the “house”) needs to cover the cost of operating the game, and wants to make a profit from it. The money needed for these costs and profits comes from having a negative average payoff per play.

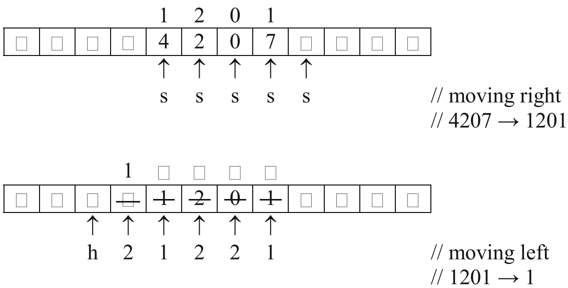
Chapter 11

1. (a) (i)



(b) $n \rightarrow 3n$

2. (a)



(b) $n \rightarrow n$ MOD 3

4. Let $\boldsymbol{L}$ denote the "language" of strings of $\boldsymbol{a}$'s and $\boldsymbol{b}$'s described. Then $L = \{\boldsymbol{a}^n \mathbf{b}^n : n \in \mathbf{N}\}$. Furthermore, $w \in \boldsymbol{L}$ if and only if w is the empty string or $w = \boldsymbol{axb}$ where $x \in \boldsymbol{L}$.

(s, a: □, R, 1)	(1, a: a, R, 1)	(2, a: □, L, No)	(3, a: a, L, 3)
(s, b: □, R, No)	(1, b: b, R, 1)	(2, b: □, L, 3)	(3, b: b, L, 3)
(s, □: □, R, Yes)	(1, □: □, L, 2)	(2, □: □, L, No)	(3, □: □, R, s)

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