Appendix A
Riemannian Geometry

In this appendix besides introducing notation and basic definitions we recollect some of the more technical results in Riemannian geometry we exploited in these lecture notes. We freely refer to the excellent exposition of the theory provided by [4, 8], and to [27] for further details.

A.1 Notation

In what follows, V^n will always denote a C^∞ compact n–dimensional manifold, $(n \geq 3)$. Unless otherwise stated, the manifold V^n will be supposed oriented, connected, and without boundary. Let (E, V^n, π) be a smooth vector bundle over V^n with projection map $\pi : E \longrightarrow V^n$. When no confusion arises the space of smooth sections $\{s \in C^\infty(V^n, E) \,|\, \pi \circ s = \mathrm{id}^n_V\}$ will simply be denoted by $C^\infty(V^n, E)$ instead of the standard $\Gamma(E)$ (to avoid confusion (Fig. A.1) with the already too numerous Γs appearing in Riemannian geometry: Christoffel symbols, harmonic coordinates, …). If $U \subset V^n$ is an open set, we let $\{x^i\}^n_{i=1}$ be local coordinates for the points $p \in U$. The vector fields $\{\partial_i := \partial/\partial x^i\}^n_{i=1} \in C^\infty(U, TV^n)$ provide the local (positively oriented) coordinate basis for T_pV^n, $p \in U$. The corresponding dual basis for the cotangent spaces $T^*_pV^n$, is provided by the set of 1–forms $\{dx^i\}^n_{i=1} \in C^\infty(U, T^*V^n)$. In particular, the metric tensor $g \in C^\infty(V^n, \otimes^2_S T^*V^n)$, where $C^\infty(V^n, \otimes^2_S T^*V^n)$ denotes the set of smooth symmetric bilinear forms over V^n, has the local coordinates representation $g = g_{ik}dx^i \otimes dx^k$, where $g_{ik} := g(\partial_i, \partial_k)$, and the Einstein summation convention is in effect. The Riemannian volume form is $d\mu_g = \sqrt{\det g}\, dx^1 \wedge \ldots \wedge dx^n$, and we denote by $\mathrm{Vol}_g(V^n) := \int_{V^n} d\mu_g$ the volume of (the compact) (V^n, g). The distance between points $x, y \in V^n$, characterizing (V^n, g) as a metric space, is defined by setting

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Fig. A.1 . . . Riemannian geometry may have unexpected sides. . . and must be used responsibly!

$$d_g(x, y) := \inf_{\gamma \in \mathscr{C}(x,y)} \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt, \tag{A.1}$$

where $\mathscr{C}(x, y)$ is the space of (piecewise C^1) curves $\gamma : [0, 1] \to V^n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. If

$$|U|_g^2 := g^{i_1 h_1} \ldots g^{i_r h_r}\, U^{k_1 \ldots k_q}_{i_1 \ldots i_p}\, U^{j_1 \ldots j_q}_{h_i \ldots h_r}\, g_{k_1 j_1} \ldots g_{k_q j_q} \tag{A.2}$$

denotes the pointwise (squared) g–norm in the tensor bundle $T^{(r,q)}V^n := \otimes^r T^* V^n \otimes^q TV^n$, then we can endow the space of smooth (r, q)–tensor fields $C^\infty(V^n, T^{(r,q)}V^n)$ on V^n with the corresponding $L^p(d\mu_g)$ and $\mathscr{H}^{p,s}(d\mu_g)$ Sobolev norms

$$||U||_{L^p(d\mu_g)} \doteq \left[\int_{V^n} |U|_g^p\, d\mu_g \right]^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{A.3}$$

$$||U||_{\mathscr{H}^{p,s}(d\mu_g)} \doteq \sum_{i=0}^{s} \left[\int_{V^n} \left| \nabla^{(i)} U \right|_g^p d\mu_g \right]^{\frac{1}{p}}, \tag{A.4}$$

where s is an integer ≥ 0, and $\nabla^{(i)} U$ denotes the ith covariant derivative of the tensor field U. The completions of $C^\infty(V^n, T^{(r,q)}V^n)$ in these norms, define

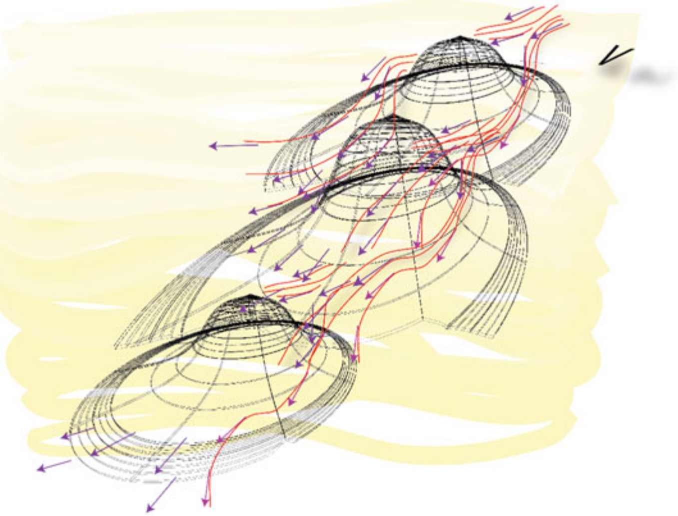

Fig. A.2 The group of diffeomorphisms

the Sobolev spaces of sections $\mathscr{H}^{p,s}(V^n, T^{(r,q)}V^n)$. In particular, for $p = 2$, we denote by $\mathscr{H}^{s}(V^n, T^{(r,q)}V^n)$ the Hilbert space $\mathscr{H}^{2,s}(V^n, T^{(r,q)}V^n)$. According to Sobolev embedding theorem, if $k \geq 0$ and $s > \frac{n}{2} + k$, then $\mathscr{H}^{s}(V^n, T^{(r,q)}V^n) \subset C^k(V^n, T^{(r,q)}V^n)$. Similarly, if we denote by $\mathscr{H}^{s}(V^n, V^n)$ the Sobolev space of maps $\phi : V^n \longrightarrow V^n$, then $\mathscr{H}^{s}(V^n, V^n) \subset C^k(V^n, V^n)$ is a dense continuous inclusion for $s > (n/2) + k$.

Let

$$\mathrm{Diff}^k(V^n) := \left\{\phi \in C^k(V^n, V^n) \middle|\ \phi \text{ is a bijection and } \phi^{-1} \in C^k(V^n, V^n)\right\} \qquad \text{(A.5)}$$

be the group of C^k diffeomorphisms of V^n, endowed with the compact–open C^k topology. The group of smooth diffeomorphisms $\mathrm{Diff}(V^n)$ is defined by the sequence of natural inclusions (Fig. A.2)

$$\mathrm{Diff}^1(V^n) \supset \mathrm{Diff}^2(V^n) \supset \ldots \mathrm{Diff}^k(V^n) \supset \ldots\ , \qquad \text{(A.6)}$$

according to

$$\mathrm{Diff}(V^n) = \cap_{k=1}^{\infty} \mathrm{Diff}^k(V^n). \qquad \text{(A.7)}$$

Recall that under a diffeomorphism $\phi : V^n \longrightarrow V^n$ the pull back from $(\phi(V^n), g)$ to $(V^n, \phi^* g)$ is given for $p \in V^n$ by

$$\left(\phi^* g\right)(V, W)\Big|_p := g\left(\phi_* V, \phi_* W\right)|_{\phi(p)}\,, \qquad \text{(A.8)}$$

where $\phi_* V$ denotes the push forward of V defined by the tangent map $\phi_* : T_p V^n \longrightarrow T_{\phi(p)} V^n$. In local coordinates (U, x^i) and $(\phi(U), y^a)$, $U \ni x^i \longmapsto \phi^a(x^i) := y^a(\phi(p)) \in \phi(U)$, we write

$$\left(\phi^* g\right)_{ik}(p) = \frac{\partial \phi^a}{\partial x^i} \frac{\partial \phi^b}{\partial x^k} g_{ab}(\phi(p)). \tag{A.9}$$

It is important to stress that the pull–back action of $\mathrm{Diff}(V^n)$ on the metric g is a right action since under composition of any two diffeomorphisms ϕ and ψ we have $(\psi \circ \phi)^* g = \phi^*(\psi^* g)$. This latter remark plays a role when, for technical reasons, it is necessary to enlarge $\mathrm{Diff}(V^n)$ to the (Hilbert manifold of the) diffeomorphisms of Sobolev class $\mathscr{H}^s$, for $s > (n/2) + 1$,

$$\mathscr{D}^s(V^n) := \left\{ \phi \in \mathscr{H}^s(V^n, V^n) |\ \phi \text{ is a bijection and } \phi^{-1} \in \mathscr{H}^s(V^n, V^n) \right\}. \tag{A.10}$$

In such a framework the group of smooth diffeomorphisms can be recovered from $\mathscr{D}^s(V^n)$ by characterizing the smooth topology on $\mathrm{Diff}(V^n)$ as the limit of the topologies of $\mathscr{D}^s(V^n)$, by viewing $\mathrm{Diff}(V^n)$ as an Inverse Limit Hilbert group in the sense of Omori [26], i.e. $\mathrm{Diff}(V^n) = \cap_{s>n/2} \mathscr{D}^s(V^n)$. Since right multiplication is smooth, but left multiplication is not, $\mathscr{D}^s(V^n)$ is not "an infinite–dimensional" Lie group but only a topological group [11, 21].

A.2 Curvature and Scaling

Besides diffeomorphisms, the metric g is naturally acted upon also by overall rescalings according to for $\lambda \in \mathbb{R}_{>0}$ (Fig. A.3)

$$g \longmapsto \lambda\, g \tag{A.11}$$

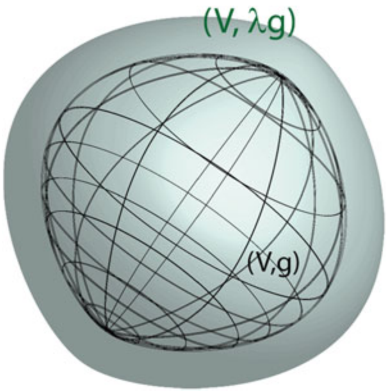

Fig. A.3 Besides diffeomorphisms, the metric g is naturally acted upon also by overall rescalings

$$g^{-1} \longmapsto \lambda^{-1} g^{-1},$$

where $g^{-1} \in C^\infty(V^n, \otimes_S^2 TV^n)$ denotes the inverse metric providing the inner product in $T_p^* V^n$, $p \in V^n$. In local coordinates (U, x^i), we write $g_{ik} \longmapsto \lambda\, g_{ik}$ and $g^{ik} \longmapsto \lambda^{-1} g^{ik}$, respectively. Under the scaling (A.11) we immediately get an obvious dilation or contraction effect on distances and volume

$$\begin{aligned} d_g(p,q) &\longmapsto d_{\lambda g}(p,q) = \lambda^{\frac{1}{2}} d_g(p,q) \\ \mathrm{Vol}_g(V^n) &\longmapsto \mathrm{Vol}_{\lambda g}(V^n) = \lambda^{\frac{n}{2}} \mathrm{Vol}_g(V^n), \end{aligned} \tag{A.12}$$

which is often useful to factor out in the study of Riemannian functional. In this connection it is worthwhile to recall how the basic Riemannian definitions interact with (A.11). We start with the Levi–Civita connection ∇^g of (V^n, g), ("∇" if there is no danger of confusion), defined by

$$\begin{aligned} 2\,g\,(\nabla_X Y, Z) := {} & X\,(g(Y,Z)) + Y\,(g(Z,X)) - Z\,(g(X,Y)) \\ & + g\,([X,Y],Z) - g\,([Y,Z],X) + g\,([Z,X],Y)\,, \end{aligned} \tag{A.13}$$

along with the torsion–free and metric compatibility conditions

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= [X, Y] \\ X\,(g(Y,Z)) &= g\,(\nabla_X Y, Z) + g\,(Y, \nabla_X Z)\,, \end{aligned} \tag{A.14}$$

for all vector fields $X, Y, Z \in C^\infty(V^n, TV^n)$. In a local coordinate neighborhood $(U, \{x^i\})$ we write $\nabla_i \partial_j := \nabla_{\partial_i} \partial_j := \Gamma_{ij}^k(g)\, \partial_k$, where the Christoffel symbols $\Gamma_{ij}^k(g)$ are given by

$$\Gamma_{ij}^k(g) = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right). \tag{A.15}$$

If $(V^n, \phi^* g)$ is the pull–back of $(\phi(V^n), g)$ under a diffeomorphism $\phi : V^n \longrightarrow V^n$, then the Christoffel symbols $\Gamma_{ij}^k(g)$ and $\Gamma_{ij}^k(\phi^* g)$ are related by

$$\Gamma_{ij}^k(\phi^* g) \frac{\partial \phi^c}{\partial x^k} = \left(\Gamma_{ab}^c(g) \circ \phi\right) \frac{\partial \phi^a}{\partial x^i} \frac{\partial \phi^b}{\partial x^j} + \frac{\partial^2 \phi^c}{\partial x^i \partial x^j}, \tag{A.16}$$

whereas, under the metric rescaling (A.11), we easily compute from (A.15) $\Gamma_{ij}^k(\lambda g) = \Gamma_{ij}^k(g)$, hence

$$\nabla^{\lambda g} = \nabla^g. \tag{A.17}$$

Let us recall that if $\mathrm{Hess}_g f := \nabla df$ denotes the Hessian of f, $f \in C^\infty(V^n, \mathbb{R})$, then the Laplace–Beltrami operator $\triangle_g$ on (V^n, g) is defined by

$$\triangle_g f \;:=\; \mathrm{tr}_g\, \left(Hess_g f\right) \;=\; g^{ik}\,\nabla_i\nabla_k f, \tag{A.18}$$

where we have used the geometer's sign convention according to which, on Euclidean space $(\mathbb{R}^n,\, \delta)$, $\triangle_\delta \;=\; \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. It is useful to have the expression of $\triangle_g$ available in local coordinates

$$\begin{aligned} \triangle_g \;&=\; \frac{1}{\sqrt{\det\, g}}\,\frac{\partial}{\partial x^i}\,\left(\sqrt{\det\, g}\, g^{ij}\,\frac{\partial}{\partial x^j}\right) \\ &=\; g^{ij}\,\frac{\partial^2}{\partial x^i \partial x^j} \;+\; \frac{1}{\sqrt{\det\, g}}\,\frac{\partial}{\partial x^i}\,\left(\sqrt{\det\, g}\, g^{ij}\right)\,\frac{\partial}{\partial x^j} \\ &=\; g^{ij}\,\frac{\partial^2}{\partial x^i \partial x^j} \;-\; \Gamma^j\,\frac{\partial}{\partial x^j}, \end{aligned} \tag{A.19}$$

where we have defined

$$\Gamma^j \;:=\; g^{hk}\,\Gamma^j_{hk} \;=\; -\,\frac{\partial}{\partial x^i}\, g^{ji} - g^{ij}\,\frac{\partial}{\partial x^i}\,\left(\log \sqrt{\det\, g}\right). \tag{A.20}$$

From the definitions of Hess_g and $\triangle_g$ it immediately follows that under the metric rescaling (A.11), we have

$$\begin{aligned} \mathrm{Hess}_{\lambda g} \;&=\; \mathrm{Hess}_g, \\ \triangle_{\lambda\, g} \;&=\; \lambda^{-1}\,\triangle_g. \end{aligned} \tag{A.21}$$

The Riemann, the sectional, the Ricci and the scalar curvature operators (Fig. A.4) associated to (V^n, g) are respectively defined by

$$\mathrm{Rm}(g)(X,Y)Z \;:=\; \left(\nabla_X\nabla_Y \;-\; \nabla_Y\nabla_X \;-\nabla_{[X,Y]}\right)\, Z, \tag{A.22}$$

$$\mathscr{S}ec(g)(X,Y) \;:=\; \frac{g\,(\mathrm{Rm}(g)(X,Y)Y,\; X)}{g(X,X)g(Y,Y) - (g(X,Y))^2}, \tag{A.23}$$

$$\mathrm{Ric}(g)(Y,Z) \;:=\; \mathrm{tr}_g\; (X \longmapsto \mathrm{Rm}(g)(X,Y)Z)\,, \tag{A.24}$$

$$\mathrm{R}(g) \;:=\; \mathit{trace}\,\mathrm{Ric}(g), \tag{A.25}$$

for all vector fields $X, Y, Z \;\in\; C^\infty(V^n, TV^n)$. They are naturally equivariant under the action of $\mathrm{Diff}(V^n)$, whereas under the scaling action (A.11) we get

$$\begin{aligned} \mathrm{Rm}(\lambda g) \;&=\; \mathrm{Rm}(g) \\ \mathrm{Sec}(\lambda g)(X,Y) \;&=\; \lambda^{-1}\,\mathrm{Sec}(g)(X,Y) \\ \mathrm{Ric}(\lambda g) \;&=\; \mathrm{Ric}(g) \end{aligned} \tag{A.26}$$

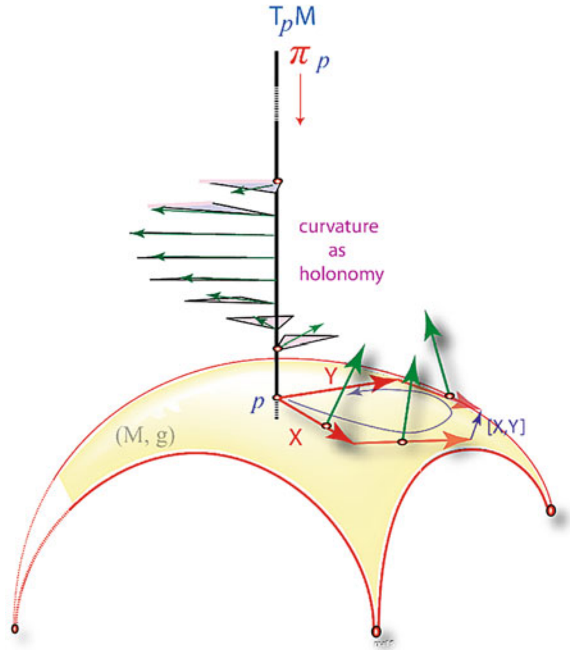

Fig. A.4 Curvature and parallel transport

$$\mathrm{R}(\lambda g) \;=\; \lambda^{-1}\,\mathrm{R}(g),$$

as easily follows from their definition and (A.17). Insight in the curvature operators is often provided by the expression of their components in a judiciously chosen coordinate system (U, x^i). To this end, let us state first our conventions for the components of $\mathrm{Rm}(g)$ and $\mathrm{Ric}(g)$, i.e.

$$\mathrm{Rm}(g)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} \;:=\; \mathrm{R}^h{}_{ijk}\,\frac{\partial}{\partial x^h}, \tag{A.27}$$

$$\mathrm{R}_{ik} \;:=\; \sum_{h=1}^{n} \mathrm{R}^h{}_{hik}, \tag{A.28}$$

where

$$\mathrm{R}^h{}_{ijk} \;=\; \frac{\partial\,\Gamma^h_{jk}}{\partial x^i} - \frac{\partial\,\Gamma^h_{ki}}{\partial x^j} + \Gamma^h_{ir}\Gamma^r_{jk} - \Gamma^r_{ki}\Gamma^h_{jr}, \tag{A.29}$$

$$\mathrm{R}_{ik} \;=\; \frac{\partial\,\Gamma^h_{ik}}{\partial x^h} - \frac{\partial\,\Gamma^h_{hk}}{\partial x^i} + \Gamma^h_{ik}\Gamma^r_{rh} - \Gamma^r_{hi}\Gamma^h_{kr}. \tag{A.30}$$

Notice that when considering the Riemann tensor components in covariant form, the upper index on the Riemann tensor is lowered into the 4–th position according to $\mathrm{R}_{ijk\ell} \;:=\; \mathrm{R}^h{}_{ijk}\,g_{h\ell}$. We recall that, given a point $p \in (V^n, g)$, the exponential map $exp_p \;:\; T_pV^n \;\longrightarrow\; V^n$ is defined by (Fig. A.5)

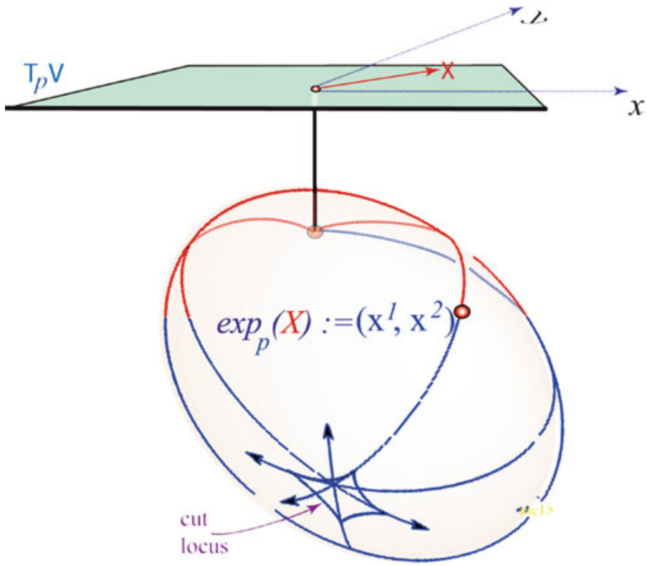

Fig. A.5 A two–dimensional rendering of the exponential map

$$X \longmapsto \exp_p(X) := \gamma_X(1),$$

where $\gamma_X : [0, \infty) \longrightarrow V^n$ is the constant speed geodesic issued from p with initial velocity $\dot{\gamma}_X(t = 0) = X$. If Cut($p$) denotes the cut locus of p in (V^n, g), namely the set of points $q \in V^n$ such that any minimal geodesic $\gamma(p, q)$ connecting p to q is not properly contained in another minimal geodesic issued from p, then normal geodesic coordinates $\{x^i\} : V^n - \mathrm{Cut}(p) \longrightarrow \mathbb{R}^n$ (based at the point $p \in V^n$) are defined by

$$\{x^i\}(q) := \left(\{X^i\} \circ \exp_p^{-1}\right)(q),$$

where $\{X^i\}$ are the components, in an orthonormal frame $\{e_{(i)}\}_{i=1}^n$ at T_pV^n, of the vector $\exp_p^{-1}(q) \in T_pV^n$. In normal coordinates one gets the classical Bertrand-Diguet-Puiseaux formulas providing a geometric interpretation, (see e.g. [4]), of the sectional, Ricci and scalar curvature, *viz.*

$$\ell(C(r)) = 2\pi r \left(1 - \frac{\mathscr{S}ec(X, Y)(p)}{6} r^2 + O\left(r^3\right)\right),$$

$$\sqrt{\det g(q)} = 1 - \frac{1}{6} \mathrm{R}_{ik}(p)\, x^i(q) x^k(q) + O\left(r^3\right), \tag{A.31}$$

$$\mathrm{Vol}\ (B(p, r)) = \omega^n r^n \left(1 - \frac{\mathrm{R}(p)}{6(n+2)} r^2 + O(r^3)\right),$$

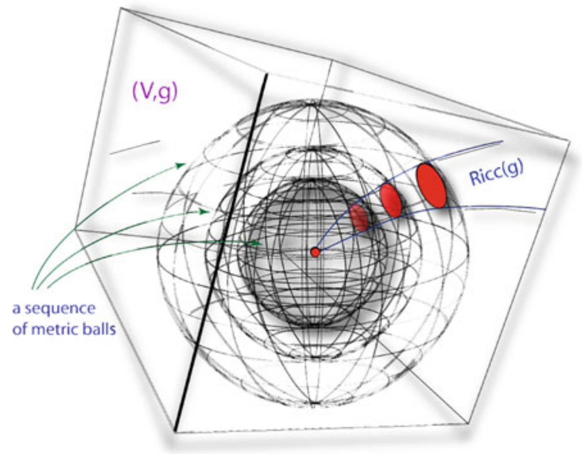

Fig. A.6 The geometrical meaning of Ricci curvature

where: (i) $r := d_{(g)}(q, p)$ is the Riemannian distance between q and p; (ii) $\ell(C(r))$ is the length of the circle of radius r in the geodesic surface generated by the unit vectors $X, Y \in T_pV^n$; (iii) $B(p, r)$ is the metric ball in (V^n, g) of radius r centered at p, and ω^n is the volume of the unit ball in Euclidean space. From these well–known expressions it follows that the sectional curvature measures the defect of the length of small circles $C(r)$ with respect to the Euclidean $2\pi r$. Similarly, the Ricci curvature measures the distortion of the solid angle, subtended by a small spherical sector in the direction $X = \exp_p^{-1}(q)$, with respect to the corresponding Euclidean value (Fig. A.6). Finally, the scalar curvature provides, as the radius varies, the distortion of the volume of small metric balls with respect to their Euclidean volume.

A.3 Some Properties of the Space of Riemannian Metrics

In the applications of Riemannian geometry discussed in these lecture notes we are not interested in a specific Riemannian metric g on V^n but rather in classes of metrics inducing on V^n the same geometric properties up to diffeomorphisms and rescalings [4]. This naturally calls into play the space of all smooth Riemannian metrics on V^n, the open convex cone (in the compact–open topology on $C^\infty(V^n, \otimes_S^2 T^*V^n)$) defined by (Fig. A.7)

$$\mathscr{M}et(V^n) := \left\{ g \in C^\infty(V^n, \otimes_S^2 T^*V^n) \middle|\ \text{g is positive definite} \right\}. \tag{A.32}$$

On $\mathscr{M}et(V^n)$ there is a natural action both of the group of diffeomorphisms $\mathrm{Diff}(V^n)$ and of the multiplicative group $\mathbb{R}^* := (\mathbb{R}_{>0}, \times)$ of positive real numbers. The former acting by pull–back

$$\begin{aligned} \mathrm{Diff}(V^n) \times \mathscr{M}et(V^n) &\longrightarrow \mathscr{M}et(V^n) \\ (\phi, g) &\longmapsto \phi^* g, \end{aligned} \tag{A.33}$$

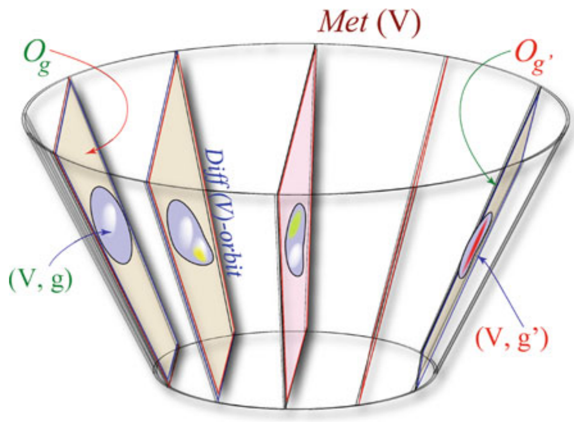

Fig. A.7 The ∞–dimensional cone of Riemannian metrics with its Diff(V^n) orbits $\mathscr{O}_g$

the latter acting by homotetic rescaling

$$\begin{aligned} \mathbb{R}^* \times \mathscr{M}et(V^n) &\longrightarrow \mathscr{M}et(V^n) \\ (\lambda,\, g) &\longmapsto \lambda\, g. \end{aligned} \tag{A.34}$$

The quotient space $\mathscr{M}et(V^n)/\mathrm{Diff}(V^n)$ under (A.33) is the space of smooth Riemannian structures on V^n describing Riemannian manifolds with the same geometric properties, whereas the quotient $\mathscr{M}et(V^n)/(\mathbb{R}^* \times \mathrm{Diff}(V^n))$, associated with the composition of Diff(V^n) with the homotheties (A.34), describes Riemannian manifolds with the same metric properties up to an overall rescaling. Note that by normalizing volume we can write

$$\frac{\mathscr{M}et(V^n)}{\mathbb{R}^* \times \mathrm{Diff}(V^n)} \simeq \left\{ g \in \mathscr{M}et(V^n) | \int_{V^n} d\mu_g = 1 \right\} \Big/ \mathrm{Diff}(V^n), \tag{A.35}$$

where $d\mu_g$ denotes the Riemannian measure on (V^n, g).

Most of our exploitation of the geometry of $\mathscr{M}et(V^n)$ is confined to its tangent space $\mathscr{T}_{(V^n,g)}\mathscr{M}et(V^n)$ at (V^n, g). Let us recall that an element of $\mathscr{T}_{(V^n,g)}\mathscr{M}et(V^n)$ is an equivalence class of curves of Riemannian metrics

$$\begin{aligned} (-\epsilon, \epsilon) &\longrightarrow \quad \mathscr{M}et(V^n) \\ \lambda \quad &\longmapsto g(\lambda)\,, \qquad \text{with } g(0) = g, \end{aligned} \tag{A.36}$$

where any two such a curve $\lambda \mapsto g_1(\lambda)$ and $\lambda \mapsto g_2(\lambda)$ are equivalent if they have the same velocity at $\lambda = 0$, i.e.

$$[g_1] \simeq [g_2] \tag{A.37}$$

$$\Leftrightarrow \left. \frac{dg_1(\lambda)}{d\lambda} \right|_{\lambda=0} = \left. \frac{dg_2(\lambda)}{d\lambda} \right|_{\lambda=0} = h \in C^\infty(V^n, \otimes_S^2 T^* V^n).$$

Hence we have the identification of the tangent space $\mathscr{T}_{(V^n,g)}\mathscr{M}et(V^n)$ with the space of sections $C^\infty(V^n, \otimes^2_S T^*V^n)$,

$$\mathscr{T}_{(V^n,g)}\mathscr{M}et(V^n) \simeq C^\infty(V^n, \otimes^2_S T^*V^n), \tag{A.38}$$

which serves as the local model for the infinite–dimensional (Frechét) manifold $\mathscr{M}et(V^n)$. To be somewhat informal, given $g \in \mathscr{M}et(V^n)$, the elements h of $\mathscr{T}_{(V^n,g)}\mathscr{M}et(V^n)$ can be interpreted as the infinitesimal variations of the given metric,

$$\begin{aligned} (-\epsilon,\epsilon) &\longrightarrow \quad \mathscr{M}et(V^n) \\ \lambda \quad &\longmapsto (g(\lambda))_{ik} = g_{ik} + \lambda\, h_{ik}\,, \quad 0 < \epsilon << 1. \end{aligned} \tag{A.39}$$

Note that $\mathscr{M}et(V^n)$, being an open cone in $C^\infty(V^n, \otimes^2_S T^*V^n)$, is contractible and the tangent bundle $\mathscr{T}\mathscr{M}et(V^n)$ of $\mathscr{M}et(V^n)$ is simply

$$\mathscr{T}\mathscr{M}et(V^n) = \mathscr{M}et(V^n) \times C^\infty(V^n, \otimes^2_S T^*V^n). \tag{A.40}$$

Applications of Riemannian geometry to quantum field theory often exploit the Berger–Ebin decomposition [3] of $\mathscr{T}_{(V^n,g)}\mathscr{M}et(V^n)$. This decomposition allows us to discriminate between trivial (infinitesimal) variations of a metric, associated with the action of the diffeomorphisms group, and essential variations describing a change in the Riemannian structure of (V^n, g) (in this latter case one speaks of (infinitesimal) deformations of the given g: see [4] for the proper use of the term deformation *vs.* variation). The construction of this decomposition is somewhat delicate and owing to its importance in the applications discussed in these lecture notes, it is worthwhile to describe the various steps involved in its proof.

To begin with, let us recall that both $\mathscr{M}et(V^n)$ and $\mathrm{Diff}(V^n)$ are locally modeled on Frechét spaces of smooth sections and maps, where the necessary tools from analysis on Banach spaces, (basically a Picard iteration scheme), are not easily available. We need to enlarge $\mathscr{M}et(V^n)$ and $\mathrm{Diff}(V^n)$ to their Sobolev counterparts. To this end we consider the open subset of $\mathscr{H}^s(V^n, \otimes^2 T^*V^n)$ defining the Riemannian metrics $\mathscr{M}et^{\,s}(V^n)$ of Sobolev class $s > \frac{n}{2}$, and the (topological) group, $\mathscr{D}^{\,s'}(V^n)$, defined by the set of diffeomorphisms which, as maps $V^n \to V^n$ are an open subset of the Sobolev space of maps $\mathscr{H}^{\,s'}(V^n, V^n)$, with $s' \geq s+1$, (see (A.10)). There is a natural projection map from $\mathscr{D}^{\,s'}(V^n)$ into $\mathscr{M}et^{\,s}(V^n)$

$$\begin{aligned} \pi : \mathscr{D}^{\,s'}(V^n) &\longrightarrow \quad \mathscr{O}^s_g \subset \mathscr{M}et^{\,s}(V^n) \\ \phi \quad &\longmapsto \pi(\phi) \doteq \phi^*\, g, \end{aligned} \tag{A.41}$$

where

$$\mathscr{O}^s_g := \left\{ \tilde{g} \in \mathscr{M}et^{\,s}(V^n) \,|\, \tilde{g} = \phi^*\, g,\, \phi \in \mathscr{D}^{\,s'}(V^n) \right\} \tag{A.42}$$

is the $\mathscr{D}^{s'}(V^n)$–orbit of a given metric $g \in \mathscr{M}et^{\,s}(V^n)$, and $\phi^*\, g$ is the pull–back under $\phi \in \mathscr{D}^{s'}(V^n), s' \geq s+1$. If $\mathscr{T}_{(V^n,g)}\mathscr{O}^{\,s}_g$ denotes the tangent space to any such an orbit, then $\mathscr{T}_{(V^n,g)}\mathscr{O}^{\,s}_g$ is the image of the operator

$$\begin{aligned} \delta^*_g : \mathscr{H}^{s+1}(V^n, TV^n) &\to \mathscr{H}^{\,s}(V^n, \otimes^2 T^* V^n) \\ w \quad &\mapsto \delta^*_g\,(w) \doteq \frac{1}{2}\,\mathscr{L}_w\, g, \end{aligned} \tag{A.43}$$

where $\mathscr{L}_w\, g$ denotes the Lie derivative of the metric tensor g along the vector field w,

$$(\mathscr{L}_w\, g)_{ik} \;=\; \nabla_i w_k \;+\; \nabla_k w_i \;=\; w^h \partial_h g_{ik} \;+\; g_{hk}\partial_i w^h \;+\; g_{ih}\partial_k w^h. \tag{A.44}$$

Its (total) symbol in the (co)direction $\xi \in T^*_p V^n$ is provided by the injective bundle homomorphism

$$\begin{aligned} V^n_\xi[\delta^*_g] \;:\; T^*_p V^n \;&\longrightarrow\; \otimes^2_S T^*_p V^n \\ X_i\, dx^i \;\longmapsto\; V^n_\xi[\delta^*_g](X) \;&=\; \frac{1}{2}\;(\xi_h X_k + \xi_k X_h)\, dx^h \otimes dx^k. \end{aligned} \tag{A.45}$$

Note that the L^2 adjoint δ_g of δ^*_g, with respect to the inner product (A.3), is provided by (minus) the divergence operator

$$\begin{aligned} \delta_g : \mathscr{H}^{\,s}(V^n, \otimes^2 T^*\, V^n) \to &\quad \mathscr{H}^{s-1}(V^n, T^*\, V^n) \\ h \qquad\qquad &\mapsto \delta_g\, h \doteq -\, g^{ij}\, \nabla_i h_{jk}\, dx^k, \end{aligned} \tag{A.46}$$

with total symbol given by

$$\begin{aligned} V^n_\xi[\delta_g] \;:\; \otimes^2_S T^*_p V^n \;&\longrightarrow\; T^*_p V^n \\ w_{hk}\, dx^h \otimes dx^k \;\longmapsto\; V^n_\xi[\delta_g](w) \;&=\; -\, g^{ih}\xi_i w_{hk} dx^k. \end{aligned} \tag{A.47}$$

Since δ^*_g has injective symbol, Banach space theory implies that $\delta_g \circ \delta^*_g$ is an elliptic operator [3]. To expand in more detail, we compute the symbol of the composition $\delta_g \circ \delta^*_g$ according to

$$\begin{aligned} V^n_\xi[\delta_g \circ \delta^*_g] \;\; &: T^* V^n \;\longrightarrow\; \otimes^2_S T^* V^n \;\longrightarrow\; T^* V^n \\ X_i\, dx^i \;&\longmapsto\; V^n_\xi[\delta^*_g](X) \;\longmapsto\; V^n_\xi[\delta_g] \circ V^n_\xi[\delta^*_g](X) \\ &=\; V^n_\xi[\delta_g] \Big(V^n_\xi[\delta^*_g](X) \Big) \;=\; -\,\frac{1}{2}\,\big(|\xi|^2_g \delta^i_k \;+\; \xi^i \xi_k\big)\, X_i\, dx^k, \end{aligned} \tag{A.48}$$

and the ellipticity of the operator $\delta_g \circ \delta^*_g$ immediately follows by observing that the matrix $\big(|\xi|^2_g \delta^i_k \;+\; \xi^i \xi_k\big)$ is non–singular with positive eigenvalues. By standard

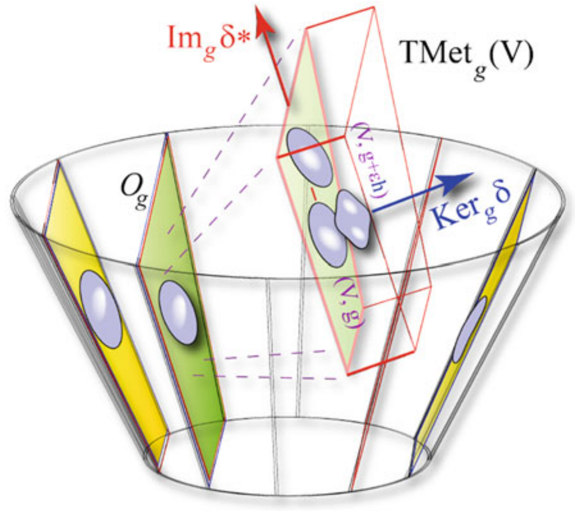

Fig. A.8 The geometry of the Berger–Ebin decomposition of $\mathscr{T}_{(V^n,g)}\mathscr{M}et(V^n)$, the tangent space to $\mathscr{M}et(V^n)$ at the metric g

elliptic theory the L^2–orthogonal subspace to Im δ_g^* in $\mathscr{T}_{(V^n,g)}\mathscr{M}et^{\,s}(V^n)$ is spanned by the (infinite–dimensional) kernel of δ_g. This entails the well–known Berger–Ebin $L^2(V^n, d\mu_g)$–orthogonal splitting [3, 8, 21] of the tangent space $\mathscr{T}_{(V^n,g)}\mathscr{M}et^s(V^n)$ (Fig. A.8),

$$\mathscr{T}_{(V^n,g)}\mathscr{M}et^s(V^n) \cong \left[\mathscr{T}_{(V^n,g)}\mathscr{M}et^s(V^n) \cap \ker\, \delta_g\right] \oplus \mathrm{Im}\, \delta_g^* \left[\mathscr{H}^{s+1}(V^n, TV^n)\right] \tag{A.49}$$

$$\cong \left[\mathscr{T}_{(V^n,g)}\mathscr{M}et^s(V^n) \cap \ker\, \delta_g\right] \oplus \mathscr{T}_{(V^n,g)}\mathscr{O}_g^s,$$

according to which, for any given tensor $h \in \mathscr{T}_{(V^n,g)}\mathscr{M}et^s(V^n)$, we can write

$$h_{ab} = h_{ab}^T + \mathscr{L}_w\, g_{ab}, \tag{A.50}$$

where h_{ab}^T denotes the div–free part of h, $(\nabla^a h_{ab}^T = 0)$, and where the vector field w is characterized as the solution, (unique up to the Killing vectors of (V^n, g)), of the elliptic PDE

$$\delta_g\, \delta_g^*\, w \;=\; \delta_g\, h. \tag{A.51}$$

We can also take into account the role of the scaling transformation (A.11) and refine (A.49) to the tangent space $\mathscr{T}_{(V^n,g)}\mathscr{M}et_1^s(V^n)$ to the space of metrics $\mathscr{M}et_1^s(V^n) \;:=\; \left\{g \in \mathscr{M}et^s(V^n) \middle|\; \int_V^n d\mu_g \;=\; 1\right\}$ of normalized total volume (cf. (A.35)). Since

$$\mathscr{T}_{(V^n,g)}\mathscr{M}et_1^s(V^n) \;=\; \left\{h \;\in\; \mathscr{H}^s(V^n, \otimes_S^2 T^*V^n) \middle|\; \int_V^n \mathrm{tr}_g\, h\, d\mu_g = 0\right\}, \tag{A.52}$$

where $tr_g\, h \;:=\; g^{ik}\, h_{ik}$, we have

$$\mathscr{T}_{(V^n,g)}\mathscr{M}et_1^s(V^n) \cong \left[\mathscr{T}_{(V^n,g)}\mathscr{M}et_1^s(V^n) \cap \ker\, \delta_g\right] \oplus \mathrm{Im}\, \delta_g^* \left[\mathscr{H}^{s+1}(V^n, TV^n)\right] \tag{A.53}$$

$$\cong \left[\mathscr{T}_{(V^n,g)}\mathscr{M}et_1^s(V^n) \cap \ker\, \delta_g\right] \;\oplus\; \mathscr{T}_{(V^n,g)}\mathscr{O}_g^s.$$

A.4 Ricci Curvature

In applications of Riemannian geometry to the quantum geometry of polyhedral surface (and its interplay with Ricci flow), Ricci curvature plays a central role. A deep insight in its geometric meaning is provided by the expression of its components in local harmonic coordinates $(U, \{x^i\})$. These latter are defined by requiring that each coordinate function $x^k \;:\; U \;\longrightarrow\; \mathbb{R}$ is harmonic. In particular, local solvability for elliptic PDE's implies, (compare with [7], Corollary 3.30), that for any given point $p \in (V^n, g)$ there always exists a neighborhood $U_p \subset V^n$ of p such that, (see (A.20)),

$$\triangle_g\, x^i \;=\; g^{jk}\left(\frac{\partial^2}{\partial x^j \partial x^k} \;-\; \Gamma^{\ell}_{jk}\frac{\partial}{\partial x^{\ell}}\right) x^i \;=\; -\Gamma^i \;=\; 0. \tag{A.54}$$

When passing from normal geodesic coordinates to harmonic coordinates we gain control on the components of the metric tensor in terms of the Ricci curvature rather than of the full Riemann tensor (cf. [14] for a particularly clear comparison of these two coordinate systems from the point of view of geometric analysis). In particular, as first stressed by C. Lanczos [20], in harmonic coordinates the components of the Ricci tensor take the suggestive form, (see e.g. [27] Lemma 2.6 for an elegant computation),

$$\mathrm{R}_{ik} \underset{(harmonic)}{=} -\frac{1}{2}\,\triangle_g\, (g_{ik}) \;+\; Q_{ik}\left(g^{-1},\, \partial g\right), \tag{A.55}$$

where for each fixed pair of indexes (i, k), the expression $\triangle_g\, (g_{ik})$ denotes the Laplace–Beltrami operator applied component wise to each g_{ik} as it were a *scalar function*, (in particular $\triangle_g g_{ik}$ is not the tensorial (rough) Laplacian $g^{ab}\nabla_a \nabla_b\, g_{ik}$ of the metric tensor g, a quantity that obviously vanishes identically for the Levi–Civita connection). In (A.55) $Q\left(g^{-1},\, \partial g\right)$ denotes a sum of terms quadratic in the components of g, g^{-1} and their first derivatives, and whose explicit expression does not concern us here (compare with [27]). Hence in harmonic coordinates the Ricci curvature acts as a quasi–linear elliptic operator on the components of the metric, an observation that can be exploited to prove [9] that the metric tensor g, if not smooth, has maximal regularity in harmonic coordinates. The expression (A.55) also plays an inspiring role in Ricci flow theory, where the minus sign in front of

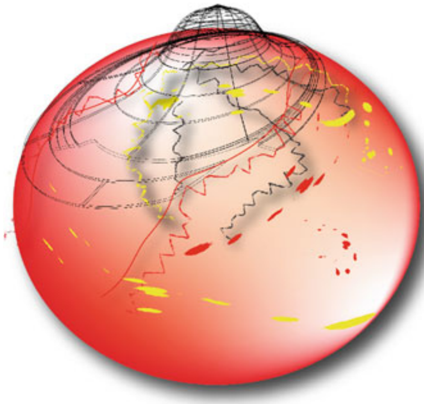

Fig. A.9 The Ricci curvature in harmonic coordinates as the generator of Brownian diffusion on the manifold (V^n, g)

the scalar Laplace–Beltrami $\triangle_g\ (g_{ik})$ suggests the right PDE candidate for the Ricci flow among the many (apparently) possible ones. It is also worthwhile to observe that the factor $1/2$ appearing in (A.55), motivating the conventional "2" in the Ricci flow generator $-2\,\mathrm{Ric}(g)$, is not as incidental as is typically assumed. This is related to the fact that $\frac{1}{2}\triangle_g$ and not $\triangle_g$ is the generator of Brownian motion on (V^n, g) [15] (Fig. A.9).

Appendix B
A Capsule of Moduli Space Theory

In this Appendix we summarize notation and a few basic definitions of Riemann Moduli theory that we have been freely using in these lecture notes, (for details we mainly refer to [12, 13, 16]).

B.1 Riemann Surfaces with Marked Points and Divisors

We shall denote by $(M_g, \mathscr{C})$ a Riemann surface of genus g, (most often we simply write $(M, \mathscr{C})$ if the genus is clear from the context). Recall that $(M, \mathscr{C})$ is characterized by an atlas of local coordinate charts (U_k, φ_k) defined by maps $\varphi_k : U_k \to \mathbb{C}$ whose transition functions, $\varphi_h \circ \varphi_k^{-1} : \varphi_k(U_k \cap U_h) \to \varphi_h(U_h \cap U_k)$, are holomorphic maps between open subsets of $\mathbb{C}$. Any such a chart (U, φ) is said to provide a *local conformal parameter*, and for the generic $p \in U$ one sets $\varphi(p) := z = x + \sqrt{-1}\, y$. A N_0–*pointed surface* $((M; N_0), \mathscr{C})$, or surface with N_0 marked points, is an oriented closed, (connected), surface of genus g decorated with a distinguished set of N_0 pairwise distinct points $\{p_1, \ldots, p_{N_0}\}$. (Note that $((M; N_0), \mathscr{C})$ is distinct from the open Riemann surface $(M', \mathscr{C}) := (M, \mathscr{C}) \setminus \{p_1, \ldots, p_{N_0}\}$ obtained by removing from $(M, \mathscr{C})$ the points $\{p_1, \ldots, p_{N_0}\}$) (Fig. B.1).

The tangent and cotangent spaces at $p \in (M, \mathscr{C})$ are naturally obtained by tensoring with $\mathbb{C}$ the tangent T_pM and cotangent space T_pM^* of the underlying real surface M, i.e., $T_{\mathbb{C},p}M := T_pM \otimes \mathbb{C}$ and $T_{\mathbb{C},p}M^* := T_pM^* \otimes \mathbb{C}$. The respective basis induced by the local conformal parameter z are provided by the usual expressions

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\,\frac{\partial}{\partial y}\right), \tag{B.1}$$

$$d\,z := dx + \sqrt{-1}\,dy\,, \quad d\,\overline{z} := dx - \sqrt{-1}\,dy, \tag{B.2}$$

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https://doi.org/10.1007/978-3-319-67937-2

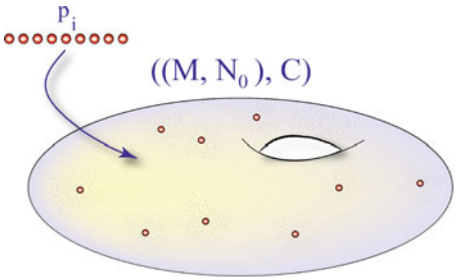

Fig. B.1 A N_0–pointed Riemann surface $((M, N_0), \mathscr{C})$ of genus g obtained by injecting a string of N_0 pairwise distinct points in $(M, \mathscr{C})$. Here $g = 1$ and $N_0 = 9$, we typically assume that $2g - 2 + N_0 > 0$, in such a case the automorphism group, $Aut\,(M, N_0)$, of the resulting pointed Riemann surface is finite. Recall that $Aut\,(M, N_0)$ is the largest group of conformal automorphisms that the Riemann surface $((M, N_0), \mathscr{C})$ can admit

$$dx \wedge dy = \frac{\sqrt{-1}}{2}\, dz \wedge d\bar{z}. \tag{B.3}$$

One can naturally split $T_{\mathbb{C},p}M$ into the holomorphic $T'_pM := \mathbb{C}\{\partial/\partial z\}$ and antiholomorphic $T''_pM := \mathbb{C}\{\partial/\partial\bar{z}\}$ tangent spaces at $p \in (M, \mathscr{C})$ according to $T_{\mathbb{C},p}M = T'_pM \oplus T''_pM$, with $T''_pM = \overline{T'_pM}$, (where the overline $\overline{\cdot}$ denotes complex conjugation). In this connection it is worthwhile recalling that a smooth map $f : M \to N$ between two surfaces M and N is holomorphic if and only if the corresponding tangent map $f_* : T_{\mathbb{C},p}M \longrightarrow T_{\mathbb{C},f(p)}N$ is such that $f_*\left(T'_pM\right) \subset T'_{f(p)}N$. This implies that there is a linear isomorphism between T_pM and the holomorphic tangent space $T'_pM = \mathbb{C}\{\partial/\partial z\}$, an isomorphism, this latter, which provides a natural dictionary between geometrical quantities on the surface M and their corresponding realizations on $(M, \mathscr{C})$. Holomorphic maps also preserve the holomorphic–antiholomorphic decomposition of the cotangent space $T_{\mathbb{C},p}M^* = T'_pM^* \oplus T''_pM*$, and the corresponding splitting $\Omega(M) = \bigoplus_{p+q\leq 2} \Omega^{(p,q)}(M)$ of the space of differential forms on $(M, \mathscr{C})$. Explicitly, the spaces $\Omega^{(1,0)}(M)$, $\Omega^{(0,1)}(M)$, and $\Omega^{(1,1)}(M)$ of forms of type (p, q) are locally generated by the monomials $\varphi(z)\,dz$, $\varphi(z)\,d\bar{z}$, and $\varphi(z)\,dz \wedge d\bar{z}$, respectively. Corresponding to such a splitting we have the Dolbeault operators $\partial : \Omega^{(p,q)}(M) \to \Omega^{(p+1,q)}(M)$ and $\overline{\partial} : \Omega^{(p,q)}(M) \to \Omega^{(p,q+1)}(M)$, with $\partial\partial = 0$, $\overline{\partial}\,\overline{\partial} = 0$, $\partial\overline{\partial} = -\overline{\partial}\partial$, and $d = \partial + \overline{\partial}$, where d denotes the exterior derivative. Locally,

$$\partial := \frac{\partial}{\partial z}\, dz\,, \quad \overline{\partial} := \frac{\partial}{\partial \bar{z}}\, d\bar{z}, \tag{B.4}$$

In particular, a form $\psi \in \Omega^{(p,0)}$ is holomorphic if $\overline{\partial}\,\psi = 0$. A metric on a Riemann surface $(M, \mathscr{C})$ is conformal if locally it can be written as

$$ds^2 = h^2(z)\, dz \otimes d\bar{z}, \tag{B.5}$$

for some smooth function $h(z) > 0$. The corresponding Kähler form is given by

$$\varpi := \frac{\sqrt{-1}}{2} h^2(z)\, dz \wedge d\bar{z}. \tag{B.6}$$

The Gaussian curvature of the (smooth) metric $ds^2 = h^2(z)\, dz \otimes d\bar{z}$ is provided by

$$K := -\Delta_{ds^2} \ln h, \tag{B.7}$$

where Δ_{ds^2}, (Δ if is clear from the context which conformal metric we are using), is the Laplace–Beltrami operator with respect to the metric ds^2, *viz.*

$$\Delta := \frac{1}{h^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{4}{h^2(z)} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}. \tag{B.8}$$

Let us recall, (see e.g. the very readable presentation in [13]), that a divisor D on a Riemann surface $(M, \mathscr{C})$ is a formal linear combination

$$D = \sum n_k\, z_k \tag{B.9}$$

where $n_k \in \mathbb{Z}$ and the z_k are points $\in M$. D is required to be locally finite, namely for any $q \in M$ there is a neighborhood $U_q \subset M$ of q such that $U_q \cap \{z_k\}$ contains only a finite number of the $\{z_k\}$ appearing in D. Since we typically deal with compact Riemann surfaces $(M, \mathscr{C})$, this implies that the formal sum defining D is finite. The set of divisors on $(M, \mathscr{C})$ naturally forms an additive (abelian) group $\mathscr{D}iv(M)$. The degree of a divisor $D = \sum n_k\, z_k$, (on a compact surface Riemann surface $(M, \mathscr{C})$), is defined according to

$$deg\, D := \sum n_k. \tag{B.10}$$

Let f be a meromorphic function, i.e. the local quotient of two holomorphic functions on $(M, \mathscr{C})$. Let $k \in \mathbb{N}$, if f has a zero of order k at $z \in (M, \mathscr{C})$, define the order of f at z to be $ord_z f := k (> 0)$, conversely, if f has a pole of order k at $z \in (M, \mathscr{C})$ we define $ord_z f := -k$. The order of f at all other points is defined to be 0. The divisor of a meromorphic function f, $f \not\equiv 0$, is conventionally denoted by (f), and defined as

$$(f) := \sum (ord_{z_h} f)\, z_h, \tag{B.11}$$

where the sum is over all zeros and all poles of f. A divisor D is called a principal divisor if it is the divisor (f) of a meromorphic function $f \not\equiv 0$. Since a meromorphic function f on $(M, \mathscr{C})$ has the same number of zeros and poles (counted with multiplicity) it immediately follows that the degree of its divisor (f) satisfies

$$deg\, (f) = 0. \tag{B.12}$$

More generally, if M^n is a (not necessarily compact) complex manifold of dimension n, a divisor can be naturally associated with hypersurfaces (i.e. $n-1$–dimensional submanifolds) of M^n. Explicitly, if V_i are irreducible analytic hypersurfaces of M^n, (i.e., for each point $p \in V_i \subset M^n$, V_i can be given in a neighborhood of p as the zeros of a single holomorphic function f, and it cannot be written as $V_i = \widetilde{V}_1 \cup \widetilde{V}_2$, where $\widetilde{V}_\alpha$, $\alpha = 1, 2$ are analytic hypersurfaces of M^n), then a divisor on M^n is a locally finite formal linear combination of irreducible analytic hypersurfaces of M^n

$$D = \sum_{V_k} n_k \, V_k. \tag{B.13}$$

Note that given any such a divisor, we can always find an open cover $\{U_\alpha\}$ of M^n such that in U_α, with $U_\alpha \cap V_i \neq \emptyset$, the hypersurface V_i has a local defining equation $V_i = \{g_{i\alpha}(z) = 0\}$ where $g_{i\alpha}(z)$ is a holomorphic function in U_α. In each U_α the set of $\{g_{i\alpha}(z)\}$, associated with all V_i for which $U_\alpha \cap V_i \neq \emptyset$, characterizes a meromorphic function f_α according to

$$f_\alpha := \prod_i g_{i\alpha}^{n_i}, \tag{B.14}$$

which are the local defining functions for the divisor D. The set of local defining functions $\{f_\alpha\}$ associated with an open covering $\{U_\alpha\}$ of M^n can be also used for characterizing the line bundle $[D]$ associated with the given divisor D, this is the bundle over M^n defined by the transition functions $\{\psi_{\alpha\beta} := f_\alpha / f_\beta\}$ for any $U_\alpha \cap U_\beta \neq \emptyset$. Note that the line bundle $[D]$ is trivial if and only if D is the divisor of a meromorphic function.

Recall that if $H^{2k}_{DR}(M^n, \mathbb{R})$ denotes the $2\,k$–th DeRham cohomology group of M^n, and $V \subset M^n$ is an analytic subvariety of (complex) dimension k, then the fundamental class (V) in the homology group $H_{2k}(M^n, \mathbb{R})$ is defined by the pairing

$$\begin{aligned} H^{2k}_{DR}(M^n, \mathbb{R}) \times H_{2k}(M^n, \mathbb{R}) &\longrightarrow \mathbb{R} \\ \eta, \ (V) \qquad\qquad &\longmapsto \int_V \eta. \end{aligned} \tag{B.15}$$

By Poincaré duality this determines the fundamental class $\eta_V \in H^{2n-2k}_{DR}(M^n, \mathbb{R})$ of V. Thus, given a divisor $D = \sum_{V_k} n_k \, V_k$, we can introduce its fundamental class (D) and its Poincaré dual $\eta_D \in H^2_{DR}(M^n)$ according to

$$(D) := \sum n_k \, (V_k) \,, \tag{B.16}$$

$$\eta_D := \sum n_k \, \eta_{V_k}. \tag{B.17}$$

Given a line bundle L over M^n, locally defined by transition functions $\{\psi_{\alpha\beta}\}$ relative to a covering $\{U_\alpha\}$ of M^n, let

$$c_1(L) = \frac{\sqrt{-1}}{2\pi}\, \Xi, \tag{B.18}$$

be the (first) Chern class of L, where $\Xi|_{U_\alpha} = d\,\omega_\alpha$ denotes the curvature of L associated with any locally given connection 1–form ω_α. A basic result connecting line bundles, divisors and Chern classes is the observation that if L is the line bundle associated with a divisor D, then, (see [13] for a nicely commented proof),

$$c_1(L) = c_1([D]) = \eta_D \in H^2_{DR}(M^n). \tag{B.19}$$

B.2 The Teichmüller Space $\mathfrak{T}_{g,N_0}(M)$

In discussing the connection between polyhedral surfaces and Riemann surfaces we are naturally led to consider the relation between the space $POL_{g,N_0}(M)$ and $\mathfrak{M}_{g,N_0}$, the moduli space of N_0–pointed Riemann surfaces of genus g. This latter features as a basic object of study in a large variety of mathematical and physical applications of Riemann surface theory, and as such it is susceptible of many possible characterizations. From our perspective it is appropriate to adopt a Riemannian geometry viewpoint and define $\mathfrak{M}_{g,N_0}$ as a suitable quotient of the space of conformal classes of riemannian metrics the surface M can carry. For details and proofs we refer to [25].

Let us consider the set of all (smooth) Riemannian metrics on the genus–g surface M, (see Sect. 1.6),

$$\mathcal{M}et(M) \doteq \{g \in S_2(M)|\; g(x)(u,u) > 0 \; if \; u \neq 0\}\,, \tag{B.20}$$

where $S_2(M)$ is the space of symmetric bilinear forms on M, and let us denote by $C^\infty(M,\, \mathbb{R}_+)$ the group of positive smooth functions acting on metrics $g \in \mathcal{M}et(M)$ by pointwise multiplications. This action is free, smooth (and proper), and the quotient

$$\mathcal{C}onf(M) := \frac{\mathcal{M}et(M)}{C^\infty(M,\, \mathbb{R}_+)}, \tag{B.21}$$

is the Fréchet manifold of conformal structures. Note that $\mathcal{C}onf(M)$ naturally extends to the pointed surface $(M; N_0)$, obtained by decorating M with N_0 marked points $\{p_1, \ldots, N_0\}$, as long as the conformal class contains a smooth representative metric [29]. Let

$$\mathcal{M}et_{-1}(M; N_0) \hookrightarrow \mathcal{M}et\,(M; N_0) \tag{B.22}$$

be the set of metrics of constant curvature -1, describing the hyperbolic structures on $(M; N_0)$. If $\mathscr{D}iff_+(M)$ is the group of all orientation preserving diffeomorphisms then

$$\mathscr{D}iff_+(M; N_0) = \left\{ \psi \in Diff_+(M) : \psi \text{ preserves setwise } \{p_i\}_{i=1}^{N_0} \right\} \tag{B.23}$$

acts by pull-back on the metrics in $\mathscr{M}et_{-1}(M; N_0)$. Let $\mathscr{D}iff_0(M; N_0)$ be the subgroup consisting of diffeomorphisms which when restricted to $(M; N_0)$ are isotopic to the identity, then the Teichmüller space $\mathfrak{T}_{g,N_0}(M)$ associated with the genus g surface with N_0 marked points M is defined by

$$\mathfrak{T}_{g,N_0}(M) = \frac{\mathscr{M}et_{-1}(M; N_0)}{\mathscr{D}iff_0(M; N_0)}. \tag{B.24}$$

From a complex function theory perspective, $\mathfrak{T}_{g,N_0}(M)$ is characterized by fixing a reference complex structure $\mathscr{C}_0$ on $(M; N_0)$ (a marking) and considering the set of equivalence classes of complex structures $(\mathscr{C}, f)$ where $f : \mathscr{C}_0 \to \mathscr{C}$ is an orientation preserving quasi-conformal map, and where any two pairs of complex structures $(\mathscr{C}_1, f_1)$ and $(\mathscr{C}_2, f_2)$ are considered equivalent if $h \circ f_1$ is homotopic to f_2 via a conformal map $h : \mathscr{C}_1 \to \mathscr{C}_2$. Let us assume that the reference complex structure $\mathscr{C}_0$ admits an antiholomorophic reflection $j : \mathscr{C}_0 \to \mathscr{C}_0$. Since any orientation reversing diffeomorphism $\widetilde{\varphi}$ can be written as $\varphi \circ j$ for some orientation preserving diffeomorphism φ, the (extended) mapping class group

$$\mathfrak{Map}(M; N_0) \doteq \mathscr{D}iff(M; N_0)/\mathscr{D}iff_0(M; N_0) \tag{B.25}$$

acts naturally on $\mathfrak{T}_{g,N_0}(M)$ according to

$$\mathfrak{Map}(M; N_0) \times \mathfrak{T}_{g,N_0}(M) \longrightarrow \mathfrak{T}_{g,N_0}(M) \tag{B.26}$$

$$\begin{cases} (\varphi, (\mathscr{C}, f)) \longmapsto (\mathscr{C}, f \circ \varphi^{-1}), & \varphi \in \mathscr{D}iff_+(M; N_0) \\ (\widetilde{\varphi}, (\mathscr{C}, f)) \longmapsto (\mathscr{C}^*, f \circ j \circ \varphi^{-1}) & \widetilde{\varphi} \in \mathscr{D}iff(M; N_0) - \mathscr{D}iff_+(M; N_0) \end{cases},$$

where the conjugate surface $\mathscr{C}^*$ is the Riemann surface locally described by the complex conjugate coordinate charts associated with $\mathscr{C}$. It follows that the Teichmüller space $\mathfrak{T}_{g,N_0}(M)$ can be also seen as the universal cover of the moduli space $\mathfrak{M}_{g,N_0}$ of genus g Riemann surfaces with $N_0(T)$ marked points defined by

$$\mathfrak{M}_{g,N_0} = \frac{\mathfrak{T}_{g,N_0}(M)}{\mathfrak{Map}(M; N_0)} \tag{B.27}$$

This characterization represents the moduli associated to $((M; N_0), \mathscr{C})$ by an equivalence class of Riemannian metrics $[ds^2]$ on M, where two metrics $ds^2_{(1)}$ and $ds^2_{(2)}$ define the same point $[ds^2]$ of $\mathfrak{M}_{g,N_0}$ if and only if there exists $f \in C^\infty(M, \mathbb{R}_+)$

and an orientation preserving diffeomorphism $\psi \in \mathcal{D}iff_{g,N_0}$ fixing each marked point p_k individually such that $ds^2_{(2)} = f\,\Psi^*\left(ds^2_{(1)}\right)$. According to the remark above, the class $[ds^2]$ may contain singular metrics provided that they are conformal to a smooth one. As emphasized by M. Troyanov, this implies that we can represent $[ds^2] \in \mathfrak{M}_{g,N_0}$ by conical metrics, as well.

For a genus g Riemann surfaces with $N_0(T) > 3$ marked points the complex vector space $Q_{N_0}(M)$ of (holomorphic) quadratic differentials is defined by tensor fields ψ described, in a locally uniformizing complex coordinate chart (U, ζ), by a holomorphic function $\mu : U \to \mathbb{C}$ such that $\psi = \mu(\zeta)d\zeta \otimes d\zeta$. Away from the discrete set of the zeros of ψ, we can locally choose a canonical conformal coordinate $z_{(\psi)}$ (unique up to $z_{(\psi)} \mapsto \pm z_{(\psi)} + const$) by integrating the holomorphic 1-form $\sqrt{\psi}$, i.e.,

$$z_{(\psi)} = \int^{\zeta} \sqrt{\mu(\zeta')d\zeta' \otimes d\zeta'}, \tag{B.28}$$

so that $\psi = dz_{(\psi)} \otimes dz_{(\psi)}$. If we endow $Q_{N_0}(M)$ with the L^1-(Teichmüller) norm

$$||\psi|| \doteq \int_M |\psi|, \tag{B.29}$$

then the Banach space of integrable quadratic differentials on M,

$$Q_{N_0}(M) \doteq \{\psi|,\ ||\psi|| < +\infty\}, \tag{B.30}$$

is non-empty and consists of meromorphic quadratic differentials whose only singularities are (at worst) simple poles at the N_0 distinguished points of (M, N_0). $Q_{N_0}(M)$ is finite dimensional and, (according to the Riemann-Roch theorem), it has complex dimension

$$\dim_{\mathbb{C}} Q_{N_0}(M) = 3g - 3 + N_0(T), \tag{B.31}$$

From the viewpoint of Riemannian geometry, a quadratic differential is basically a transverse-traceless two tensor deforming a Riemannian structure to a nearby inequivalent Riemannian structure. Thus a quadratic differential $\psi = \mu(\zeta)d\zeta \otimes d\zeta$ also encodes information on possible deformations of the given complex structure. Explicitly, by performing an affine transformation with constant dilatation $K > 1$, one defines a new uniformizing variable $z'_{(\psi)}$ associated with $\psi = \mu(\zeta)d\zeta \otimes d\zeta$ by deforming the variable $z_{(\psi)}$ defined by (B.28) according to

$$z'_{(\psi)} = KRe(z_{(\psi)}) + \sqrt{-1}Im(z_{(\psi)}). \tag{B.32}$$

The new metric associated with such a deformation is provided by

$$\left|dz'_{(\psi)}\right|^2 = \frac{(K+1)^2}{4}\left|dz_{(\psi)} + \frac{K-1}{K+1}d\bar{z}_{(\psi)}\right|^2. \tag{B.33}$$

Since $dz^2_{(\psi)}$ is the given quadratic differential $\psi = \mu(\zeta)d\zeta \otimes d\zeta$, we can equivalently write $\left|dz'_{(\psi)}\right|^2$ as

$$\left|dz'_{(\psi)}\right|^2 = \frac{(K+1)^2}{4}\,|\mu|\left|d\zeta + \frac{K-1}{K+1}\left(\frac{\overline{\mu}}{|\mu|^{1/2}}\right)d\overline{\zeta}\right|^2, \tag{B.34}$$

where $(\overline{\mu}/\,|\mu|)\; d\overline{\zeta} \otimes d\zeta^{-1}$ is the (Teichmüller-) Beltrami form associated with the quadratic differential ψ. If we consider quadratic differentials $\psi = \mu(\zeta)d\zeta \otimes d\zeta$ in the open unit ball $Q^{(1)}_{N_0}(M) \doteq \{\psi|,\ ||\mu(\zeta)|| < 1\}$ in the Teichmüller norm $\tilde{(}$B.29), then there is a natural choice for the constant K provided by

$$K = \frac{1 + ||\mu(\zeta)||}{1 - ||\mu(\zeta)||}. \tag{B.35}$$

In this latter case we get

$$\left|dz'_{(\psi)}\right|^2 = \frac{||\mu(\zeta)||^2}{(||\mu(\zeta)||-1)^2}\,|\mu|\left|d\zeta + \frac{1}{||\mu(\zeta)||}\left(\frac{\overline{\mu}}{|\mu|^{1/2}}\right)d\overline{\zeta}\right|^2. \tag{B.36}$$

According to Teichmüller's existence theorem any complex structure on can be parametrized by the metrics (B.36) as $\psi = \mu(\zeta)d\zeta \otimes d\zeta$ varies in $Q^{(1)}_{N_0}(M)$. This is equivalent to saying that for any given (M, g), (with $(M, N_0; [g])$ defining a reference complex structure $\mathscr{C}_0$ on (M, N_0)), and any diffeomorphism $f \in \mathscr{D}iff_0(M; N_0)$ mapping (M, g) into[1] (M, g_1), there is a quadratic differential $\psi \in Q^{(1)}_{N_0}(M)$ and a biholomorphic map $F \in \mathscr{D}iff_0(M; N_0)$, homotopic to f such that $[F^* g_1]$ is given by the conformal class associated with (B.36). This is the familiar point of view which allows to identify Teichmüller space with the open unit ball $Q^{(1)}_{N_0}(M)$ in the space of quadratic differentials $Q_{N_0}(M)$. It is also worthwhile noticing that (B.36) allows us to consider the open unit ball $Q^{(1)}_{N_0}(M)$ in the space of quadratic differential as providing a slice for the combined action of $\mathscr{D}iff_0(M; N_0)$ and of the conformal group $\mathscr{C}onf^s(M; N_0)$ on the space of Riemannian metrics $\mathscr{M}et(M; N_0)$, i.e.

$$Q^{(1)}_{N_0}(M) \hookrightarrow \frac{\mathscr{M}et(M; N_0)}{\mathscr{D}iff_0(M; N_0)} \simeq \mathscr{C}onf^s(M; N_0) \times \mathfrak{T}_{g,N_0}(M) \tag{B.37}$$

$$\left[\left|dz'_{\psi}\right|^2\right] \longmapsto \frac{||\mu||^2}{(||\mu||-1)^2}\,|\mu| \cdot \left|d\zeta + \frac{1}{||\mu||}\left(\frac{\overline{\mu}}{|\mu|^{1/2}}\right)d\overline{\zeta}\right|^2,$$

where

$$\mathscr{C}onf^s(M; N_0) \doteq \left\{f : M \to \mathbb{R}^+ \,|\, f \in H^s(M, \mathbb{R})\right\} \tag{B.38}$$

[1]With $(M, N_0; [g_1])$ a complex structure distinct from $(M, N_0; [g])$.

denotes the (Weyl) space of conformal factors defined by of all positive (real valued) functions on $M, f \in H^s(M, \mathbb{R})$, whose derivatives up to the order s exist in the sense of distributions and are represented by square integrable functions.

In line with the above remarks, one introduces also the space $B_{N_0}(M)$ of (L^∞ measurable) Beltrami differentials, i.e. of tensor fields $\varpi = \nu(\zeta)d\overline{\zeta} \otimes d\zeta^{-1}$, sections of $k^{-1} \otimes \overline{k}$, ($k$ being the holomorphic cotangent bundle to M), with $\sup_M |\nu(\zeta)| < \infty$. The space of Beltrami differentials is naturally identified with the tangent space to $\mathfrak{T}_{g,N_0}(M)$, i.e.,

$$\varpi = \nu(\zeta)d\overline{\zeta} \otimes d\zeta^{-1} \in T_{\mathscr{C}}\mathfrak{T}_{g,N_0}(M), \tag{B.39}$$

(with $\mathscr{C}$ a complex structure in $\mathfrak{T}_{g,N_0}(M)$). The two spaces $Q_{N_0}(M)$ and $B_{N_0}(M)$ can be naturally paired according to

$$\langle \psi | \varpi \rangle = \int_M \mu(\zeta)\nu(\zeta)d\zeta d\overline{\zeta}. \tag{B.40}$$

In such a sense $Q_{N_0}(M)$ is $\mathbb{C}$-anti-linear isomorphic to $T_{\mathscr{C}}\mathfrak{T}_{g,N_0}(M)$, and can be canonically identified with the cotangent space $T^*_{\mathscr{C}}\mathfrak{T}_{g,N_0}(M)$ to $\mathfrak{T}_{g,N_0}(M)$. On the cotangent bundle $T^*_{\mathscr{C}}\mathfrak{T}_{g,N_0}(M)$ we can define the Weil-Petersson metric as the inner product between quadratic differentials corresponding to the L^2- norm provided by

$$\|\psi\|^2_{WP} \doteq \int_M h^{-2}(\zeta)\, |\psi(\zeta)|^2\, |d\zeta|^2\,, \tag{B.41}$$

where $\psi \in Q_{N_0}(M)$ and $h(\zeta)\,|d\zeta|^2$ is the hyperbolic metric on M. Note that $\frac{\psi}{h}$ is a Beltrami differential on M, thus if we introduce a basis $\{\frac{\partial}{\partial \mu_\alpha}\}_{\alpha=1}^{3g-3+N_0}$ of the vector space of harmonic Beltrami differentials on (M, N_0), we can write

$$G_{\alpha\overline{\beta}} = \int_M \frac{\partial}{\partial \mu_\alpha}\, \frac{\partial}{\partial \overline{\mu}_\beta}\, h(\zeta)\, |d\zeta|^2 \tag{B.42}$$

for the components of the Weil-Petersson metric on the tangent space to $\mathfrak{T}_{g,N_0}(M)$. We can introduce the corresponding Weil-Petersson Kähler form according to

$$\omega_{WP} := \sqrt{-1} G_{\alpha\overline{\beta}} dZ^\alpha \wedge d\overline{Z}^\beta, \tag{B.43}$$

where $\{dZ^\alpha\}$ are the basis, in $Q_{N_0}(M)$, dual to $\{\frac{\partial}{\partial \mu_\alpha}\}$ under the pairing (B.40). Such Kähler potential is invariant under the mapping class group $\mathfrak{Map}(M; N_0)$ to the effect that the Weil-Petersson volume 2-form ω_{WP} on $\mathfrak{T}_{g,N_0}(M)$ descends on $\mathfrak{M}_{g,N_0}$, and it has a (differentiable) extension, in the sense of orbifold, to $\overline{\mathfrak{M}}_{g,N_0}$.

B.3 Some Properties of the Moduli Spaces $\mathfrak{M}_{g,N_0}$

It is well-known that the moduli space $\mathfrak{M}_{g,N_0}$ is a connected orbifold space of complex dimension $3g - 3 + N_0(T)$ and that, although in general non complete, it admits a stable compactification (Deligne-Mumford) $\overline{\mathfrak{M}}_{g,N_0}$. This latter is, by definition, the moduli space of stable N_0-pointed surfaces of genus g, where a stable surface is a compact Riemann surface with at most ordinary double points such that its parts are hyperbolic. Topologically, a stable pointed surface (or, equivalently a stable curve, in the complex sense), is obtained by considering a finite collection of embedded circles in $M' := M/\{p_1, \ldots p_{N_0}\}$, each in a distinct isotopy class relative to M', and such that none of these circles bound a disk in M containing at most one p_k. By contracting each such a circle, and keeping track of the marked points in such a way that any component of the resulting surface can support a hyperbolic metric, we get the stable pointed surface of genus g. Thus, the closure $\partial\mathfrak{M}_{g,N_0}$ of $\mathfrak{M}_{g,N_0}$ in $\overline{\mathfrak{M}}_{g,N_0}$ consists of stable surfaces with double points, and gives rise to a stratification decomposing $\overline{\mathfrak{M}}_{g,N_0}$ into subvarieties. By definition, a stratum of codimension k is the component of $\overline{\mathfrak{M}}_{g,N_0}$ parametrizing stable surfaces (of fixed topological type) with k double points. The orbifold $\overline{\mathfrak{M}}_{g,N_0}$ is endowed with $N_0(T)$ natural line bundles $\mathscr{L}_i$ defined by the cotangent space to M at the i-th marked point.

A basic observation in moduli space theory is the fact that any point p on a stable curve $((M; N_0), \mathscr{C}) \in \overline{\mathfrak{M}}_{g,N_0}$ defines a natural mapping

$$((M; N_0), \mathscr{C}) \longrightarrow \overline{\mathfrak{M}}_{g,N_0+1} \tag{B.44}$$

that determines a stable curve $((M; N_0 + 1), \mathscr{C}') \in \overline{\mathfrak{M}}_{g,N_0+1}$. Explicitly, as long as the point p is disjoint from the set of marked points $\{p_k\}_{k=1}^{N_0}$ one simply defines $((M; N_0+1), \mathscr{C}')$ to be $((M; N_0, \{p\}), \mathscr{C})$. If the point $p = p_h$ for some $p_h \in \{p_k\}_{k=1}^{N_0}$, then: (i) for any $1 \leq i \leq N_0$, with $i \neq h$, identify $p'_i \in ((M; N_0 + 1), \mathscr{C}')$ with the corresponding p_i; (ii) take a thrice–pointed sphere $\mathbb{CP}^1_{(0,1,\infty)}$, label with a sub-index h one of its marked points $(0, 1, \infty)$, say ∞_h, and attach it to the given $p_h \in [((M; N_0), \mathscr{C})]$; (iii) relabel the remaining two marked points $(0, 1) \in \mathbb{CP}^1_{(0,1,\infty)}$ as p'_h and p'_{N_0+1}. In this way, we get a genus g noded surface

$$s_h\left[((M; N_0), \mathscr{C})\right] = \left[((M; N_0 + 1), \mathscr{C}')\right] \doteq ((M; N_0), \mathscr{C})_h \cup \mathbb{CP}^1_{(0,1,\infty_h)} \tag{B.45}$$

with a rational tail and with a double point corresponding to the original marked point p_h. Finally if p happens to coincide with a node, then $[((M; N_0 + 1), \mathscr{C}')]$ results from setting $p'_j \doteq p_j$ for any $1 \leq i \leq N_0$ and by: (i) normalizing $[((M; N_0), \mathscr{C})]$ at the node (i.e., by separating the branches of $[((M; N_0), \mathscr{C})]$at p); (ii) inserting a copy of $\mathbb{CP}^1_{(0,1,\infty)}$with $\{0, \infty\}$ identified with the preimage of p and with $p'_{N_0+1} \doteq 1 \in \mathbb{CP}^1_{(0,1,\infty)}$. Conversely, let

$$\pi : \overline{\mathfrak{M}}_{g,N_0+1} \longrightarrow \overline{\mathfrak{M}}_{g,N_0} \tag{B.46}$$

$$[((M;N_0+1),\mathscr{C})] \vdash \begin{matrix} \textit{forget} \\ \& \\ \textit{collapse} \end{matrix} \longrightarrow \left[((M;N_0),\mathscr{C}')\right]$$

the projection which forgets the (N_0+1)st marked point and collapse to a point any irreducible unstable component of the resulting curve. The fiber of π over $((M;N_0),\mathscr{C})$ is parametrized by the map (B.44), and if $((M;N_0),\mathscr{C})$ has a trivial automorphism group $Aut[((M;N_0),\mathscr{C})]$ then $\pi^{-1}((M;N_0),\mathscr{C})$ is by definition the surface $((M;N_0),\mathscr{C})$, otherwise it is identified with the quotient

$$((M;N_0),\mathscr{C})/Aut[((M;N_0),\mathscr{C})]. \tag{B.47}$$

Thus, under the action of π, we can consider $\overline{\mathfrak{M}}_{g,N_0+1}$ as a family (in the orbifold sense) of Riemann surfaces over $\overline{\mathfrak{M}}_{g,N_0}$ and we can identify $\overline{\mathfrak{M}}_{g,N_0+1}$ with the universal curve $\overline{\mathscr{C}}_{g,N_0}$,

$$\pi : \overline{\mathscr{C}}_{g,N_0} \longrightarrow \overline{\mathfrak{M}}_{g,N_0}\,. \tag{B.48}$$

Note that, by construction, $\overline{\mathscr{C}}_{g,N_0}$ (but for our purposes is more profitable to think in terms of $\overline{\mathfrak{M}}_{g,N_0+1}$) comes endowed with the N_0 natural sections $s_1,\ldots,s_{N_0}$

$$\begin{aligned} s_h : \overline{\mathfrak{M}}_{g,N_0} &\longrightarrow \overline{\mathscr{C}}_{g,N_0} \\ [((M;N_0),\mathscr{C})] &\longmapsto s_h\left[((M;N_0),\mathscr{C})\right] \doteq ((M;N_0),\mathscr{C})_h \cup \mathbb{CP}^1_{(0,1,\infty_h)}, \end{aligned} \tag{B.49}$$

defined by (B.45).

The images of the sections s_i characterize a divisor $\{D_i\}_{i=1}^{N_0}$ in $\overline{\mathscr{C}}_{g,N_0}$ which has a great geometric relevance in discussing the topology of $\overline{\mathfrak{M}}_{g,N_0}$. Such a study exploits the properties of the tautological classes over $\overline{\mathscr{C}}_{g,N_0}$ generated by the sections $\{s_i\}_{i=1}^{N_0}$ and by the corresponding divisors $\{D_i\}_{i=1}^{N_0}$. To define such classes, recall that the cotangent bundle (in the orbifold sense) to the fibers of the universal curve $\pi : \overline{\mathscr{C}}_{g,N_0} \longrightarrow \overline{\mathfrak{M}}_{g,N_0}$ gives rise to a holomorphic line bundle $\omega_{g,N_0} \doteq \omega_{\overline{\mathscr{C}}_{g,N_0}/\overline{\mathfrak{M}}_{g,N_0}}$ over $\overline{\mathscr{C}}_{g,N_0}$ (the relative dualizing sheaf of $\pi : \overline{\mathscr{C}}_{g,N_0} \longrightarrow \overline{\mathfrak{M}}_{g,N_0}$), this is essentially the sheaf of 1-forms with a natural polar behavior along the possible nodes of the Riemann surface describing the fiber of π. It is worthwhile to discuss the behavior of the relative dualizing sheaf ω_{g,N_0} restricted to the generic divisor D_h generated by the section s_h. To this end, let $z_1(h)$ and $z_2(\infty_h)$ denote local coordinates defined in the disks $\Delta_{p_h} \doteq \{|z_1(h)| < 1\}$ and $\Delta_{\infty_h} \doteq \{|z_2(\infty_h)| < 1\}$ respectively centered around the marked points $p_h \in ((M;N_0),\mathscr{C})$, and $\infty \in \mathbb{CP}^1_{(0,1,\infty)}$. Let $\Delta_{t_h} = \{t_h \in \mathbb{C} : |t_h| < 1\}$. Consider the analytic family $s_h(t_h)$ of surfaces of genus g defined over Δ_{t_h} and obtained by removing the disks $|z_1(h)| < |t_h|$ and $|z_2(\infty_h)| < |t_h|$ from $((M;N_0),\mathscr{C})$ and $\mathbb{CP}^1_{(0,1,\infty)}$ and gluing the resulting surfaces through the annulus $\{(z_1(h),z_2(\infty_h))|\ z_1(h)z_2(\infty_h) = t_h,\ t_h \in \Delta_{t_h}\}$ by identifying the points of coordinate $z_1(h)$ with the points of coordinates $z_2(\infty_h) = t_h/z_1(h)$. The family $s_h(t_h) \to \Delta_{t_h}$ opens the node $z_1(h)z_2(\infty_h) = 0$ of the section $s_h|_{((M;N_0),\mathscr{C})}$. Note that in such a way we can independently and holomorphically open the distinct nodes of the various sections $\{s_k\}_{k=1}^{N0}$. More generally, while opening the node we can also vary the complex structure of $((M;N_0),\mathscr{C})$ by introducing local complex coordinates $(\tau_\alpha)_{\alpha=1}^{3g-3+N_0}$ for $\mathfrak{M}_{g,N_0}$ around $((M;N_0),\mathscr{C})$. If

$$s_h(\tau_\alpha, t_h) \to \mathfrak{M}_{g,N_0} \times \Delta_{t_h} \tag{B.50}$$

denotes the family of surfaces opening of the node, then in the corresponding coordinates (τ_α, t_h) the divisor D_h, image of the section s_h, is locally defined by the equation $t_h = 0$. Similarly, the divisor $D \doteq \sum_{h=1}^{N_0} D_h$ is characterized by the locus of equation $\prod_{h=1}^{N_0} t_h = 0$.

The elements of the dualizing sheaf $\omega_{g,N_0}|_{s_h(t_h)} \doteq \omega_{g,N_0}(D_h)$ are differential forms $u(h) = u_1 dz_1(h) + u_2 dz_2(\infty_h)$ such that $u(h) \wedge dt_h = f dz_1(h) \wedge dz_2(\infty_h)$, where f is a holomorphic function of $z_1(h)$ and $z_2(\infty_h)$. By differentiating $z_1(h)z_2(\infty_h) = t_h$, one gets $f = u_1 z_1(h) - u_2 z_2(\infty_h)$ which is the defining relation for the forms in $\omega_{g,N_0}(D_h)$. In particular, by choosing $u_1 = f/2z_1(h)$, and $u_2 = f/2z_2(\infty_h)$ we get the local isomorphism between the sheaf of holomorphic functions $\mathscr{O}_{s_h(t_h)}$ over $s_h(t_h)$ and $\omega_{g,N_0}(D_h)$

$$f \longmapsto u(h) = f\left(\frac{1}{2}\frac{dz_1(h)}{z_1(h)} - \frac{1}{2}\frac{dz_2(\infty_h)}{z_2(\infty_h)}\right). \tag{B.51}$$

If we set $f = f_0 + f_1(z_1(h)) + f_2(z_2(\infty_h))$, where f_0 is a constant and $f_1(0) = 0 = f_2(0)$, then on the noded surface s_h, $(t_h = 0)$, we get from the relation $z_2(\infty_h)dz_1(h) + z_1(h)dz_2(\infty_h) = 0$,

$$\begin{aligned} u_h|_{z_2(\infty_h)=0} &= \frac{f_0 + f_1(z_1(h))}{z_1(h)} dz_1(h), \\ u_h|_{z_1(h)=0} &= -\frac{f_0 + f_2(z_2(\infty_h))}{z_2(\infty_h)} dz_2(\infty_h), \end{aligned} \tag{B.52}$$

on the two branches $\Delta_{p_h} \cap ((M;N_0),\mathscr{C})$ and $\Delta_{\infty_h} \cap \mathbb{CP}^1_{(0,1,\infty)}$ of the node where $z_1(h)$ and $z_2(\infty_h)$ are a local coordinate (i.e. $z_2(\infty_h) = 0$ and $z_1(h) = 0$, respectively). Thus, near the node of s_h, $\omega_{g,N_0}(D_h)$ is generated by $\frac{dz_1(h)}{z_1(h)}$ and $\frac{dz_2(\infty_h)}{z_2(\infty_h)}$ subjected to the relation $\frac{dz_1(h)}{z_1(h)} + \frac{dz_2(\infty_h)}{z_2(\infty_h)} = 0$. Stated differently, a section of the sheaf $\omega_{g,N_0}(D_h)$ pulled back to the smooth normalization $((M;N_0),\mathscr{C})_{p_h} \bigsqcup \mathbb{CP}^1_{(0,1,\infty_h)}$ of s_h can be identified with a meromorphic 1-form with at most simple poles at the marked points p_h and ∞_h which are identified under the normalization map, and with opposite residues at such marked points. By extending such a construction to all N_0 sections $\{s_h\}_{h=1}^{N_0}$, we can define the line bundle

$$\omega_{g,N_0}(D) \doteq \omega_{g,N_0}\left(\sum_{i=1}^{N_0} D_i\right) \longrightarrow \overline{\mathscr{C}}_{g,N_0} \tag{B.53}$$

as ω_{g,N_0} twisted by the divisor $D \doteq \sum_{h=1}^{N_0} D_h$, *viz.* the line bundle locally generated by the differentials $\frac{dz_1(h)}{z_1(h)}$ for $z_1(h) \neq 0$ and $-\frac{dz_2(\infty_h)}{z_2(\infty_h)}$ for $z_2(\infty_h) \neq 0$, with $z_1(h)z_2(\infty_h) = 0$, and $h = 1, \ldots, N_0$. As above, $\{z_1(h)\}_{h=1}^{N_0}$ are local variables at the marked points $\{p_h\}_{i=1}^{N_0} \in ((M; N_0), \mathscr{C})$, whereas $z_2(\infty_h)$ is the corresponding variable in the thrice–pointed sphere $\mathbb{CP}^1_{(0,1,\infty)}$.

If, roughly speaking, we interpret $\overline{\mathfrak{M}}_{g,N_0+1}$, as a bundle of surfaces over $\overline{\mathfrak{M}}_{g,N_0}$, then the stratification of $\overline{\mathfrak{M}}_{g,N_0+1}$ into subvarieties $\{V_k\}$ provides an intuitive motivation for the existence of characteristic classes ($\mathbb{Q}$-Poincaré duals of the $\{V_k\}$) that describe the topological properties of $\overline{\mathfrak{M}}_{g,N_0}$. Two such families of classes have proven to be particularly relevant. The first is obtained by pulling back ω_{g,N_0} to $\overline{\mathfrak{M}}_{g,N_0}$ by means of the sections s_k. In this way one gets the line bundle

$$s_k^* \omega_{g,N_0} \doteq \mathscr{L}_k \to \overline{\mathfrak{M}}_{g,N_0} \tag{B.54}$$

whose fiber at the moduli point $((M; N_0), \mathscr{C})$ is defined by the cotangent space $T^*_{(M,p_k)}$ to $((M; N_0), \mathscr{C})$ at the marked point p_k. By taking the first Chern class $c_1(\mathscr{L}_k)$ of the resulting bundles $\{\mathscr{L}_k\}_{k=1}^{N_0}$ one gets the Witten classes $\psi_{(g,N_0),k} \in H^2(\overline{\mathfrak{M}}_{g,N_0}; \mathbb{Q})$

$$\psi_{(g,N_0),k} \doteq c_1(\mathscr{L}_k)\,. \tag{B.55}$$

Conversely, if we take the Chern class $c_1\left(\omega_{g,N_0}(D)\right)$ of the line bundle $\omega_{g,N_0}(D)$ and intersect it with itself $j \geq 0$ times, then one can define the Mumford $k_{(g,N_0),j}$ classes $\in H^{2j}(\overline{\mathfrak{M}}_{g,N_0}; \mathbb{Q})$ according to [1]

$$k_{(g,N_0),j} \doteq \pi_* \left(\left(c_1 \left(\omega_{g,N_0}(D) \right) \right)^{j+1} \right), \tag{B.56}$$

where π_* denotes fiber integration. If $\widehat{\psi}_{N_0+1} \doteq \psi_{(g,N_0+1),N_0+1}$ denotes the Witten class in $H^2(\overline{\mathfrak{M}}_{g,N_0+1}; \mathbb{Q})$ associated with the last marked point, then $k_{(g,N_0),j}$ can be also defined as

$$k_{(g,N_0),j} \doteq \pi_* \left(\left(\widehat{\psi}_{N_0+1} \right)^{j+1} \right) \tag{B.57}$$

It is worthwhile recalling that $k_{(g,N_0),1}$ is the class of the Weil-Petersson Kähler form ω_{W-P} on $\overline{\mathfrak{M}}_{g,N_0}$, [1],

$$k_{(g,N_0),1} = \pi_* \left(\left(c_1 \left(\omega_{g,N_0}(D) \right) \right)^2 \right) = \frac{1}{2\pi^2} [\omega_{W-P}]. \tag{B.58}$$

B.4 Strebel Theorem

A basic result in phrasing the correspondence between Riemann surfaces and combinatorial structures is provided by Strebel's theory of holomorphic (and meromorphic) quadratic differentials [28]. Recall that these objects are the holomorphic (meromorphic) sections of $T'M \otimes T'M$, i.e. the tensor fields on $(M, \mathscr{C})$ that can be locally written as $\Phi = \phi(z)\, dz \otimes dz$, for some holomorphic (meromorphic) $\phi(z)$. Geometrically the role of holomorphic quadratic differential stems from the basic observation that for a given point $p \in U$, the function $\zeta(q) := \int_p^q \sqrt{\phi(z)\, dz^2}$ provides a local conformal parameter in a neighborhood $U' \subset U$ of p if $\phi(p) \neq 0$. In terms of the coordinate ζ we can write $\Phi = d\zeta \otimes d\zeta$, and the sets $\zeta^{-1}\{z \mid \Im z = const\}$, and $\zeta^{-1}\{z \mid \Re z = const\}$ foliate U' in the standard $\zeta = X + \sqrt{-1}\, Y$ orthogonal way. In general, the structure of the foliation induced by Φ around its zeros and poles is quite more sophisticated, however the case relevant to ribbon graphs and polyhedral surfaces can be fully described by the Strebel theorem. For future reference, it is worthwhile having it handy in the elegant formulation provided by Mulase and Penkawa [24]:

Theorem B.1 *Let* $((M; N_0), \mathscr{C})$ *be a closed Riemann surface of genus* $g \geq 0$ *with* $N_0 \geq 1$ *marked points* $\{p_k\}_{k=1}^{N_0}$*, where* $2-2g-N_0 < 0$*. Let us denote by* $(L_1, \ldots, L_{N_0})$ *an ordered* N_0*–tuple of positive real numbers. Then there is a unique (Jenkins–Strebel) meromorphic quadratic differential* Φ *on* $((M; N_0), \mathscr{C})$ *such that:*

(i) Φ *is holomorphic on* $M \setminus \{p_1, \ldots, p_{N_0}\}$ *;*
(ii) Φ *has a double pole at each* p_k*,* $k = 1, \ldots, N_0$ *;*
(iii) The union of all non–compact horizontal leaves $\zeta^{-1}\{z \mid \Im z = const\}$ *form a closed subset* $\subset ((M; N_0), \mathscr{C})$ *of measure zero ;*
(iv) Every compact horizontal leaf λ *is a simple loop circling around one of the poles. In particular, if* λ_k *is the loop around the pole* p_k*, then* $L_k = \oint_{\lambda_k} \sqrt{\Phi}$*, (the branch of* $\sqrt{\Phi}$ *is chosen so that the integral is positive when the circuitation along* λ *is along the positive orientation induced by* $((M; N_0), \mathscr{C})$*) ;*
(v) Every non–compact horizontal leaf of Φ *is bounded by zeros of* Φ*. Conversely, every zero of degree* m *of* Φ *bounds* $m+2$ *horizontal leaves;*
(vi) If N_2 *denotes the number of zeros of the quadratic differential* Φ*, then* Φ *induces a unique cell–decomposition of* $((M; N_0), \mathscr{C})$ *with* N_0 *2–cells,* N_1 *1–cells, and* N_2 *0–cells where* $N_1 = N_0 + N_2 - 2 + 2g$*. The* 1*–skeleton of this cell decomposition is a metric ribbon graph with vertex–valency* ≥ 3.

It is straightforward to check that Strebel's theorem characterizes a map $\mathscr{S}$: $\mathfrak{M}_{g,N_0} \times \mathbb{R}_+^{N_0} \to RG^{met}_{g,N_0}$ which, given a sequence of N_0 positive real numbers $\{L(k)\}$, associates to a pointed Riemann surface $((M; N_0), \mathscr{C}) \in \mathfrak{M}_{g,N_0}$ a metric ribbon graph $\Gamma \in RG^{met}_{g,N_0}$ with N_0 labelled boundary components $\{\partial\Gamma(k)\}$ of perimeters $\{L(k)\}$. This map is actually a bijection since, given a metric ribbon graph $\Gamma \in \mathbb{R}_+^{|e(\Gamma)|}$ with N_0 labelled boundary components $\{\partial\Gamma(k)\}$ of perimeters $\{L(k)\}$, we can, out of such combinatorial data, construct a decorated Riemannian surface

$((M;N_0),\mathscr{C};\,\{L(K)\}) \in \mathfrak{M}_{g,N_0} \times \mathbb{R}_+^{N_0}$. The characterization of the correspondance $\mathscr{S}^{-1} : \Gamma \mapsto ((M;N_0),\mathscr{C};\,\{L(K)\})$ is described in a cristal clear way in [24], and one eventually establishes [19] the

Theorem B.2 *Strebel theory defines a natural bijection*

$$\mathfrak{M}_{g,N_0} \times \mathbb{R}_+^{N_0} \simeq RG^{met}_{g,N_0}, \tag{B.59}$$

between the decorated moduli space $\mathfrak{M}_{g,N_0} \times \mathbb{R}_+^{N_0}$ *and the space of all metric ribbon graphs* RG^{met}_{g,N_0} *with* N_0 *labelled boundary components.*

B.5 The Teichmüller Space of Surfaces with Boundaries

This is also the appropriate place for a few remarks on moduli space theory for surfaces with boundaries. The elements of the Teichmüller space $\mathfrak{T}^{\partial}_{g,\,N_0}$ of hyperbolic surfaces Ω with N_0 geodesic boundary components are marked Riemann surface modelled on a surface S_{g,N_0} of genus g with complete finite-area metric of constant Gaussian curvature -1, (and with N_0 geodesic boundary components $\partial S = \sqcup \partial S_j$), i.e., a triple $(S_{g,N_0}, f,\, \Omega)$ where $f : S_{g,N_0} \to \Omega$ is a quasiconformal homeomorphism, (the marking map), which extends uniquely to a homeomorphism from $S_{g,N_0} \cup \partial S$ onto $\Omega \cup \partial\Omega$. Any two such a triple $(S_{g,N_0}, f_1,\, \Omega_{(1)})$ and $(S_{g,N_0}, f_2,\, \Omega_{(2)})$ are considered equivalent iff there is a biholomorphism $h : \Omega_{(1)} \to \Omega_{(2)}$ such that $f_2^{-1} \circ h \circ f_1 : S_{g,N_0} \cup \partial S \to S_{g,N_0} \cup \partial S$ is homotopic to the identity via continuous mappings pointwise fixing ∂S.

For a given string $L = (L_1, \ldots, L_{N_0}) \in \mathbb{R}^{N_0}_{\geq 0}$, we let $\mathfrak{T}^{\partial}_{g,\,N_0}(L)$ denote the Teichmüller space of hyperbolic surfaces Ω with geodesic boundary components of length

$$(|\partial\Omega_1|, \ldots, |\partial\Omega_{N_0}|) = (L_1, \ldots, L_{N_0}) \doteq L \in \mathbb{R}^{N_0}_{\geq 0}. \tag{B.60}$$

This characterizes the boundary length map

$$\begin{aligned} \mathfrak{L} : \mathfrak{T}^{\partial}_{g,N_0} &\longrightarrow \mathbb{R}^{N_0}_{\geq 0} \\ \Omega &\longmapsto \mathfrak{L}(\Omega) = (|\partial\Omega_1|, \ldots, |\partial\Omega_{N_0}|)\,, \end{aligned} \tag{B.61}$$

and we can write $\mathfrak{T}^{\partial}_{g,N_0}(L) := \mathfrak{L}^{-1}(L)$. Note that, by convention, a boundary component such that $|\partial\Omega_j| = 0$ is a cusp and moreover $\mathfrak{T}_{g,N_0}(L = 0) = \mathfrak{T}_{g,N_0}$, where $\mathfrak{T}_{g,N_0}$ is the Teichmüller space of hyperbolic surfaces with N_0 punctures, (with $6g - 6 + 2N_0 \geq 0$). For each given string $L = (L_1, \ldots, L_{N_0})$ there is a natural action on $\mathfrak{T}_{g,N_0}(L)$ of the mapping class group $\mathfrak{Map}^{\partial}_{g,N_0}$ defined by the group of all the isotopy classes of orientation preserving homeomorphisms of Ω which leave each boundary component $\partial\Omega_j$ pointwise (and isotopy-wise) fixed. This action changes the marking f of S_{g,N_0} on Ω, and characterizes the quotient space

$$\mathfrak{M}_{g,N_0}(L) \doteq \frac{\mathfrak{T}_{g,N_0}(L)}{\mathfrak{Map}^{\partial}_{g,N_0}} \tag{B.62}$$

as the moduli space of Riemann surfaces (homeomorphic to S_{g,N_0}) with N_0 boundary components of length $|\partial\Omega_k| = L_k$. Note again that when $\{L_k \to 0\}$, $\mathfrak{M}_{g,N_0}(L)$ reduces to the usual moduli space $\mathfrak{M}_{g,N_0}$ of Riemann surfaces of genus g with N_0 marked points. It is worthwhile noticing that, since the boundary components of a surface $\Omega_{g,N_0} \in \mathfrak{T}^{\partial}_{g,N_0}$ are left pointwise fixed, any surface Ω_{g,N_0} in $\mathfrak{T}^{\partial}_{g,N_0}$, for $N_0 > 1$, can be embedded into a surface $\Omega'_{g+1,N_0-1} \in \mathfrak{T}^{\partial}_{g+1,N_0-1}$ by glueing the two legs of a pair of pants onto two of the boundary components of Ω_{g,N_0}. The attachment of a torus with two boundary components allows also to include $\Omega_{g,1}$ in $\Omega_{g+1,1}$. Such a chain of embeddings induces a corresponding chain of embeddings of the mapping class group $\mathfrak{Map}^{\partial}_{g,N_0}$ into $\mathfrak{Map}^{\partial}_{g+1,N_0-1}$. Under direct limit, this gives rise to a notion of stable mapping class group playing a basic role in the study of the cohomology of the moduli space.

We conclude this notational capsule by specializing to the boundary case the characterization of the Weil–Petersson inner product. The reader will find many more details in the remarkable and very informative papers by G. Mondello, (in particular [22]). As in Sect. B.2, we introduce the real vector space of holomorphic quadratic differentials $Q^{\partial}_{N_0}(\Omega)$ whose restrictions to $\partial\Omega$ is real. The corresponding space of Beltrami differentials, identified with the tangent space $T_{\mathscr{C}}\,\mathfrak{T}^{\partial}_{g,\,N_0}$, will be denoted by $B_{N_0}(\Omega)$ and the they are paired according to

$$(\mu\, d\zeta \otimes d\zeta, \nu\, d\overline{\zeta} \otimes d\zeta^{-1}) \longmapsto \int_{\Omega} \mu\, \nu\, d\zeta\, d\overline{\zeta}, \tag{B.63}$$

where $\mu\, d\zeta \otimes d\zeta \in Q^{\partial}_{N_0}(\Omega)$ and $\nu\, d\overline{\zeta} \otimes d\zeta^{-1}) \in B_{N_0}(\Omega)$. As in the boundaryless case, by noticing that the ratio between a quadratic differential and the hyperbolic metric $h\, d\zeta \otimes d\overline{\zeta}$ on Ω is a Beltrami differential, we can define the Weil–Petersson inner product on $T_{\mathscr{C}}\,\mathfrak{T}^{\partial}_{g,\,N_0}$ according to

$$G^{\partial}_{\alpha\overline{\beta}} = \int_M \frac{\partial}{\partial\mu_\alpha}\, \frac{\partial}{\partial\overline{\mu}_\beta}\, h(\zeta)\, |d\zeta|^2 \tag{B.64}$$

where $\{\frac{\partial}{\partial\mu_\alpha}\}_{\alpha=1}^{3g-3+2N_0}$ is a basis of the vector space of harmonic Beltrami differentials on Ω. The corresponding Weil-Petersson form is provided by

$$\omega_{WP} := \sqrt{-1}G^{\partial}_{\alpha\overline{\beta}} dZ^{\alpha} \wedge d\overline{Z}^{\beta}, \tag{B.65}$$

where $\{dZ^{\alpha}\}$ are the basis, in $Q^{\partial}_{N_0}(M)$, dual to $\{\frac{}{\mu_{\alpha}}\}$ under the pairing (B.63). Note in particular that

$$\eta^{\alpha\,\overline{\beta}} \;:=\; \sqrt{-1}\,\int_{\Omega} h^{-2}\,dZ^{\alpha} d\overline{Z}^{\beta}\;|d\zeta|^{2}, \tag{B.66}$$

defines the Weil–Petersson Poisson tensor associated with ω_{WP}. It is important to stress that, (because of the presence of the boundaries), the Weil–Petersson form is degenerate (in particular it is not Kähler) on $\mathfrak{T}^{\partial}_{g,N_0}$. However, the Poisson structure defined by $\eta^{\alpha\,\overline{\beta}}$ induces a foliation in $\mathfrak{T}^{\partial}_{g,N_0}$ whose symplectic leaves are the spaces $\mathfrak{T}^{\partial}_{g,N_0}(L)$, of Riemann surfaces with given boundary length vector $L \in \mathbb{R}^{N_0}_{\geq 0}$, endowed with ω_{WP}. G. Mondello [22] has provided a nice geometrical characterization of the Poisson structure $\eta^{\alpha\,\overline{\beta}}$ in terms of ideal hyperbolic triangulations of the surfaces $\Omega \in \mathfrak{T}^{\partial}_{g,N_0}$. As we have seen, this characterization plays an important role in Chap. 3, and it hints to even deeper connections between the geometry of the space of polyhedral surfaces and hyperbolic geometry.

Appendix C
Spectral Theory on Polyhedral Surfaces

In this appendix we briefly discuss the basic facts of spectral theory of Laplace type operators on polyhedral surfaces that we have exploited in these lecture notes.

C.1 Kokotov's Spectral Theory on Polyhedral Surfaces

Spectral theory for cone manifolds has a long standing tradition, (a fine sample of classical works is provided by [2, 5, 6, 10, 18]). Here we shall mainly refer to the elegant results recently obtained by A. Kokotov [17]. They provide a rather complete analysis of the determinant of the Laplacian on polyhedral surfaces. It must be stressed that whereas the study of the determinant of the Laplacian in the smooth setting is a well–developed subject, results in the polyhedral case have been sparse and often subjected to quite restrictive hypotheses [10, 21].

As a consequence of the presence of the conical points $\{p_1, \ldots, N_0\}$, the Laplacian Δ on the Riemann surface $((M, N_0), \mathscr{C}_{sg})$ is not an essentially self-adjoint operator. There are (infinitely) many possible self–adjoint extensions of Δ, with domains typically determined by the behavior of functions (formally) harmonic at the conical points. To take care of this extension problem in a natural way, let us recall that the minimal domain $\mathscr{D}_{min}$ of the Laplacian on $C_0^\infty(M', \mathbb{R})$ consists of the graph closure on the set $C_0^\infty(M', \mathbb{R})$, where a function $u \in \mathscr{D}_{min}$ if there is a sequence $\{u_k\} \in C_0^\infty(M', \mathbb{R})$ and a function $w \in L^2(M)$ such that $u_k \to u$ and $\Delta\, u_k \to w$ in $L^2(M)$, where $L^2(M)$ denotes the space of square summable functions on $((M, N_0), \mathscr{C}_{sg})$. Similarly, the maximal domain $\mathscr{D}_{max}$ for Δ, corresponding to the domain of the adjoint operator to Δ on $\mathscr{D}_{min}$, is the subspace of functions $v \in L^2(M)$ such that for all $u \in \mathscr{D}_{min}$ there is a function $f \in L^2(M)$ with $\langle v, \Delta u\rangle = \langle f, u\rangle$, where $\langle \cdot, \cdot\rangle$ denotes the $L^2(M)$ pairing on $((M, N_0), \mathscr{C}_{sg})$. All possible distinct self–adjoint extensions of Δ on $((M, N_0), \mathscr{C}_{sg})$ are parametrized by a domain $\mathscr{D}_\Delta$, with $\mathscr{D}_{min} \subseteq \mathscr{D}_\Delta \subseteq \mathscr{D}_{max}$ [23]. Explicitly, we can consider without loss of generality the

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case in which we have just one conical point $\{p\}$ with conical angle θ, and denote by z the local conformal parameter in a neighborhood of p. One introduces[17] the functions (formally harmonic on $((M,p),\,\mathscr{C}_{sg})$), defined by

$$V^k_{\pm}(z) := |z|^{\pm\frac{2\pi k}{\theta}}\ \exp\left\{\sqrt{-1}\,\frac{2\pi\,k}{\theta}\ \arg z\right\}\ ,\qquad k>0, \tag{C.1}$$

$$V^0_+ := 1\ ,\qquad V^0_- := \ln|z|. \tag{C.2}$$

These functions are in $L^2(M)$ as long as $k<\frac{\theta}{2\pi}$, and we can consider the linear subspaces $\mathscr{E}$ of $L^2(M)$ generated by the functions $\varphi\,V^k_{\pm}(z)$, with $0\leq k<\frac{\theta}{2\pi}$, where φ is a C^∞–mollifier of the characteristic function of the cone with vertex at p, e.g., $\varphi(z):=c\,\exp\left(|z|^2-1\right)^{-1},\ |z|<1$, and $\varphi(z)=0$, for $|z|\geq 1$, with $\varphi(z=0)=1$, (assuming that the region isometric to the cone corresponds to $|z|<1$). The self–adjoint extension of Δ are then parametrized by the subspaces of functions $\mathscr{E}$ such that[17]

$$\langle\Delta\,u\,,v\rangle-\langle u\,,\Delta\,v\rangle = \lim_{\varepsilon\searrow 0+}\oint\left(u\,\frac{\partial v}{\partial r}-v\,\frac{\partial u}{\partial r}\right)=0, \tag{C.3}$$

for all $u,\ v\ \in\mathscr{E}$, and where $r:=|z|$, (note that in the above characterization one exploits the delicate fact that any $u\in\mathscr{D}_{min}$ is such that $u(z)=O(r)$ as $r\searrow 0$–see section 3.2.1 of [17] for details). The Friedrics extension, which is the relevant one for the results to follow, is associated with the subspace $\mathscr{E}$ generated by the functions $\varphi\,V^k_+(z)$, $0\leq k<\frac{\theta}{2\pi}$. The associated domain $\mathscr{D}_\Delta:=\mathscr{D}_{min}+\mathscr{E}$ comprises functions which are bounded near the apex of the cone.

Denote by $\mathscr{H}(z,z';\eta)$ the heat kernel for the Friedrichs extension of the Laplacian Δ. This is a distribution on $M\times M\times[0,\infty)$ such that, for all $u\in\mathscr{D}_\Delta:=\mathscr{D}_{min}+\mathscr{E}$ away from the conical points, the convolution $\int_M\mathscr{H}(z,z';\eta)\,u(z')\,dA_{z'}$ is smooth for all $\eta>0$, and

$$\left(\frac{\partial}{\partial\eta}+\Delta^{(z)}\right)\mathscr{H}(z,z';\eta)=0, \tag{C.4}$$

$$\lim_{\eta\searrow 0+}\mathscr{H}(z,z';\eta)\ =\ \delta(z,z'),$$

where $(z,z';\eta)\ \in\ (M\times M\backslash Diag\,(M\times M))\times[0,\infty)$, and $\Delta^{(z)}$ denotes the Laplacian with respect to the variable z. The Dirac initial condition is understood in the distributional sense, i.e., for any smooth $u\in C^\infty_0(M',\mathbb{R})\cap\mathscr{D}_\Delta$, $\int_M\mathscr{H}(z,z';\eta)\,u(z')\,dA_{z'}\ \rightarrow\ u(z)$, as $\eta\searrow 0^+$, where the limit is meant in the uniform norm on $C^\infty_0(M',\mathbb{R})$. We are now ready to state Kokotov's main results

Theorem C.1 (Kokotov[17], Th.1) *Let* $((M,N_0),\,\mathscr{C}_{sg})$ *be the Riemann surface, with conical singularities* $Div(T):=\sum_{k=1}^{N_0}\left(\frac{\Theta(k)}{2\pi}-1\right)$*, associated with the polyhe-*

dral manifold (P_T, M)*, and let* Δ *be (the Friedrichs extension) of the corresponding Laplace operator. Then*

(i) Δ *has a discrete spectral resolution, the eigenvalues* $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$ *have finite multiplicities, and the associated spectral counting function* $N(\lambda) := \mathrm{Card}\ [k \geq 1\ :\ \lambda_k \leq \lambda] = O(\lambda)$*, as* $\lambda \to \infty$.

(ii) If $Tr\ e^{\eta\Delta}$ *denotes the heat trace of the heat kernel* $\mathscr{H}(z, z'; \eta)$ *associated with* $(\Delta, \mathscr{D}_\Delta)$*, then, for some* $\varepsilon > 0$*, the asymptotics*

$$Tr\ e^{\eta\Delta} := \int_M \mathscr{H}(z, z; \eta)\, dA = \frac{Area\,((M, N_0),\, \mathscr{C}_{sg})}{4\pi\,\eta} \tag{C.5}$$

$$+ \frac{1}{12}\sum_{k=1}^{N_0}\left\{\frac{2\pi}{\Theta(k)} - \frac{\Theta(k)}{2\pi}\right\} + O(e^{-\varepsilon/\eta}),$$

holds in the uniform norm on $C^\infty(M', \mathbb{R})$.

Let

$$\zeta_\Delta(s) := \sum_{\lambda_k > 0}\frac{1}{\lambda_k^s} \tag{C.6}$$

denote the ζ–function associated with the positive part of the spectrum of the operator $(\Delta, \mathscr{D}_\Delta)$, then we have

Theorem C.2 (Kokotov[17]) *The function* $\zeta_\Delta(s)$ *is holomorphic in the half–plane* $\{\Re\ s > 1\}$*, and there is an entire function* $e(s)$ *such that*

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)}\left\{\frac{Area\,((M, N_0),\, \mathscr{C}_{sg})}{4\pi\,(s-1)}\right. \tag{C.7}$$

$$\left. + \left[\frac{1}{12}\sum_{k=1}^{N_0}\left\{\frac{2\pi}{\Theta(k)} - \frac{\Theta(k)}{2\pi}\right\} - 1\right]\frac{1}{s} + e(s)\right\},$$

where $\Gamma(s)$ *denotes the Gamma function.* $\zeta_\Delta(s)$ *is a regular at* $s = 0$*, and one can define the Ray–Singer* ζ*–regularized determinate of* $(\Delta, \mathscr{D}_\Delta)$ *according to*

$$\det{}'\,\Delta := \exp\left\{-\zeta'_\Delta(s = 0)\right\}. \tag{C.8}$$

As a direct consequence of this representation, one has[17] the

Corollary C.1 *Let* $((M, N_0),\, \mathscr{C}_{sg})$ *and* $(\widetilde{(M, N_0)},\, \mathscr{C}_{sg})$ *two homothetic Riemann surfaces, with (the same) conical singularities* $Div(T) := \sum_{k=1}^{N_0}\left(\frac{\Theta(k)}{2\pi} - 1\right)\sigma^0(k)$*, and let* ds^2 *and* $\widetilde{ds}^2 = \kappa\, ds^2$*, with* $k > 0$ *a positive constant, be the respective* *conical metrics, (see Theorem 2.1). If we denote by* $\det'\ \Delta$ *and* $\det'\ \widetilde{\Delta}$ *the associated* ζ*–regularized determinants, then one has the rescaling*

$$\det{}'\ \widetilde{\Delta} \;=\; \kappa^{-\left(\frac{\chi(M)}{6}-1\right)-\frac{1}{12}\sum_{k=1}^{N_0}\left\{\frac{2\pi}{\Theta(k)}+\frac{\Theta(k)}{2\pi}-2\right\}}\ \det{}'\ \Delta. \tag{C.9}$$

For $\eta \in [0, 1]$, let us consider two distinct families of polyhedral surfaces

$$\eta \mapsto (T_{(1)}, M)_\eta \in POL_{g,N_0}(M) \tag{C.10}$$

$$\eta \mapsto (T_{(2)}, M)_\eta \in POL_{g,\widehat{N}_0}(M), \tag{C.11}$$

(note that generally $N_0 \neq \widehat{N}_0$, and that η may be allowed to vary on a smooth parameter manifold [17]). We assume that the corresponding vertex sets $\{\sigma^0_{(1)}(k, ; \eta)\}_{k=1}^{N_0}$ and $\{\sigma^0_{(2)}(h,\ \eta)\}_{h=1}^{\widehat{N}_0}$ are disjoint for all $\eta \in [0, 1]$, and that they support distinct η–independent conical singularities $\{\Theta_{(1)}(k)\}$ and $\{\Theta_{(2)}(h)\}$. We also assume that $(T_{(1)}, M)_\eta$, and $(T_{(2)}, M)_\eta$, $\eta \in [0, 1]$, define the same (η–independent) conformal structure $((M, N_0),\ \mathscr{C}^{(1)}_{sg}) \simeq ((M, \widehat{N}_0),\ \mathscr{C}^{(2)}_{sg})$. Let $\{p_k(\eta)\}_{k=1}^{N_0} \in M$ and $\{q_h(\eta)\}_{h=1}^{\widehat{N}_0} \in M$ be the disjoint sets of points associated with the divisors

$$Div(T_{(1)},\ \eta) := \sum_{k=1}^{N_0}\left(\frac{\Theta_{(1)}(k)}{2\pi}-1\right) p_k(\eta), \tag{C.12}$$

and

$$Div(T_{(2)},\ \eta) := \sum_{h=1}^{\widehat{N}_0}\left(\frac{\Theta_{(2)}(h)}{2\pi}-1\right) q_h(\eta). \tag{C.13}$$

According to (2.70) the conical metric $ds^2_{T_{(1)}}$ of $((M, N_0),\ \mathscr{C}^{(1)}_{sg})$ around the generic conical point $p_k(\eta)$ is given, in term of a local conformal parameter $t(k,\ \eta)$, by

$$ds^2_{T_{(1)},\,(k)} := \frac{[L(k)]^2}{4\pi^2\ |t(k,\ \eta)|^2}\,|t(k,\ \eta)|^{2\left(\frac{\Theta_{(1)}(k)}{2\pi}\right)}\ |d\,t(k,\ \eta)|^2\,, \tag{C.14}$$

whereas the conical metric $ds^2_{T_{(2)}}$ of $((M, \widehat{N}_0),\ \mathscr{C}^{(2)}_{sg})$ around the generic conical point $q_h(\eta)$ is given, in term of a local conformal parameter $z(h,\ \eta)$, by

$$ds^2_{T_{(2)},\,(k)} := \frac{[L'(h)]^2}{4\pi^2\ |z(h,\ \eta)|^2}\,|z(h,\ \eta)|^{2\left(\frac{\Theta_{(2)}(h)}{2\pi}\right)}\ |d\,z(h,\ \eta)|^2\,. \tag{C.15}$$

Since in the metric $ds^2_{T_{(1)}}$ the points $\{q_h(\eta)\}_{h=1}^{\widehat{N}_0} \in M$, supporting the conical singularities of $ds^2_{T_{(2)}}$, are regular points, we can assume that there are smooth functions $g_{(2,\,h)}(\eta)$ of $z(h,\ \eta)$ such that in a neighborhood of $q_h(\eta)$ the metric $ds^2_{T_{(1)}}$ takes the form

$$ds^2_{T_{(1)},}\Big|_{q_h(\eta)} := \left|g_{(2,\,h)}(z(h,\ \eta))\right|^2\ |d\,z(h,\ \eta)|^2 . \tag{C.16}$$

Similarly, we can assume that there are smooth functions $f_{(1,\,k)}(\eta)$ such that in a neighborhood of $p_k(\eta)$ the metric $ds^2_{T_{(2)}}$ takes the form

$$ds^2_{T_{(2)},}\Big|_{p_k(\eta)} := \left|f_{(1,\,k)}(t(k,\ \eta))\right|^2\ |d\,t(k,\ \eta)|^2 . \tag{C.17}$$

With these notational remarks along the way, we have the following

Theorem C.3 (Kokotov [17]) *If* det $\Delta^{(1)}$ *and* det $\Delta^{(2)}$ *respectively denote the* ζ*–regularized determinants of the (Friedrichs extension of the) Laplacian associated with the conical metrics* $ds^2_{T_{(1)}}$ *and* $ds^2_{T_{(2)}}$*, then there is a constant C independent of* $\eta \in [0, 1]$ *such that*

$$\frac{\det'\ \Delta^{(1)}}{\det'\ \Delta^{(2)}} = C\ \frac{Area\,((M, N_0),\ \mathscr{C}^{(1)}_{sg})}{Area\,((M, \widehat{N}_0),\ \mathscr{C}^{(2)}_{sg})}\ \frac{\prod_{h=1}^{\widehat{N}_0}\left|\mathfrak{g}_{(2,\,h)}\right|^{\frac{1}{6}\left(\frac{\Theta^{(2)}(h)}{2\pi}-1\right)}}{\prod_{k=1}^{N_0}\left|\mathfrak{f}_{(1,\,k)}\right|^{\frac{1}{6}\left(\frac{\Theta^{(1)}(k)}{2\pi}-1\right)}}, \tag{C.18}$$

where $Area\,((M, N_0),\ \mathscr{C}^{(j)}_{sg})$, $j = 1, 2$, *denotes the area of the Riemann surface* $((M, N_0),\ \mathscr{C}^{(1)}_{sg})$ *in the corresponding conical metric* $ds^2_{T_{(j)}}$*, and where we have set* $\mathfrak{f}_{(1,\,k)} := f_{(1,\,k)}(t(k,\ \eta) = 0)$ *and* $\mathfrak{g}_{(2,\,h)} := g_{(2,\,h)}(z(h,\ \eta) = 0)$.

As emphasized by Kokotov, this results extends to polyhedral surfaces Polyakov's formula describing the scaling, under a conformal transformation, of the determinant of the Laplacian on a smooth Riemann surface.

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