Concluding Remarks

It is considered that French mathematician and political scientist J.-C. de Borda first attempted to solve the multicriteria problem. In particular, he suggested the ranked preferential voting system with the linear combination of criteria, see [1]. In the middle of the 19th century Irish economist F. Edgeworth [8] introduced the so-called "Edgeworth box," which actually involved the notion of a locally Pareto optimal alternative in terms of two criteria long before V. Pareto. The general notion of Pareto optimality appeared at the junction of the 19th and 20th centuries, but its intensive usage started in the 1940s–1950s. The research of that period was mostly dedicated to different generalizations of the well-known results on optimization theory, i.e., the development of necessary and sufficient optimality conditions and also existence conditions for certain optimality concepts, as well as to the duality issues in multicriteria programming (the problems with constraints defined by the solution set of system of equalities and/or inequalities). Nowadays this direction of investigations continues its evolvement, parallel to the corresponding branches of single-criterion optimization theory. In this context, also mention the research works suggesting different algorithms (including approximate ones) for Pareto set construction. For example, for the linear problems was developed the multicriteria analog of the simplex method, which yields all facets of the Pareto set.

On the other hand, following the vital demands of economics and engineering and the associated multicriteria optimization problems, many authors started suggesting different "best" solutions of the multicriteria problems using certain heuristic considerations. The pioneering results in this field belong to de Borda, see above. The scientific literature of the 1970s–1980s provides numerous examples illustrating how the linear combination of criteria (and other scalarization methods) can be used to solve various economic and engineering problems. In the 1980s it became finally clear that the "best" alternative choice cannot be justified without involving additional information (not including the collection of criteria and the set of feasible alternatives). That period was remarkable for the development and usage of the so-called "decision rules" that allow to extract the "best" solutions of the multicriteria problems in a certain sense. In the USSR (by then, with a considerable community of researchers focused on multicriteria optimization), different authors

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introduced decision rules by designing some "resulting" binary relations. Note that a similar trend showed up since the 1950s in the western countries after the appearance of Arrow's impossibility theorem (a Nobel Prize winner in economics). Subsequently, this trend yielded the general theory of alternative choice. An endeavor to translate the result of this theorem into the multicriteria language gives the following: generally, the multicriteria problem is not reducible to the single-criterion problem, since these problems are qualitatively different. After the impossibility theorem, hundreds of papers continued further analysis of the theoretical aspects and constructive ways to "aggregate" some general relation from a finite collection of partial binary relations. In terms of multicriteria optimization this means the reduction of the multicriteria problem to the single-criterion one (i.e., the scalarization of the multicriteria problem).

As mentioned, in the 1980s it became clear that the "best" alternative choice cannot be justified without involving additional information. For instance, such information may specify certain parameters (e.g., the weight coefficients of the linear combination of criteria) that participate in the corresponding scalarization approach. A series of authors proposed "the best alternative" based on some analogies or general considerations. As an example, refer to the center of gravity for the set of Pareto optimal vectors (by analogy with the Shapley value from game theory) or the Pareto optimal vector having the shortest distance to a certain ideal unattainable vector (like in goal programming).

According to the gradually maturing idea, the researchers started believing that the final choice is performed by an individual interested in the solution of the multicriteria problem (called the decision-maker). Each human has the right to consider its own "best" alternatives. Therefore, all attempts to suggest a universal rule or notion of the "best" alternative are doomed from the start. Embedding the binary preference relation in the multicriteria problem gave an opportunity to take into account the specifics of certain DM. However, the difficulty is that a human dealing with the choice problem often has a hazy idea of its preferences. In any case, the DM is unable to describe completely its preference relation. And the path of further development laid towards the consideration of some "fragmentary" information about the DM's preference relation, with minimum assumptions imposed on it. Such information was represented by the pairs of incomparable vectors in terms of the Pareto relation: in each pair, the DM surely prefers one vector to the other. And such information was later called the information quanta about the DM's preference relation.

The preference relation was explicitly incorporated into the multicriteria problem statement in the author's report presented in 1982 ([29]). The cited report also included the requirements in the form of axioms imposed on this relation (irreflexivity, the Pareto axiom, transitivity and invariance). The role of additional information about the DM's preference relation was played by a finite collection of the pairs of incomparable vectors (in terms of the Pareto relation) where one vector is preferable to the other. The axiomatic approach originated in this report. Later on, the axiomatic approach was developed in detail first in 1986 for the bicriteria problems [30] and then in 1991 for the multicriteria problems with an arbitrary finite number of criteria [31]. The multicriteria problem supplemented by the DM's binary preference relation was subsequently called the multicriteria choice problem in order to underline its connection to general choice theory (especially, to the paired dominant choice, i.e., the choice based on a certain binary relation). Next, the theorem on taking into account a general information quantum was proved in [33], and the Edgeworth-Pareto principle was logically justified in [37].

Year 2003 saw the monograph by the author on the quantitative approach to decision-making in multicriteria environment, which systematized the results obtained by then. The second edition of the monograph was published in 2005, see [38]. As a matter of fact, a series of new interesting results was published since that time. Presently, the axiomatic approach to reduce the Pareto set can be considered well-developed and, as the author believes, this book gives a rather complete description of the theory.

The interconnection between the original axiomatic approach and other methods and approaches was discussed in [39].

In conclusion, note that in some papers (see, for example, [16]) the authors avoid the axiomatic characterization of the preference relation and operate the terminology of cone-based approach with allowable tradeoffs among criteria. In fact, they deal with information quanta in the special case where all parameters w_i are 1.

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