Appendix A
Euler-MacLaurin and Euler-Boole Formulas

A.1 A Taylor Formula

The classical Taylor formula

$$f(x) = \sum_{k=0}^{m} \partial^k f(0)\frac{x^k}{k!} + \int_0^x \frac{(x-t)^m}{m!}\partial^{m+1}f(t)dt$$

can be generalized if we replace the polynomial $\frac{x^k}{k!}$ by other polynomials (Viskov 1988; Bourbaki 1959).

Definition If μ is a linear form on $C^0(\mathbb{R})$ such that $\mu(1) = 1$, we define the polynomials (P_n) by:

$$P_0 = 1$$
$$\partial P_n = P_{n-1}\ ,\ \mu(P_n) = 0 \text{ for } n \geq 1$$

The Generating Function $\sum_{k\geq 0} P_k(x)z^k$

We have formally

$$\partial_x(\sum_{k\geq 0} P_k(x)z^k) = \sum_{k\geq 1} P_{k-1}(x)z^k = z(\sum_{k\geq 0} P_k(x)z^k)$$

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thus

$$\sum_{k\geq 0} P_k(x)z^k = C(z)e^{xz}$$

To evaluate $C(z)$ we use the notation μ_x for μ and by definition of (P_n) we can write

$$\mu_x(\sum_{k\geq 0} P_k(x)z^k) = \sum_{k\geq 0} \mu_x(P_k(x))z^k = 1$$

$$\mu_x(\sum_{k\geq 0} P_k(x)z^k) = \mu_x(C(z)e^{xz}) = C(z)\mu_x(e^{xz})$$

this gives $C(z) = \frac{1}{\mu_x(e^{xz})}$. Thus the generating function of the sequence (P_n) is

$$\sum_{n} P_n(x)z^n = e^{xz}/M_\mu(z)$$

where the function M_μ is defined by $M_\mu(z) = \mu_x(e^{xz})$.

Examples

(1) $\mu(f) = f(0),\ P_n(x) = \frac{x^n}{n!}$, $M_\mu(z) = 1,\ \sum_{n\geq 0} P_n(x)z^n = e^{xz}$
(2) $\mu(f) = \int_0^1 f(t)dt$, $P_n(x) = \frac{B_n(x)}{n!}$, $M_\mu(z) = \int_0^1 e^{zx}dx = \frac{1}{z}(e^z - 1)$

$$\sum_{n\geq 0} \frac{B_n(x)}{n!} z^n = \frac{ze^{xz}}{e^z - 1}$$

The $B_n(x)$ are the Bernoulli polynomials and the $B_n = B_n(0)$ the Bernoulli numbers. With the generating function we verify that $B_0 = 1$, $B_1 = -1/2$, $B_{2n+1} = 0$ if $n \geq 1$, $B_n(1-x) = (-1)^n B_n(x)$.

(3) $\mu(f) = \frac{1}{2}(f(0) + f(1))$, $P_n(x) = \frac{E_n(x)}{n!}$

$$\sum_{n\geq 0} \frac{E_n(x)}{n!} z^n = \frac{2e^{xz}}{e^z + 1}$$

The $E_n(x)$ are the Euler polynomials and we set $E_n = E_n(0)$.

With the generating function we verify that $E_0 = 1, E_1 = -1/2$ if $n \geq 1$, $E_n(1-x) = (-1)^n E_n(x)$.

The Taylor Formula

Let f be a function in $C^\infty(\mathbb{R})$, then we have

$$f(x) = f(y) + \int_y^x \partial P_1(x+y-t)\partial f(t)dt$$

and by integration by parts we get for every $m \geq 1$

$$f(x) = f(y) + \sum_{k=1}^{m}(P_k(x)\partial^k f(y) - P_k(y)\partial^k f(x))$$
$$+ \int_y^x P_m(x+y-t)\partial^{m+1} f(t)dt$$

Applying μ to this function as a function of y gives a **general Taylor formula**: for every $m \geq 0$

$$f(x) = \sum_{k=0}^{m} \mu_y(\partial^k f(y))P_k(x) + \mu_y(\int_y^x P_m(x+y-t)\partial^{m+1} f(t)dt)$$

A.2 The Euler-MacLaurin Formula

We can transform the Taylor formula to get a summation formula. Taking $x = 0$ we get

$$f(0) = \sum_{k=0}^{m} \mu_y(\partial^k f(y))P_k(0) - \mu_y(\int_0^y P_m(y-t)\partial^{m+1} f(t)dt)$$

In the case of $\mu : f \mapsto \int_0^1 f(t)dt$ we have

$$f(0) = \sum_{k=0}^{m} \frac{B_k}{k!}\partial^{k-1} f]_0^1 - \int_0^1 (\int_0^y \frac{B_m(y-t)}{m!}\partial^{m+1} f(t)dt)dy$$

Replacing m by $2m$ and with $B_1 = -1/2$ and $B_{2k+1} = 0$, we get

$$f(0) = \int_0^1 f(t)dt + \frac{1}{2}(f(0) - f(1)) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\partial^{2k-1} f]_0^1$$
$$- \int_0^1 (\int_0^y \frac{B_{2m}(y-t)}{(2m)!}\partial^{2m+1} f(t)dt)dy$$

The last integral can easily be evaluated by Fubini's theorem, we get

$$f(0) = \int_0^1 f(t)dt + \frac{1}{2}(f(0) - f(1)) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\partial^{2k-1}f]_0^1$$
$$+ \int_0^1 \frac{B_{2m+1}(t)}{(2m+1)!}\partial^{2m+1}f(t)dt$$

Let j be a positive integer, by replacing f by $x \mapsto f(j+x)$ in the last formula, we have

$$f(j) = \int_j^{j+1} f(t)dt + \frac{1}{2}(f(j) - f(j+1)) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\partial^{2k-1}f]_j^{j+1}$$
$$+ \int_j^{j+1} \frac{b_{2m+1}(t)}{(2m+1)!}\partial^{2m+1}f(t)dt$$

where $b_{2m+1}(t) = B_{2m+1}(t - [t])$.

Summing these relations for j from 1 to $n-1$, we get for $f \in C^\infty(]0, \infty[)$ the **Euler-MacLaurin formula**

$$f(1) + \ldots + f(n) = \int_1^n f(x)dx + \frac{f(1) + f(n)}{2}$$
$$+ \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}[\partial^{2k-1}f]_1^n$$
$$+ \int_1^n \frac{b_{2m+1}(x)}{(2m+1)!}\partial^{2m+1}f(x)dx$$

A.3 The Euler-Boole Formula

In the case of the Euler polynomials, the formula

$$f(0) = \sum_{k=0}^{m} \mu_y(\partial^k f(y))P_k(0) - \mu_y(\int_0^y P_m(y-t)\partial^{m+1}f(t)dt)$$

gives

$$f(0) = \sum_{k=0}^{m} \frac{1}{2}(\partial^k f(0) + \partial^k f(1))\frac{E_k}{k!} - \frac{1}{2}\int_0^1 \frac{(-1)^m E_m(t)}{m!}\partial^{m+1}f(t)dt$$

Let j be a positive integer, by replacing f by $x \mapsto f(j + x)$ in the last formula we obtain

$$f(j) = \sum_{k=0}^{m} \frac{1}{2}(\partial^k f(j) + \partial^k f(j+1)) \frac{E_k}{k!} - \frac{1}{2} \int_j^{j+1} \frac{(-1)^m E_m}{m!} (t - j)\, \partial^{m+1} f(t)\ dt$$

Let's define

$$e_m(t) = (-1)^{[t]}(-1)^m E_m (t - [t])$$

we obtain by summation on j the **Euler-Boole summation formula**

$$f(1) - f(2) + \ldots + (-1)^{n-1} f(n) = \frac{1}{2} \sum_{k=0}^{m} \partial^k f(1) \frac{E_k}{k!} + \frac{(-1)^{n-1}}{2} \sum_{k=0}^{m} \partial^k f(n+1) \frac{E_k}{k!} + \frac{1}{2} \int_1^{n+1} \frac{1}{m!} e_m(t)\, \partial^{m+1} f(t)\ dt$$

Appendix B
Ramanujan's Interpolation Formula and Carlson's Theorem

We give a proof of the following theorem.

Carlson's Theorem *Let f be an analytic function in the half-plane $Re(z) > -d$ where $0 < d < 1$. Let's assume there exist $a > 0$ and $b < \pi$ such that*

$$|f(z)| \le ae^{b|z|} \text{ for every } z \text{ with } Re(z) > -d$$

Then the condition $f(n) = 0$ for $n = 0, 1, 2, \ldots,$ implies $f = 0$.

We prove this theorem by the use of an interpolation formula which is related to Ramanujan's interpolation formula.

First in Theorem 1 below we get an integral formula for the function

$$g : x \mapsto \sum_{n=0}^{+\infty} f(n)(-1)^n x^n$$

which is

$$g(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi u)} f(-u) x^{-u} du$$

Then in Theorem 2 we prove the interpolation formula

$$f(z) = \frac{-\sin(\pi z)}{\pi} \int_0^{+\infty} g(x) x^{-z-1} dx$$

Since for the definition of the function g we only need to know the values $f(n)$, $n = 0, 1, 2, \ldots$., we see that this interpolation formula determines the function f in the half-plane $\mathrm{Re}(z) < -d$ when we only know the values $f(0), f(1), f(2), \ldots$, thus we have a proof of Carlson's theorem.

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Theorem 1 *Let f be an analytic function in the half-plane $Re(z) > -d$ where $0 < d < 1$. Let's assume there exist $a > 0$ and $b < \pi$ such that*

$$|f(z)| \leq ae^{b|z|} \text{ for every } z \text{ with } Re(z) > -d$$

Then the series $\sum_{n\geq 0} f(n)(-1)^n x^n$ is convergent in

$$D(0, e^{-b}) = \{x \in \mathbb{C}, |x| < e^{-b}\}$$

and defines an analytic function g in $D(0, e^{-b})$.

This function g has an analytic continuation in $S_b = \{|Arg(z)| < \pi - b\}$ which is defined by

$$g : z \mapsto \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi u)} f(-u) z^{-u} du$$

Proof For $0 \leq \alpha < e^{-b}$ we have

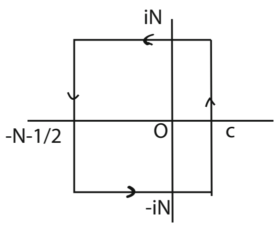
$$|f(n)(-1)^n x^n| \leq ae^{bn}\alpha^n$$

Thus the series $\sum_{n\geq 0} f(n)(-1)^n x^n$ is normally convergent in $D(0, \alpha)$ and defines a function $g : z \mapsto \sum_{n=0}^{+\infty} f(n)(-1)^n z^n$ that is analytic in $D(0, e^{-b})$.

For $0 < x < e^{-b}$ and $0 < c < d$, the function $u \mapsto f(-u)x^{-u}$ is analytic in the half-plane $\text{Re}(u) < d$, and we consider the integral

$$\frac{1}{2i\pi} \int_{\gamma_N} \frac{\pi}{\sin(\pi u)} f(-u) x^{-u} du$$

where γ_N is the path



The function $u \mapsto \frac{\pi}{\sin(\pi u)} f(-u) x^{-u}$ has simple poles at $0, -1, -2, \ldots, -n, \ldots$ with

$$\text{Res}(\frac{\pi}{\sin(\pi u)}; -n) = (-1)^n f(n) x^n$$

thus we get

$$\frac{1}{2i\pi}\int_{\gamma_N}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du = \sum_{n=0}^{N} f(n)(-1)^n x^n$$

Lemma 1 *When $N \to +\infty$ we have for $0 < x < e^{-b}$*

$$\frac{1}{2i\pi}\int_{\gamma_N}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du \to \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du$$

By the preceding lemma we get for $0 < x < e^{-b}$

$$\lim_{N\to+\infty}\frac{1}{2i\pi}\int_{\gamma_N}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du = \lim_{N\to+\infty}\frac{1}{2i\pi}\int_{c-iN}^{c+iN}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du$$

Thus we have for $0 < x < e^{-b}$

$$\sum_{n=0}^{+\infty} f(n)(-1)^n x^n = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du$$

Lemma 2 *For $0 < c < d$, the function*

$$g :\mapsto \int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)z^{-u}du$$

is defined and analytic in $S_b = \{|Arg(z)| < \pi - b\}$.

The function

$$z \mapsto \sum_{n=0}^{+\infty} f(n)(-1)^n z^n$$

is defined and analytic in $D(0, e^{-b}) = \{|z| < e^{-b}\}$ and is equal to g in the interval $[0, e^{-b}]$. By analytic continuation we get

$$\sum_{n=0}^{+\infty} f(n)(-1)^n z^n = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)z^{-u}du \text{ if } z \in D(0, e^{-b}) \cap S_b.$$

□

The Mellin Inversion We now show that the formula

$$g(z) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)z^{-u}du$$

can be inverted to give

$$\frac{\pi}{\sin(\pi z)} f(-z) = \int_0^{+\infty} g(x) x^{z-1} dx$$

Theorem 2 *Let f be an analytic function in the half-plane $Re(z) > -d$ where $0 < d < 1$. Let's assume there exist $a > 0$ and $b < \pi$ such that*

$$|f(z)| \leq a e^{b|z|} \text{ for every } z \text{ with } Re(z) > -d$$

For $0 < Re(z) < d$ we get

$$\frac{\pi}{\sin(\pi z)} f(-z) = \int_0^{+\infty} g(x) x^{z-1} dx$$

where g is the analytic continuation of the function $z \to \sum_{n=0}^{+\infty} f(n)(-1)^n z^n$ in $S_b = \{|Arg(z)| < \pi - b\}$.

We have in the half-plane $Re(z) > -d$ the interpolation formula

$$f(z) = \frac{-\sin(\pi z)}{\pi} \int_0^{+\infty} g(x) x^{-z-1} dx$$

Proof We consider $0 < c_1 < c_2 < d$, and z such that $c_1 < \text{Re}(z) < c_2$.

(a) First we evaluate $\int_0^1 g(x) x^{z-1} dx$.

We have

$$\begin{aligned}\int_0^1 g(x) x^{z-1} dx &= \int_0^1 \left(\frac{1}{2i\pi} \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{\pi}{\sin(\pi u)} f(-u) x^{-u} du\right) x^{z-1} dx \\ &= \int_0^1 \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(-c_1 - it) x^{-c_1 - it} \frac{\pi}{\sin(\pi(c_1 + it))} dt\right) x^{z-1} dx\end{aligned}$$

Since

$$\left| f(-c_1 - it) \frac{\pi x^{-c_1 - it} x^{z-1}}{\sin(\pi(c_1 + it))} \right| \leq 2\pi a e^{bc} x^{-c_1} x^{\text{Re}(z)-1} e^{b|t|} \left| \frac{1}{e^{i\pi c_1} e^{-\pi t} - e^{-i\pi c_1} e^{\pi t}} \right|$$

with $Re(z) - c_1 - 1 > -1$ we get the integrability for $(t, x) \in \mathbb{R} \times [0, 1]$.

Thus by Fubini's theorem we get

$$\begin{aligned}\int_0^1 g(x) x^{z-1} dx &= \frac{1}{2i\pi} \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{\pi}{\sin(\pi u)} f(-u) \left(\int_0^1 x^{-u+z-1} dx\right) du \\ &= \frac{1}{2i\pi} \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{\pi}{\sin(\pi u)} f(-u) \frac{-1}{u - z} du\end{aligned}$$

(b) Then we evaluate $\int_1^{+\infty} g(x)x^{z-1}dx$.

This is

$$\int_1^{+\infty} g(x)x^{z-1}dx = \int_1^{+\infty} (\frac{1}{2i\pi}\int_{c_2-i\infty}^{c_2+i\infty} \frac{\pi}{\sin(\pi u)} f(-u)x^{-u}du)x^{z-1}dx$$

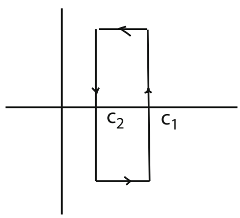
As in (a) we see that for $\mathrm{Re}(z) - c_2 - 1 < -1$ we can apply Fubini's theorem to get

$$\int_1^{+\infty} g(x)x^{z-1}dx = \frac{1}{2i\pi}\int_{c_2-i\infty}^{c_2+i\infty} \frac{\pi}{\sin(\pi u)} f(-u)(\int_1^{+\infty} x^{-u+z-1}dx)du$$
$$= \frac{1}{2i\pi}\int_{c_2-i\infty}^{c_2+i\infty} \frac{\pi}{\sin(\pi u)} f(-u)\frac{1}{u-z}du$$

Finally we have for $c_1 < \mathrm{Re}(z) < c_2$

$$\int_0^{+\infty} g(x)x^{z-1}dx = \frac{1}{2i\pi}\int_{c_2-i\infty}^{c_2+i\infty} \frac{\pi}{\sin(\pi u)} f(-u)\frac{1}{u-z}du$$
$$-\frac{1}{2i\pi}\int_{c_1-i\infty}^{c_1+i\infty} \frac{\pi}{\sin(\pi u)} f(-u)\frac{1}{u-z}du$$

We then apply Cauchy's formula with the path



to get

$$\int_0^{+\infty} g(x)x^{z-1}dx = \frac{\pi}{\sin(\pi z)} f(-z)$$

By the preceding result we have for $-d < \mathrm{Re}(z) < 0$

$$f(z) = \frac{-\sin(\pi z)}{\pi}\int_0^{+\infty} g(x)x^{-z-1}dx$$

where g is the analytic continuation of the function $z \to \sum_{n=0}^{+\infty} f(n)(-1)^n z^n$.

The function

$$z \mapsto \frac{-\sin(\pi z)}{\pi} \int_0^{+\infty} g(x) x^{-z-1} dx$$

is defined and analytic in the half-plane $\mathrm{Re}(z) > -d$ since by the integral formula for $g(x)$ $(x > 0)$ we can write

$$\left| f(-c-it) x^{-c-it} \frac{\pi}{\sin(\pi(c+it))} \right| \leq 2\pi a e^{bc} x^{-c} e^{b|t|} \left| \frac{1}{e^{i\pi c} e^{-\pi t} - e^{-i\pi} e^{\pi t}} \right|$$

to get $g(x) = O(x^{-c})$ for $x \to +\infty$, for $0 < c < d$.

Thus by analytic continuation we get the *interpolation formula* in the half-plane $\mathrm{Re}(z) > -d$

$$f(z) = \frac{-\sin(\pi z)}{\pi} \int_0^{+\infty} g(x) x^{-z-1} dx$$

□

Remark: Ramanujan's Interpolation Formula

Let's consider the function $f : x \mapsto \frac{1}{\Gamma(x+1)}$. This function is analytic in the half-plane $Re(z) > -1$. The function g defined by the analytic continuation of

$$z \to \sum_{n=0}^{+\infty} (-1)^n \frac{z^n}{n!}$$

is simply the function $z \mapsto e^{-z}$. Thus by the preceding theorem we get for $0 < \mathrm{Re}(z) < 1$

$$\frac{\pi}{\sin(\pi z)} \frac{1}{\Gamma(1-z)} = \int_0^{+\infty} e^{-x} x^{z-1} dx = \Gamma(z)$$

With $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ we get for $0 < \mathrm{Re}(z) < d$

$$\int_0^{+\infty} g(x) x^{z-1} dx = \Gamma(z)\Gamma(1-z) f(-z)$$

Let's take $h(z) = f(z)\Gamma(z+1)$. We have

$$g(x) = \sum_{n=0}^{+\infty} h(n)(-1)^n \frac{x^n}{n!}$$

and we get *Ramanujan's interpolation formula*

$$h(-z) = \frac{1}{\Gamma(z)} \int_0^{+\infty} g(x) x^{z-1} dx$$

Proofs of the Lemmas

Lemma 1 *When* $N \to +\infty$ *we have for* $0 < x < e^{-b}$

$$\frac{1}{2i\pi} \int_{\gamma_N} \frac{\pi}{\sin(\pi u)} f(-u) x^{-u} du \to \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi u)} f(-u) x^{-u} du$$

Proof (a) The integral on the vertical line through $-N - \frac{1}{2}$ is

$$\int_{-N}^{N} f(N + \frac{1}{2} - it) x^{N+\frac{1}{2}-it} \frac{\pi}{\sin(\pi(-N - \frac{1}{2} + it))} idt$$

since

$$\frac{\pi}{\sin(\pi(-N - \frac{1}{2} + it))} = \frac{2\pi(-1)^{N+1}}{e^{\pi t} + e^{-\pi t}}$$

we get

$$\left| \int_{-N}^{N} f(N + \frac{1}{2} - it) \frac{\pi x^{N+\frac{1}{2}-it}}{\sin(\pi(-N - \frac{1}{2} + it))} idt \right| \leq 2\pi x^{N+\frac{1}{2}} \int_{-N}^{N} \frac{a e^{b(N+\frac{1}{2})} e^{b|t|}}{e^{\pi t} + e^{-\pi t}} dt$$

Since $b < \pi$ we have

$$\int_{-N}^{N} \frac{e^{b|t|}}{e^{\pi t} + e^{-\pi t}} dt \to \int_{-\infty}^{+\infty} \frac{e^{b|t|}}{e^{\pi t} + e^{-\pi t}} dt$$

For $0 < x < e^{-b}$ we have $b + Log(x) < 0$ thus

$$e^{b(N+\frac{1}{2})} x^{N+\frac{1}{2}} = e^{(N+\frac{1}{2})(b+Log(x))} \to 0$$

Thus the integral on the vertical line through $-N - \frac{1}{2}$ tends to 0 when $N \to +\infty$.

(b) The integral on the horizontal segment from $c + iN$ to $-N - \frac{1}{2} + iN$ is

$$I_N = -\int_{-N-\frac{1}{2}}^{c} f(-t-iN)x^{-t-iN}\frac{2i\pi}{e^{-\pi N}e^{i\pi t} - e^{\pi N}e^{-i\pi t}}dt$$

and we have

$$\left| f(-t-iN)x^{-t-iN}\frac{2i\pi}{e^{-\pi N}e^{i\pi t} - e^{\pi N}e^{-i\pi t}} \right| \leq ae^{b(t+N)}x^{-t}\frac{2\pi}{e^{\pi N} - e^{-\pi N}}$$

thus

$$|I_N| \leq ae^{bN}\frac{2\pi}{e^{\pi N} - e^{-\pi N}}\int_{-N-\frac{1}{2}}^{c} e^{t(b-Log(x))}dt$$

since $0 < x < e^{-b}$ we have $b - Log(x) > 2b > 0$ thus

$$\int_{-N-\frac{1}{2}}^{c} e^{t(b-Log(x))}dt \to \int_{-\infty}^{c} e^{t(b-Log(x))}dt$$

And $e^{bN}\frac{2\pi}{e^{\pi N}-e^{-\pi N}} \to 0$ since $b < \pi$.

(c) For the integral on the horizontal segment from $-N - \frac{1}{2} - iN$ to $c - iN$ the proof is similar to the preceding one. □

Lemma 2 *Let's take* $0 < c < d$, *the function*

$$g : z \mapsto \int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi u)} f(-u)z^{-u}du$$

is defined and analytic in $S_b = \{|Arg(z)| < \pi - b\}$.

Proof For this integral on the vertical line through c we can write

$$\left| f(-c-it)\frac{\pi z^{-c-it}}{\sin(\pi(c+it))}i \right| \leq 2\pi ae^{bc}|z|^{-c}e^{b|t|}e^{tArg(z)}\left| \frac{1}{e^{i\pi c}e^{-\pi t} - e^{-i\pi c}e^{\pi t}} \right|$$

For z in any compact K of S_b we have

$$e^{b|t|}e^{tArg(z)}\left| \frac{1}{e^{i\pi c}e^{-\pi t} - e^{-i\pi c}e^{\pi t}} \right| \leq k(t)$$

where $t \mapsto k(t)$ is an integrable function independent of $z \in K$ since $|Arg(z)| < \pi - b'$ with $b < b' < \pi$.

The function

$$z \mapsto f(-c-it)z^{-c-it}\frac{\pi}{\sin(\pi(c+it))}i$$

is analytic for all t, thus we get the analyticity of the function defined by the integral. □

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