

Appendix A

Historical Notes

We provide here a number of historical and other notes on the material of this work.

Section 1.3: A comprehensive treatment of stable processes and their integral representations can be found in Samorodnitsky and Taqqu [56]. Univariate stable distributions is the focus of a fascinating monograph by Uchaikin and Zolotarev [63], as well as of the classic monograph by Zolotarev [69]. For a more recent treatment of stable distributions and processes, see Rachev and Mittnik [43] for applications in finance, and Nolan [32].

We assumed in Section 1.3 and throughout the work that stable processes are $S\alpha S$. In the study of stationary stable processes and their connections to flows, *skewness* was included with “minor” modifications in Rosiński [45] and Kolodyński and Rosiński [24]. Extensions to stationary stable *fields*, that is, stable processes $X_\alpha(t)$ indexed by $t \in \mathbb{R}^d$ or $t \in \mathbb{Z}^d$, are not immediate and were considered in Rosiński [47] and Roy and Samorodnitsky [49]. Similar developments concerning skewness and random fields are expected but not available yet in the context of self-similar mixed moving averages.

Section 2.1: The definition (2.3) of minimal integral representations was given by Hardin [19, 20]. Rosiński [48] reexamined minimal integral representations in greater depth and provided alternative formulations, including those in (2.4)–(2.5) and Remark 2.1. Pipiras [34] showed that a minimal integral representation can be achieved naturally from a nonminimal one by removing a so-called nonminimal set from the space underlying the nonminimal integral representation.

Despite significant progress in the theory, determining whether a given integral representation is minimal, may not be immediate. See, for example, Examples 2.2 and 3.3 concerning stationary and stationary increments moving averages.

Section 2.2: Theorem 2.1, characterizing linear isometries of spaces L^α , appears in Banach [4] and Lamperti [29]. It is a well-known result in real analysis, for example, part of the classic by Royden [50], Chapter 15. For a more advanced and

comprehensive treatment, including Rudin's theorem (see (2.36)–(2.37)), see Fleming and Jamison [14], Section 3. Section 3.5, in particular, has interesting notes and remarks.

The statement and proof of Proposition 2.2, relating minimal integral representations and isometries, are taken from Rosiński [48].

Section 2.3: As indicated in the introduction of Section 1.1, the connection between stationary stable processes and nonsingular flows outlined in Section 2.3.1 is due to Rosiński [46]. Nonsingular flows, cocycles, and related notions (Hopf decomposition, special flows) are part of ergodic theory (Cornfeld et al. [8], Krengel [26], Zimmer [68]), and, especially, infinite¹ ergodic theory (Aaronson [1], Glasner [15]).

1- and 2-semi-additive functionals arise specifically when relating self-similar mixed moving averages and flows. The terms are coined by analogy to additive functionals $f_h(s)$ (for a flow $\phi_h(s)$) satisfying

$$f_{h_1+h_2}(s) = f_{h_1}(s) + f_{h_2}(\phi_{h_1}(s)), \quad s \in S, h_1, h_2 \in \mathbb{R}$$

(Kubo [27, 28]). See Pipiras and Taqqu [41].

Having established a connection between stable processes and flows, it is natural to ask how ergodic or other properties of flows affect those of the corresponding stable processes. For stationary stable processes, this is studied in Samorodnitsky [52, 53].

Section 3.1: The name mixed moving averages is adopted from the stationary case, where mixed moving averages are represented as

$$\int_X \int_{\mathbb{R}} G(x, t + u) M(dx, du).$$

These processes were studied in Surgailis et al. [60], and are stationary stable processes generated by dissipative flows in the decomposition established by Rosiński [46].

Using a suitable transformation, Surgailis et al. [61] associated stable processes with stationary increments with stable stationary processes, and then decomposed them based on the decomposition of stationary processes of Rosiński [46]. In this approach, mixed moving averages with stationary increments are associated with stationary mixed moving averages and can be represented by (3.1). From the perspective of these structural results, it should be noted that mixed moving averages make just a single, even if large, class of stable processes with stationary increments (namely, those generated by dissipative flows).

Section 3.2: Sections 3.2.1 and 3.2.2 are based on Pipiras and Taqqu [35], Section 3.2.3 on Pipiras and Taqqu [36, 37], Section 3.2.5 on Pipiras and Taqqu [36, 40], and Section 3.2.6 on Pipiras and Taqqu [39].

¹ “Infinite” refers here to the fact that the measure on the space underlying a flow is not finite. Another term used is “nonsingular ergodic theory.”

Several notable differences between Section 3.2 and these references are the following. Definition 3.2 found in [35, 36] does not stipulate for $j_c(x)$ to be a 2-semi-additive functional. By making this assumption, the current definition thus provides a description of the structure of $j_c(x)$ as a function in (c, x) . This is useful, for example, in establishing canonical representations of self-similar mixed moving averages (see the proof of Theorem 3.11). To account for the change in the definition, we had to modify the proofs of the fundamental Theorem 3.1 and several other results.

Proposition 3.2 on the existence of minimal mixed moving average representations was proved in [35] under the assumption $\alpha \in (1, 2)$. This assumption was then made throughout [35, 36]. The proposition was extended to the entire range $\alpha \in (0, 2)$ in Pipiras [34], making it possible to assume this range throughout Section 3.2. Finally, we also note that fixed fractional stable motions (FFSMs) were called mixed linear fractional stable motions (mixed LFSMs) in [36, 39, 40]. While the latter term was motivated by the canonical representation in Theorem 3.11, the term FFSM used in Section 3.2 is in line with the nature of the underlying (that is, fixed) flow, and the analogous terms cLFSM for cyclic fractional stable motion and PFSM for periodic fractional stable motion.

Other notes: Natural connections to nonsingular flows are available not only for stable processes but also for the so-called max-stable processes. If stable distributions arise as limits of normalized partial sums, max-stable distributions arise as limits of normalized partial maxima. Max-stable processes thus play an important role in extreme value theory (Coles [7], de Haan and Ferreira [9]).

Minimal representations, max-linear isometries, and glimpses of flows in the context of max-stable processes date back at least to de Haan and Pickands [10]. A flow-based decomposition of stationary max-stable processes, parallel to that of Rosiński [46] for stationary stable processes, was established by Wang and Stoev [67]. Parallels between max-stable and stable frameworks are studied in Kabluchko [21] and Wang and Stoev [66].

Appendix B

Standard Lebesgue Spaces and Projections

A measure space (S, \mathcal{S}, ν) is called a *standard Lebesgue space* when (S, \mathcal{S}) is a *standard Borel space*, equipped with a σ -finite measure μ . In a standard Borel space, S can be thought as a Borel subset of a Polish space, and the σ -field \mathcal{S} is the σ -field of Borel sets $\mathcal{B}(S)$ defined as $\mathcal{B}(S) = \sigma\{A : A \subset S \text{ is open}\}$. An example of a standard Lebesgue space is the Euclidean space \mathbb{R}^n , with a measure consisting of Lebesgue measure and discrete point masses.

Standard Lebesgue spaces (or standard Borel spaces) are convenient to work with, have nice properties and are widely used in ergodic theory (Walters [65], Petersen [33]) and in other areas of mathematics (Mackey [31], Arveson [3], Zimmer [68]).

In this book, we used a number of times the notion of projection. If S_1 and S_2 are two spaces, the projection of a set $E \in S_1 \times S_2$ onto S_1 is defined as

$$\text{proj}_{S_1} E = \{s_1 \in S_1 : \exists s_2 \in S_2 : (s_1, s_2) \in E\}.$$

The projection $\text{proj}_{S_2} E$ is defined in a similar way. When $S_1 = S_2 = S$, proj_S is understood as the projection onto the first variable.

When $(S_1, \mathcal{B}(S_1))$ and $(S_2, \mathcal{B}(S_2))$ are two standard Borel spaces, and $E \in \mathcal{B}(S_1) \times \mathcal{B}(S_2)$, it is well known that $\text{proj}_{S_1} E$ is not necessarily in $\mathcal{B}(S_1)$. This important fact has essentially given rise to the field of the so-called (classical) descriptive set theory. See, for example, the monographs of Kechris [23] and Srivastava [59].

One of the key notions in descriptive set theory is that of *projective classes* $\Sigma_n^1(S), \Pi_n^1(S), \Delta_n^1(S), n \in \mathbb{N}$, on a Polish space S . They can be defined recursively in n as follows. For $n = 1$,

$$\Sigma_1^1(S) = \{B : B = \text{proj}_S A \text{ for } A \in \mathcal{B}(S^2)\}, \tag{B.1}$$

$$\Pi_1^1(S) = \{B^c : B \in \Sigma_1^1(S)\}, \quad \Delta_1^1(S) = \Sigma_1^1(S) \cap \Pi_1^1(S).$$

The elements of $\Sigma_1^1(S)$ and $\Pi_1^1(S)$ are called *analytic* and *coanalytic* sets, respectively, and $\Delta_1^1(S) = \mathcal{B}(S)$. Then, recursively in n ,

$$\Sigma_{n+1}^1(S) = \{B : B = \text{proj}_S A \text{ for } A \in \Pi_n^1(S^2)\},$$

$$\Pi_{n+1}^1(S) = \{B^c : B \in \Sigma_{n+1}^1(S)\}, \quad \Delta_{n+1}^1(S) = \Sigma_{n+1}^1(S) \cap \Pi_{n+1}^1(S).$$

Another, more general way is to define the class Σ_{n+1}^1 as images of sets from the projective class Π_n^1 under Borel maps (projection, as in (B.1), is one such map).

Much is known about the above projective classes. For example, in the following diagram, any class is a subset of every class to the right of it:

$$\begin{array}{ccccccc} & & \Sigma_1^1(S) & & \Sigma_2^1(S) & \dots & \\ \mathcal{B}(S) = \Delta_1^1(S) & & & \Delta_2^1(S) & \dots & & \overline{\mathcal{B}}(S) \\ & & \Pi_1^1(S) & & \Pi_2^1(S) & \dots & \end{array} \quad (\text{B.2})$$

where

$$\overline{\mathcal{B}}(S) := \overline{\mathcal{B}}_\mu(S)$$

is the completion σ -field of $\mathcal{B}(S)$ under μ . The classes $\Sigma_n^1(S)$ and $\Pi_n^1(S)$ are closed under countable unions and intersections, and $\Delta_n^1(S)$ are σ -fields, and so on. Except the proof of Lemma B.1 below, the various intermediate classes in (B.2) are not used directly in the book. But they are naturally related to projections, and as discussed here, are behind a useful theory of classes in between the commonly used spaces $\mathcal{B}(S)$ and $\overline{\mathcal{B}}(S)$.

Another important idea is that of *uniformization* and *uniformizing functions* defined in Kechris [23], or measurable sections defined in Srivastava [59], or measurable selections defined in Wagner [64]. We are not going to describe the many related results here. We shall only state next an auxiliary result which was used above (e.g., Lemma 2.3). The function h appearing below is called a measurable selection.

Lemma B.1. *Let $(S_1, \mathcal{S}_1, \nu_1)$ and $(S_2, \mathcal{S}_2, \nu_2)$ be two standard Lebesgue spaces and $(S_1 \times S_2, \mathcal{S}_1 \times \mathcal{S}_2, \nu_1 \times \nu_2)$ be their Cartesian product. Let also $A \in \mathcal{S}_1 \times \mathcal{S}_2$ be a Borel set of $S_1 \times S_2$. Then, the set*

$$\text{proj}_{S_1} A := \{s_1 \in S_1 : \exists s_2 \in S_2 : (s_1, s_2) \in A\}$$

is ν_1 -measurable, that is, belongs to $\overline{\mathcal{B}}_{\mu_1}(S_1)$, and there is a ν_1 -measurable function $h : \text{proj}_{S_1} A \mapsto A$ such that $(s_1, h(s_1)) \in A$ for all $s_1 \in \text{proj}_{S_1} A$.

PROOF: The set $\text{proj}_{S_1} A$ is ν_1 -measurable because the map $\text{proj}_{S_1}(s_1, s_2) = s_1$ is continuous and the set A can be approximated $(\nu_1 \times \nu_2)$ -a.e. by rectangles whose projections are measurable. (In the case $S_1 = S_2 = S$, this also follows from the inclusions in (B.2) where all the classes, including $\Sigma_1^1(S)$, are subsets of $\overline{\mathcal{B}}(S)$.)

We will show next that there is a ν_1 -measurable map $h : \text{proj}_{S_1} A \mapsto A$ such that $(s_1, h(s_1)) \in A$ for $s_1 \in \text{proj}_{S_1} A$. To do so, we will use Theorem 3.4.3 in Arveson [3],

p. 77, which concerns the so-called cross sections of Borel maps. Consider the map $f = \text{proj}_{\mathcal{S}_1} : A \mapsto f(A) = \text{proj}_{\mathcal{S}_1} A$. The image set $f(A)$, together with the induced Borel structure

$$\mathcal{F}(A) = \{f(A) \cap B : B \in \mathcal{S}_1\},$$

is a Borel space. Moreover, this Borel space is countably separated (as defined in Arveson [3], p. 69) since the underlying standard Lebesgue space $(S_1, \mathcal{S}_1, \nu_1)$ is countably separated. The Borel set A , equipped with the Borel structure

$$\mathcal{A} = \{A \cap B : B \in \mathcal{S}_1 \times \mathcal{S}_2\},$$

is also a Borel space. It is an analytic Borel space (as defined in Arveson [3], p. 71) by using Corollary in Arveson [3], p. 65, and the fact that A is a Borel set. Since $f^{-1}(f(A) \cap B) = A \cap (B \times \mathbb{R}) \in \mathcal{A}$ for all $B \in \mathcal{S}_1$, the map

$$f : (A, \mathcal{A}) \mapsto (f(A), \mathcal{F}(A))$$

is Borel. It follows from Theorem 3.4.3 in Arveson [3] that there is a ν_1 -measurable map $g : f(A) \mapsto A$ such that $f(g(s_1)) = s_1$. Since f is a projection, we have that $g(s_1) = (s_1, h(s_1))$ for some ν -measurable map $h(s_1)$ and hence that there is a ν -measurable map $h(s_1)$ such that $(s_1, h(s_1)) \in A$. \square

Appendix C

Notation Summary

We denote by

$$X_\alpha^D, X_\alpha^C, X_\alpha^P, X_\alpha^F, X_\alpha^L, X_\alpha^{C \setminus P}$$

the processes generated respectively by dissipative (D), conservative (C), periodic (P), fixed (F), cyclic (L), and conservative but not periodic ($C \setminus P$) flows. These processes are abbreviated respectively

$$\text{DFSM, CFSM, PFSM, FFSM, cLFSM, (C \setminus P)FSM.}$$

They are characterized in terms of minimal representations. The dissipative (D) and conservative (C) sets associated with a flow can also be characterized through the structure of the kernel function G in (3.1). The processes

$$X_\alpha^P, X_\alpha^F, X_\alpha^L, X_\alpha^{C \setminus P}$$

can also be characterized through the structure of the kernel function G in (3.1), namely through respectively, the following subsets of C ,

$$C_P, C_F, C_L, C \setminus C_P.$$

Note that the linear fractional stable motion (LFSM) defined in Example 2.5 is also a FFSM as indicated in Example 3.14.

References

- [1] J. Aaronson. *An Introduction to Infinite Ergodic Theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.
(Cited on page 116.)
- [2] J. Aaronson and M. Denker. Characteristic functions of random variables attracted to 1-stable laws. *The Annals of Probability*, 26(1):399–415, 1998.
(Cited on page 4.)
- [3] W. Arveson. *An Invitation to C^* -Algebras*. Springer-Verlag, New York, 1976.
(Cited on pages 119, 120, 121.)
- [4] S. Banach. *Théorie des Opérations Linéaires*. Warsaw, 1932.
(Cited on pages 17, 115.)
- [5] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.
(Cited on pages 40, 46.)
- [6] P. J. Brockwell and R. A. Davis. *Time Series: Theory and Methods*. Springer Series in Statistics. Springer-Verlag, New York, second edition, 1991.
(Cited on page 32.)
- [7] S. Coles. *An Introduction to Statistical Modeling of Extreme Values*. Springer Series in Statistics. Springer-Verlag London Ltd., London, 2001.
(Cited on page 117.)
- [8] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai. *Ergodic Theory*. Springer-Verlag, 1982.
(Cited on page 116.)
- [9] L. de Haan and A. Ferreira. *Extreme Value Theory*. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2006. An introduction.
(Cited on page 117.)

- [10] L. de Haan and J. Pickands III. Stationary min-stable stochastic processes. *Probability Theory and Related Fields*, 72(4):477–492, 1986.
(Cited on page 117.)
- [11] J. L. Doob. *Stochastic Processes*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1953. Reprint of the 1953 original, a Wiley-Interscience Publication.
(Cited on page 49.)
- [12] H. Dym and H. P. McKean. *Fourier Series and Integrals*. Academic Press, New York, 1972.
(Cited on page 18.)
- [13] P. Embrechts and M. Maejima. *Selfsimilar Processes*. Princeton Series in Applied Mathematics. Princeton University Press, 2002.
(Cited on pages 1, 2.)
- [14] R. J. Fleming and J. E. Jamison. *Isometries on Banach Spaces: Function Spaces*, volume 129 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2003.
(Cited on page 116.)
- [15] E. Glasner. *Ergodic Theory via Joinings*, volume 101 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
(Cited on page 116.)
- [16] B. V. Gnedenko and A. N. Kolmogorov. *Limit distributions for sums of independent random variables*. Addison-Wesley, Reading, MA, 1954.
(Cited on page 4.)
- [17] K. Górska and K. A. Penson. Lévy stable two-sided distributions: Exact and explicit densities for asymmetric case. *Physical Review E*, 83:061125, 2011.
(Cited on page 3.)
- [18] P. R. Halmos. *Measure Theory*. Van Nostrand, New York, 1950.
(Cited on page 40.)
- [19] C. D. Hardin Jr. Isometries on subspaces of L^p . *Indiana University Mathematics Journal*, 30:449–465, 1981.
(Cited on pages 22, 23, 115.)
- [20] C. D. Hardin Jr. On the spectral representation of symmetric stable processes. *Journal of Multivariate Analysis*, 12:385–401, 1982.
(Cited on pages 11, 12, 115.)
- [21] Z. Kabluchko. Spectral representations of sum- and max-stable processes. *Extremes*, 12(4):401–424, 2009.
(Cited on page 117.)
- [22] M. Kanter. The L^p norm of sums of translates of a function. *Transactions of the American Mathematical Society*, 79:35–47, 1973.
(Cited on page 33.)
- [23] A. S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, New York, 1995.
(Cited on pages 119, 120.)

- [24] S. Kolodyński and J. Rosiński. Group self-similar stable processes in \mathbb{R}^d . *Journal of Theoretical Probability*, 16(4):855–876 (2004), 2003.
(Cited on page 115.)
- [25] U. Krengel. Darstellungssätze für Strömungen und Halbströmungen II. *Mathematische Annalen*, 182:1–39, 1969.
(Cited on pages 37, 38.)
- [26] U. Krengel. *Ergodic Theorems*. Walter de Gruyter, Berlin, 1985.
(Cited on pages 36, 116.)
- [27] I. Kubo. Quasi-flows. *Nagoya Mathematical Journal*, 35:1–30, 1969.
(Cited on pages 42, 116.)
- [28] I. Kubo. Quasi-flows II: Additive functionals and TQ-systems. *Nagoya Mathematical Journal*, 40:39–66, 1970.
(Cited on pages 45, 46, 48, 116.)
- [29] J. Lamperti. On the isometries of certain function-spaces. *Pacific Journal of Mathematics*, 8:459–466, 1958.
(Cited on pages 17, 115.)
- [30] G. Lindgren. *Stationary Stochastic Processes*. Chapman & Hall/CRC Texts in Statistical Science Series. CRC Press, Boca Raton, FL, 2013. Theory and applications.
(Cited on page 114.)
- [31] G. W. Mackey. Borel structure in groups and their duals. *Transactions of the American Mathematical Society*, 85:134–165, 1957.
(Cited on pages 63, 119.)
- [32] J. P. Nolan. *Stable Distributions - Models for Heavy Tailed Data*. Forthcoming, 2017.
(Cited on page 115.)
- [33] K. Petersen. *Ergodic Theory*. Cambridge University Press, Cambridge, 1983.
(Cited on page 119.)
- [34] V. Pipiras. Nonminimal sets, their projections and integral representations of stable processes. *Stochastic Processes and their Applications*, 117(9):1285–1302, 2007.
(Cited on pages 53, 115, 117.)
- [35] V. Pipiras and M. S. Taqqu. Decomposition of self-similar stable mixed moving averages. *Probability Theory and Related Fields*, 123(3):412–452, 2002.
(Cited on pages 45, 51, 53, 78, 109, 116, 117.)
- [36] V. Pipiras and M. S. Taqqu. The structure of self-similar stable mixed moving averages. *The Annals of Probability*, 30(2):898–932, 2002.
(Cited on pages 87, 94, 95, 96, 108, 109, 116, 117.)
- [37] V. Pipiras and M. S. Taqqu. Dilated fractional stable motions. *Journal of Theoretical Probability*, 17(1):51–84, 2004.
(Cited on pages 78, 81, 116.)
- [38] V. Pipiras and M. S. Taqqu. Stable stationary processes related to cyclic flows. *The Annals of Probability*, 32(3A):2222–2260, 2004.
(Cited on page 41.)

- [39] V. Pipiras and M. S. Taqqu. Integral representations of periodic and cyclic fractional stable motions. *Electronic Journal of Probability*, 12:no. 7, 181–206, 2007.
(Cited on pages 101, 104, 105, 116, 117.)
- [40] V. Pipiras and M. S. Taqqu. Identification of periodic and cyclic fractional stable motions. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, 44(4):612–637, 2008.
(Cited on pages 91, 94, 95, 109, 111, 116, 117.)
- [41] V. Pipiras and M. S. Taqqu. Semi-additive functionals and cocycles in the context of self-similarity. *Discussiones Mathematicae. Probability and Statistics*, 30(2):149–177, 2010.
(Cited on pages 48, 64, 116.)
- [42] V. Pipiras and M. S. Taqqu. *Long-Range Dependence and Self-Similarity*. Cambridge University Press, 2017.
(Cited on pages 1, 2, 4, 31, 50, 114.)
- [43] S. T. Rachev and S. Mittnik. *Stable Paretian Models in Finance*. Wiley, New York, 2000.
(Cited on page 115.)
- [44] J. Rosiński. On uniqueness of the spectral representation of stable processes. *Journal of Theoretical Probability*, 7(3):615–634, 1994.
(Cited on pages 20, 21, 24, 28, 29, 30.)
- [45] J. Rosiński. Uniqueness of spectral representations of skewed stable processes and stationarity. In H. Kunita and H.-H. Kuo, editors, *Stochastic Analysis On Infinite Dimensional Spaces.*, pages 264–273. Proceedings of the U.S.-Japan Bilateral Seminar, 1994.
(Cited on pages 5, 115.)
- [46] J. Rosiński. On the structure of stationary stable processes. *The Annals of Probability*, 23:1163–1187, 1995.
(Cited on pages 2, 36, 37, 38, 53, 63, 116, 117.)
- [47] J. Rosiński. Decomposition of stationary α -stable random fields. *The Annals of Probability*, 28(4):1797–1813, 2000.
(Cited on page 115.)
- [48] J. Rosiński. Minimal integral representations of stable processes. *Probability and Mathematical Statistics*, 26(1):121–142, 2006.
(Cited on pages 11, 15, 16, 24, 115, 116.)
- [49] P. Roy and G. Samorodnitsky. Stationary symmetric α -stable discrete parameter random fields. *Journal of Theoretical Probability*, 21(1):212–233, 2008.
(Cited on page 115.)
- [50] H. L. Royden. *Real Analysis*. Macmillan, third edition, 1988.
(Cited on pages 19, 115.)
- [51] W. Rudin. L^p -isometries and equimeasurability. *Indiana University Mathematics Journal*, 25(3):215–228, 1976.
(Cited on page 22.)

- [52] G. Samorodnitsky. Extreme value theory, ergodic theory and the boundary between short memory and long memory for stationary stable processes. *The Annals of Probability*, 32(2):1438–1468, 2004.
(Cited on page 116.)
- [53] G. Samorodnitsky. Null flows, positive flows and the structure of stationary symmetric stable processes. *The Annals of Probability*, 33(5):1782–1803, 2005.
(Cited on page 116.)
- [54] G. Samorodnitsky. *Stochastic Processes and Long Range Dependence*. Springer Series in Operations Research and Financial Engineering, Springer, 2016.
(Cited on page 2.)
- [55] G. Samorodnitsky and M. S. Taqqu. $1/\alpha$ -self-similar processes with stationary increments.. *Journal of Multivariate Analysis*, 35:308–313, 1990.
(Cited on pages 80, 84.)
- [56] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes*. Stochastic Modeling. Chapman & Hall, New York, 1994. Stochastic models with infinite variance.
(Cited on pages 2, 3, 4, 5, 6, 7, 8, 9, 12, 31, 33, 72, 79, 85, 115.)
- [57] K.-i. Sato. *Lévy Processes and Infinitely Divisible Distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013. Translated from the 1990 Japanese original, Revised edition of the 1999 English translation.
(Cited on page 4.)
- [58] R. Sikorski. *Boolean Algebras*. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 25. Springer-Verlag New York Inc., New York, 1969.
(Cited on page 20.)
- [59] S. M. Srivastava. *A Course on Borel Sets*. Springer-Verlag, New York, 1998.
(Cited on pages 119, 120.)
- [60] D. Surgailis, J. Rosiński, V. Mandrekar, and S. Cambanis. Stable generalized moving averages. *Probability Theory and Related Fields*, 97:543–558, 1993.
(Cited on page 116.)
- [61] D. Surgailis, J. Rosiński, V. Mandrekar, and S. Cambanis. On the mixing structure of stationary increment and self-similar $S\alpha S$ processes. Preprint, 1998.
(Cited on pages 80, 116.)
- [62] S. Takenaka. Integral-geometric construction of self-similar stable processes. *Nagoya Mathematical Journal*, 123:1–12, 1991.
(Cited on page 79.)
- [63] V. V. Uchaikin and V. M. Zolotarev. *Chance and Stability*. Modern Probability and Statistics. VSP, Utrecht, 1999. Stable distributions and their applications, With a foreword by V. Yu. Korolev and Zolotarev.
(Cited on pages 3, 4, 115.)

- [64] D. H. Wagner. Survey of measurable selection theorems. *SIAM Journal of Control and Optimization*, 15(5):859–903, 1977.
(Cited on page 120.)
- [65] P. Walters. *An Introduction to Ergodic Theory*. Springer-Verlag, New York, 1982.
(Cited on page 119.)
- [66] Y. Wang and S. A. Stoev. On the association of sum- and max-stable processes. *Statistics & Probability Letters*, 80(5–6):480–488, 2010.
(Cited on page 117.)
- [67] Y. Wang and S. A. Stoev. On the structure and representations of max-stable processes. *Advances in Applied Probability*, 42(3):855–877, 2010.
(Cited on page 117.)
- [68] R. J. Zimmer. *Ergodic Theory and Semisimple Groups*. Birkhäuser, Boston, 1984.
(Cited on pages 45, 116, 119.)
- [69] V. M. Zolotarev. *One-dimensional Stable Distributions*, volume 65 of “Translations of mathematical monographs”. American Mathematical Society, 1986. Translation from the original 1983 Russian edition.
(Cited on pages 3, 4, 115.)

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