Appendices

Appendix A: Integrals with Polylogarithm

The weighted integrals of the function

$$f(x) = \ln\Big(\tanh\Big(a + \operatorname{arctanh}(e^{bx})\Big)\Big)$$

are:

$$I_0(a,b;X) \equiv \int_0^X f(x)dx =$$

$$= X\ \ln(\tanh(a)) + \frac{1}{b}(\Lambda_2 - \mathrm{M}_2 + \mu_2 - \lambda_2),$$

$$I_1(a,b;X) \equiv \int_0^X f(x)xdx =$$

$$= \frac{1}{6b^2}\left[\pi^2 \ln\left(\coth(a)\right) + \ln^3(\coth(a)) + 3b^2X^2 \ln\left(\tanh(a)\right)\right] +$$

$$+ \frac{1}{b^2}(\mathrm{M}_3 - \Lambda_3) - \frac{X}{b}(\mathrm{M}_2 - \Lambda_2),$$

$$I_2(a,b;X) \equiv \int_0^X f(x)x^2dx = \frac{X^3}{3}\ln(\tanh(a)) +$$

$$+ \frac{2}{b^3}(\Lambda_4 - \mathrm{M}_4 + \mu_4 - \lambda_4) +$$

$$+ \frac{2X}{b^2}(\mathrm{M}_3 - \Lambda_3) - \frac{X^2}{b}(\mathrm{M}_2 - \Lambda_2).$$

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The following abbreviations are used:

$$\mathrm{M}_k = \mathrm{Li}_k(-\coth(a)e^{bX}), \quad \mu_k = \mathrm{M}_k|_{X=0} \equiv \mathrm{Li}_k(-\coth(a)),$$
$$\Lambda_k = \mathrm{Li}_k(-\tanh(a)e^{bX}), \quad \lambda_k = \Lambda_k|_{X=0} \equiv \mathrm{Li}_k(-\tanh(a)).$$

In these expressions is $\mathrm{Li}_k(z)$ the polylogarithm (also known as Jonquière's function) of order k and argument z (Lewin 1981).

Appendix B: Integrals with Hypergeometric Function

The weighted integrals of the function $g = (a + x^{-m})^{-1/m}$ are

$$J_p(a, m; X) \equiv \int_0^X x^p g(x)dx =$$
$$= {}_2\mathbf{F}_1\left(\frac{1}{m}, \frac{2+p}{m}; \frac{2+p+m}{m}; -aX^m\right)\frac{X^{2+p}}{2+p}, \text{ for } p \geq 0.$$

For some cases the integrals could be expressed in terms of elementary functions:

$$J_1(a, 1; X) = -\frac{X}{a^2} + \frac{X^2}{2a} + \frac{\ln(1 + aX)}{a^3},$$
$$J_2(a, 1; X) = \frac{X}{a^3} - \frac{X^2}{2a^2} + \frac{X^3}{3a} - \frac{\ln(1 + aX)}{a^4},$$
$$J_1(a, 2; X) = \frac{X\sqrt{1 + aX^2}}{2a} - \frac{\arcsin\left(X\sqrt{A}\right)}{2a^{3/2}},$$
$$J_2(a, 2; X) = \frac{2}{3a^2} - \frac{2\sqrt{1 + aX^2}}{3a^2} + \frac{X^2\sqrt{1 + aX^2}}{3a},$$
$$J_1(a, 3; X) = -\frac{1}{2a} + \frac{\left(1 + aX^3\right)^{2/3}}{2a},$$
$$J_2(a, 4; X) = -\frac{1}{3a} + \frac{\left(1 + aX^4\right)^{3/4}}{3a}.$$

Appendix C: Integrals with Incomplete Beta Function

The weighted integrals of the function $(|c-r|/r)^{1/n}$are:

$$K_n(a,b,c) \equiv \frac{1}{c}\int_a^b \left(\frac{|c-r|}{r}\right)^{1/n} dx =$$

$$-\mathbf{B}\left(\frac{a}{c};\frac{n-1}{n},\frac{n+1}{n}\right) - (-1)^{1/n}\ \mathbf{B}\left(\frac{b}{c};\frac{n-1}{n},\frac{n+1}{n}\right)$$

$$+\frac{(-1)^{-1/n}\pi}{n}\left(i+\cot\left(\frac{\pi}{2n}\right)\right)$$

and

$$L_n(a,b,c) \equiv \frac{1}{c^2}\left[\int_a^b \left(\frac{|c-r|}{r}\right)^{1/n}(c-r)dx\right] =$$
$$\mathbf{B}\left(\frac{a}{c};\frac{2n-1}{n},\frac{n+1}{n}\right) - \mathbf{B}\left(\frac{a}{c};\frac{n-1}{n},\frac{n+1}{n}\right) +$$
$$+(-1)^{-1/n}\left[\mathbf{B}\left(\frac{b}{c};\frac{2n-1}{n},\frac{n+1}{n}\right) - \mathbf{B}\left(\frac{b}{c};\frac{n-1}{n},\frac{n+1}{n}\right)\right].$$
$$+\frac{i\pi}{\exp\left(\frac{i\pi}{n}\right)-1}\frac{1+n}{n^2}$$

In these expressions

$$\mathbf{B}(x;p,q) = \int_0^x z^{p-1}(1-z)^{q-1}dz$$

is the incomplete beta-function (Pearson 1968). This functions expresses in terms of hypergeometric functions (Abramowitz and Stegun 1972) by:

$$\mathbf{B}(x;p,q) = {}_2\mathbf{F}_1(p,1-q;p+1;x)\frac{x^p}{p}.$$

Appendix D: Complete Elliptic Integrals

$$\mathbf{K}(k) = \int_0^1 \frac{1}{\sqrt{(1-k^2t^2)(1-t^2)}}dt$$ complete elliptic integrals of the first kind.

$$\mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}dt$$ complete elliptic integrals of the second kind .

$$\mathbf{\Pi}(\omega, k) = \int_0^1 \frac{1}{(1-\omega t^2)\sqrt{(1-k^2t^2)(1-t^2)}}dt$$ complete elliptic integrals of the third kind.

Appendix E: Appell Hypergeometric Function

Appell hypergeometric function of two variables (Erdélyi 1950; Kampe de Feriet 1957):

$$\mathbf{F}_1([A, B_1, B_2, C]; x, y) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(C-A)}\int_0^1 t^{A-1}(1-t)^{C-A-1}(1-tx)^{-B_1}(1-ty)^{-B_2}dt$$

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