

Appendices

Appendix A: Integrals with Polylogarithm

The weighted integrals of the function

$$f(x) = \ln\left(\tanh\left(a + \operatorname{arctanh}(e^{bx})\right)\right)$$

are:

$$\begin{aligned} I_0(a, b; X) &\equiv \int_0^X f(x) dx = \\ &= X \ln(\tanh(a)) + \frac{1}{b}(\Lambda_2 - M_2 + \mu_2 - \lambda_2), \end{aligned}$$

$$\begin{aligned} I_1(a, b; X) &\equiv \int_0^X f(x)x dx = \\ &= \frac{1}{6b^2}[\pi^2 \ln(\coth(a)) + \ln^3(\coth(a)) + 3b^2 X^2 \ln(\tanh(a))] + \\ &+ \frac{1}{b^2}(M_3 - \Lambda_3) - \frac{X}{b}(M_2 - \Lambda_2), \end{aligned}$$

$$\begin{aligned} I_2(a, b; X) &\equiv \int_0^X f(x)x^2 dx = \frac{X^3}{3} \ln(\tanh(a)) + \\ &+ \frac{2}{b^3}(\Lambda_4 - M_4 + \mu_4 - \lambda_4) + \\ &+ \frac{2X}{b^2}(M_3 - \Lambda_3) - \frac{X^2}{b}(M_2 - \Lambda_2). \end{aligned}$$

The following abbreviations are used:

$$\begin{aligned} M_k &= \text{Li}_k(-\coth(a)e^{bX}), & \mu_k &= M_k|_{X=0} \equiv \text{Li}_k(-\coth(a)), \\ \Lambda_k &= \text{Li}_k(-\tanh(a)e^{bX}), & \lambda_k &= \Lambda_k|_{X=0} \equiv \text{Li}_k(-\tanh(a)). \end{aligned}$$

In these expressions is $\text{Li}_k(z)$ the polylogarithm (also known as Jonquière's function) of order k and argument z (Lewin 1981).

Appendix B: Integrals with Hypergeometric Function

The weighted integrals of the function $g = (a + x^{-m})^{-1/m}$ are

$$\begin{aligned} J_p(a, m; X) &\equiv \int_0^X x^p g(x) dx = \\ &= {}_2F_1\left(\frac{1}{m}, \frac{2+p}{m}; \frac{2+p+m}{m}; -aX^m\right) \frac{X^{2+p}}{2+p}, \text{ for } p \geq 0. \end{aligned}$$

For some cases the integrals could be expressed in terms of elementary functions:

$$\begin{aligned} J_1(a, 1; X) &= -\frac{X}{a^2} + \frac{X^2}{2a} + \frac{\ln(1+aX)}{a^3}, \\ J_2(a, 1; X) &= \frac{X}{a^3} - \frac{X^2}{2a^2} + \frac{X^3}{3a} - \frac{\ln(1+aX)}{a^4}, \\ J_1(a, 2; X) &= \frac{X\sqrt{1+aX^2}}{2a} - \frac{\arcsin(X\sqrt{A})}{2a^{3/2}}, \\ J_2(a, 2; X) &= \frac{2}{3a^2} - \frac{2\sqrt{1+aX^2}}{3a^2} + \frac{X^2\sqrt{1+aX^2}}{3a}, \\ J_1(a, 3; X) &= -\frac{1}{2a} + \frac{(1+aX^3)^{2/3}}{2a}, \\ J_2(a, 4; X) &= -\frac{1}{3a} + \frac{(1+aX^4)^{3/4}}{3a}. \end{aligned}$$

Appendix C: Integrals with Incomplete Beta Function

The weighted integrals of the function $(|c - r|/r)^{1/n}$ are:

$$\begin{aligned}
 K_n(a, b, c) &\equiv \frac{1}{c} \int_a^b \left(\frac{|c - r|}{r} \right)^{1/n} dx = \\
 &- \mathbf{B} \left(\frac{a}{c}; \frac{n-1}{n}, \frac{n+1}{n} \right) - (-1)^{1/n} \mathbf{B} \left(\frac{b}{c}; \frac{n-1}{n}, \frac{n+1}{n} \right) \\
 &+ \frac{(-1)^{-1/n} \pi}{n} \left(i + \cot \left(\frac{\pi}{2n} \right) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 L_n(a, b, c) &\equiv \frac{1}{c^2} \left[\int_a^b \left(\frac{|c - r|}{r} \right)^{1/n} (c - r) dx \right] = \\
 &\mathbf{B} \left(\frac{a}{c}; \frac{2n-1}{n}, \frac{n+1}{n} \right) - \mathbf{B} \left(\frac{b}{c}; \frac{n-1}{n}, \frac{n+1}{n} \right) + \\
 &+ (-1)^{-1/n} \left[\mathbf{B} \left(\frac{b}{c}; \frac{2n-1}{n}, \frac{n+1}{n} \right) - \mathbf{B} \left(\frac{b}{c}; \frac{n-1}{n}, \frac{n+1}{n} \right) \right]. \\
 &+ \frac{i\pi}{\exp \left(\frac{i\pi}{n} \right) - 1} \frac{1+n}{n^2}
 \end{aligned}$$

In these expressions

$$\mathbf{B}(x; p, q) = \int_0^x z^{p-1} (1 - z)^{q-1} dz$$

is the incomplete beta-function (Pearson 1968). This functions expresses in terms of hypergeometric functions (Abramowitz and Stegun 1972) by:

$$\mathbf{B}(x; p, q) = {}_2F_1(p, 1 - q; p + 1; x) \frac{x^p}{p}.$$

Appendix D: Complete Elliptic Integrals

$\mathbf{K}(k) = \int_0^1 \frac{1}{\sqrt{(1-k^2t^2)(1-t^2)}} dt$ complete elliptic integrals of the first kind.

$\mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}} dt$ complete elliptic integrals of the second kind .

$\mathbf{\Pi}(\omega, k) = \int_0^1 \frac{1}{(1-\omega t^2)\sqrt{(1-k^2t^2)(1-t^2)}} dt$ complete elliptic integrals of the third kind.

Appendix E: Appell Hypergeometric Function

Appell hypergeometric function of two variables (Erdélyi 1950; Kampe de Fariet 1957):

$$\mathbf{F}_1([A, B_1, B_2, C]; x, y) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(C-A)} \int_0^1 t^{A-1} (1-t)^{C-A-1} (1-tx)^{-B_1} (1-ty)^{-B_2} dt$$

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