Concluding Remarks

The starting point of this book was that the only unproblematic notion of identity applied to individuals is the notion of 'same individual' within one and the same world. I set as my task to develop a systematic answer to the question of what it means to speak of one and the same object in a variety of situations. I said in Chap. 1 that the *form* of my answer constitutes the common thread running through this book. I declared that in the framework I formulate, cross-world statements are seen as systematically involving two types of components: worlds and links between suitable world-bound objects. The form of my proposal is indeed that worlds and world lines are two complementary modal unities, both of which are operative whenever we think or talk about objects in modal settings.

In Chaps. 1 and 2, I pointed out that there are different ways of understanding world lines. Viewed epistemologically, they would be seen as codifying means of reidentification: methods of finding out how one and the same individual manifests itself in a variety of circumstances. Seen metaphysically, they would amount to variable embodiments in Fine's sense [25]. An anti-realist would maintain that we can only meaningfully speak of one and the same individual in several situations if we are in a position to recognize it in those situations: world lines would emerge as epistemically flavored applicability conditions of the notion of cross-world identity. My preferred understanding of world lines is, however, none of these three options: my proposal is that it is a transcendental precondition for our speaking and thinking about individuals in many-world settings that individuals are construed as world lines. This proposal admits two variants: it can be understood along the lines of transcendental idealism or, alternatively, in accordance with conceptualist realism in the sense of Cassam [13]. In the former case, the fact that objects are conceptualized as world lines is seen as originating in the constitution of our mind: according to this view, we can think and speak of objects as they appear to us only in terms of world lines. The conceptualist realist claims that if external objects indeed are temporally extended and have modal properties, then external objects must be thought of as world lines. The presumed truth of this conditional claim is seen as an explanation of our cognitive faculty to think of objects as world lines. It was not among the goals of this book to develop an argument that would allow us to decide between these two

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variants, though I do think that unless convincing overall evidence is presented for transcendental idealism, we had better adopt conceptualist realism. All that matters for this book, however, is the thesis that in cross-world settings, objects must be viewed as world lines. The book as a whole aims to defend this thesis—on the one hand by developing a specific semantic theory that incorporates the proposed understanding of worlds and world lines and on the other hand by relating this semantic theory to alternative semantic approaches and by spelling out its usefulness for discussing a variety of philosophically interesting phenomena.

In Chap. 2, I formulated *world line semantics*, according to which 'first-order quantifiers' range over a set of world lines dependent on the context of evaluation. Once the main ideas of world line semantics were laid down, it turned out that it provides us not only a way of analyzing what it means to speak of individuals (physical objects) existing in many scenarios but also an analysis of sentences ascribing object-directed intentional states to various agents (states with intentional objects). Indeed, I showed in Chaps. 3 and 4 that we can make a distinction between physically and intentionally individuated world lines and discuss in terms of this distinction interrelations between physical and intentional objects. I proposed to semantically model contents of intentional states as structures consisting of a world representation (a set of worlds) and a sequence of object representations (a sequence of intentionally individuated world lines). The multiplicity of worlds in a world representation—due to the agent's incapacity to make very fine distinctions among possible scenarios—finds its analog in the plurality of worlds over which an intentional object is realized. This latter plurality accounts for the possibility of indeterminate intentional objects.

In Chap. 5, I took up the study of the logical properties of the modal language L. Despite its higher-order flavor, this language was shown to be translatable into two-sorted first-order logic. Quantifiers of one sort range over first-order objects playing the role of worlds, while those of the other sort range over first-order objects that play the role of world lines. While values of these quantifiers are substantially speaking sets of local objects, we need not think of them as such for the purposes of the translation. Instead of considering local objects as elements of worlds and world lines, they can be recovered as pairs of worlds and world lines. Identity formulas of L (stating that two world lines share a realization) can be translated utilizing a special ternary predicate E applied to a world and a pair of world lines. I pointed out in Chap. 5 that L is anomalous in not admitting a well-behaved notion of logical form. This is due to the behavior of atomic predicates, which are existence-entailing, unlike arbitrary predicates. I argued on the one hand that L has a natural generalization S_L that is not anomalous in this way and on the other hand that the deviant behavior of L is a fact of life that must be accepted—not a reason to dismiss the study of a language reflecting conceptually interesting distinctions.

I discussed general theoretical consequences of world line semantics in Chap. 6. I stressed that any objects of thought (propositions and intentional objects alike) are naturally modeled in terms of situated contents—structures composed of a set of worlds and a number of intentionally individuated world lines, relativized to a fixed world. The notion of singular proposition finds its analog in my framework in the notion of *singular content*, which does not suffer from metaphysical problems of the sort associated with the idea of a structured proposition having physical objects as components. Singular contents are situated contents with a single world line component that counts as specific—i.e., indeterminate to a sufficiently small degree. Finally, I explored different ways of talking about objects of intentional states and indicated how my framework offers a novel perspective on the semantics of certain intensional transitive verbs: those I termed robust intensional verbs.

My analysis offers a surprising generalization of possible world semantics, by construing objects and worlds as things of the same general type (sets of local objects). This book demonstrates that this philosophically motivated semantic theory is fruitful in allowing us to semantically model language pertaining to intentional states and to clarify the similarities and dissimilarities between physical and intentional objects within the confines of a unified framework.

Appendix A
Proofs

A.1 Internally Indistinguishable Worlds and Physical Objects

Let us consider hypothesis H5 mentioned in Sect. 4.5:

H5. If there is a map $g : w \cong w'$, then $\mathcal{P}_w = \mathcal{P}_{w'}$, and there is in particular a map $f : w \cong w'$ such that $f(\mathbf{I}(w)) = \mathbf{I}(w')$ for all $\mathbf{I} \in \mathcal{P}_w$.

Assuming that no two physically individuated world lines overlap (H1) and that every local object is the realization of a physical object (H4), the function f mentioned in H5 is actually uniquely determined by the set $\mathcal{P}_w$. Let us see why. Suppose that w and w' are internally indistinguishable—i.e., there is at least one map g such that $g : w \cong w'$. For all $a \in dom(w)$, let $\mathbf{I}_a$ be the unique element of $\mathcal{P}_w$ mapping w to a. (By H4, there is at least one such world line and by H1, at most one.) By H5, we have $\mathcal{P}_w = \mathcal{P}_{w'}$, which entails that $\mathbf{I}_a$ is realized in w' as well: all elements of $\mathcal{P}_{w'}$ are realized in w'. Define a map $h : dom(w) \to dom(w')$ by setting $h(a) = \mathbf{I}_a(w')$. Let f be any map satisfying $f : w \cong w'$ and $f(\mathbf{I}(w)) = \mathbf{I}(w')$ for all $\mathbf{I} \in \mathcal{P}_w$. By H5, there is at least one such map. I claim that in fact $f = h$. Let $a \in dom(w)$ be arbitrary. It suffices to show that $f(a) = h(a)$. Because $\mathbf{I}_a(w) = a$ and $\mathbf{I}_a \in \mathcal{P}_w$, we have $f(a) = f(\mathbf{I}_a(w)) = \mathbf{I}_a(w')$. Since $\mathbf{I}_a(w') = h(a)$, it follows that $f(a) = h(a)$.

A.2 Bound Variables and Substitutivity of Identicals

Fact 5.1 *Let ϕ be an L-formula in which y is free for x. Let ϕ' be the result of replacing at least one free occurrence of x in ϕ by y. The following is a valid formula:*

$$\bigwedge_{\mathbf{i} \in P_\phi} \psi_{\mathbf{i}} \to \mathsf{A}x\mathsf{A}y([x = y \wedge \phi] \to \phi').$$

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Proof If $\chi \in L$, write $\Sigma(\chi, x/y)$ for the set of formulas obtained from χ by substituting y for at least one free occurrence of x in χ. (If χ contains no free occurrences of x, then $\Sigma(\chi, x/y) = \{\chi\}$.) Let ϕ be any L-formula in which y is free for x.

If ϕ is non-modal and therefore $P_\phi = \varnothing$, the formula $\mathsf{A}x\mathsf{A}y([x = y \wedge \phi] \rightarrow \phi')$ is valid for all $\phi' \in \Sigma(\phi, x/y)$. A fortiori, in this case all formulas $\bigwedge_{\mathbf{i} \in P_\phi} \psi_{\mathbf{i}} \rightarrow \mathsf{A}x\mathsf{A}y([x = y \wedge \phi] \rightarrow \phi')$ are valid. Suppose, then, that $md(\phi) \geq 1$ and $M, w, g \models x = y \wedge \bigwedge_{\mathbf{i} \in P_\phi} \psi_{\mathbf{i}}$, where M, w, and g are otherwise arbitrary except that $g(x) \in \mathfrak{I}_w^{\alpha_1}$ and $g(y) \in \mathfrak{I}_w^{\alpha_2}$ for some agents α_1 and α_2. This assumption entails that for all $\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi$ and all u with $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m})(w, u)$, the following *Identity Fact* holds: if either $g(x)$ or $g(y)$ is realized in u, then they both are and $g(x)(u) = g(y)(u)$. Note that since y is free for x in ϕ, y is automatically free for x in all subformulas θ of ϕ. I prove by induction that the following claim $S(n)$ holds for all n with $0 \leq n \leq md(\phi)$:

> For all subformulas θ of ϕ with $md(\theta) = n$, formulas $\theta' \in \Sigma(\theta, x/y)$, and assignments h with $h(x) = g(x)$ and $h(y) = g(y)$, we have: $M, v, h \models \theta$ iff $M, v, h \models \theta'$, where v is any world such that $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m})(w, v)$ for some $\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi$ with $m + n \leq md(\phi)$.

The special case in which $\theta = \phi$ and $n = md(\phi)$ and $h = g$ and $m = 0$ will entail, then, that all formulas $\bigwedge_{\mathbf{i} \in P_\phi} \psi_{\mathbf{i}} \rightarrow \mathsf{A}x\mathsf{A}y([x = y \wedge \phi] \rightarrow \phi')$ with $\phi' \in \Sigma(\phi, x/y)$ are valid, given that the empty composition amounts to identity: $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m}) = \{\langle w, w\rangle : w \in dom(M)\}$ when $m = 0$.[1]

For the base case $S(0)$, suppose first that θ is an *atomic* subformula of ϕ and $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m})(w, v)$. Let $\theta' \in \Sigma(\theta, x/y)$ be arbitrary and let h be any assignment such that $h(x) = g(x)$ and $h(y) = g(y)$. If $\theta := P(\mathsf{t}_1, \ldots, \mathsf{t}_k)$, then $\theta' := P(\mathsf{s}_1, \ldots, \mathsf{s}_k)$, where the term t_i can be distinct from the term s_i only if they both are variables and in particular $\mathsf{t}_i = x$ and $\mathsf{s}_i = y$. Now, $M, v, h \models \theta$ iff all values $\mathsf{t}_1^{M,v,h}, \ldots, \mathsf{t}_k^{M,v,h}$ are defined and $\langle \mathsf{t}_1^{M,v,h}, \ldots, \mathsf{t}_k^{M,v,h}\rangle \in Int(P, v)$ iff all values $\mathsf{s}_1^{M,v,h}, \ldots, \mathsf{s}_k^{M,v,h}$ are defined and $\langle \mathsf{s}_1^{M,v,h}, \ldots, \mathsf{s}_k^{M,v,h}\rangle \in Int(P, v)$ iff $M, v, h \models \theta'$. The reason why the second equivalence holds is as follows. If t_j is a constant symbol or a variable distinct from x, then $\mathsf{t}_j = \mathsf{s}_j$, and trivially either both values $\mathsf{t}_j^{M,v,h}$ and $\mathsf{s}_j^{M,v,h}$ are undefined or else $\mathsf{t}_j^{M,v,h} = \mathsf{s}_j^{M,v,h}$. By the Identity Fact we have, moreover, either $v \notin marg(h(x)) \cup marg(h(y))$ or else $v \in marg(h(x)) \cap marg(h(y))$ and $h(x)(v) = h(y)(v)$. Therefore, if $\mathsf{t}_j = x$ and $\mathsf{s}_j \in \{x, y\}$, then either both values $\mathsf{t}_j^{M,v,h}$ and $\mathsf{s}_j^{M,v,h}$ are undefined or else $\mathsf{t}_j^{M,v,h} = \mathsf{s}_j^{M,v,h}$. By similar reasoning, it follows that $M, v, h \models \theta$ iff $M, v, h \models \theta'$ when θ is an identity formula. The remaining subformulas of ϕ having modal depth 0 are obtained from atomic formulas by applications of the operators $\wedge, \neg, \exists$, and E_{a}. By the compositionality of L (see Fact 5.9), the satisfaction of an L-formula of modal depth 0 at a world v depends only on the satisfaction of its atomic subformulas at v. Now, there is a one-one correspondence between atomic subformulas $\chi_1, \ldots, \chi_k$ of θ and atomic subformulas $\chi_1', \ldots, \chi_k'$ of θ' such that θ'

[1] In fact, the proof establishes the following stronger claim: whenever $\phi' \in \Sigma(\phi, x/y)$, the formula $\bigwedge_{\mathbf{i} \in P_\phi} \psi_{\mathbf{i}} \rightarrow \mathsf{A}x\mathsf{A}y[x = y \rightarrow (\phi \leftrightarrow \phi')]$ is valid.

is built from the formulas χ_i' in exactly the same way as θ is built from the χ_i. Since $M, v, h \models \chi_i$ iff $M, v, h \models \chi_i'$, it follows that $M, v, h \models \theta$ iff $M, v, h \models \theta'$.

Assume, then, inductively that the claim $S(n)$ holds for $n < md(\phi)$. (Recall that we are working under the hypothesis that $md(\phi) \geq 1$.) I move on to prove the claim $S(n+1)$. I begin by considering the case $\theta := \Box_{\mathsf{i}}\chi$ with $md(\chi) = n$. Let $\theta' \in \Sigma(\theta, x/y)$ and let h be any assignment such that $h(x) = g(x)$ and $h(y) = g(y)$. Consequently, there is $\chi' \in \Sigma(\chi, x/y)$ such that $\theta' = \Box_{\mathsf{i}}\chi'$. Suppose $M, v, h \models \Box_{\mathsf{i}}\chi$, where $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m})(w, v)$ with $m + md(\theta) = m + md(\chi) + 1 \leq md(\phi)$. Let u with $R_{\mathsf{i}}(v, u)$ be arbitrary. It follows that $M, u, h \models \chi$, where $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m} \circ R_{\mathsf{i}})(w, u)$. Because $(m+1) + md(\chi) \leq md(\phi)$, the inductive hypothesis yields $M, u, h \models \chi'$ and we may conclude that $M, v, h \models \Box\chi'$. Similarly, the inductive hypothesis guarantees that if $M, v, h \models \Box\chi'$, then $M, v, h \models \Box\chi$. Now, write Ξ for the class of all subformulas of ϕ that are either of modal depth at most n or else of the syntactic form $\Box_{\mathsf{i}}\chi$ with $md(\chi) = n$ for some index i. By the above reasoning, we know that the claim $S(n+1)$ holds if attention is restricted to formulas $\theta \in \Xi$. Write Ξ^* for the class of all subformulas of ϕ whose modal depth is at most $n+1$. It remains to argue why the claim $S(n+1)$ holds for an *arbitrary* formula $\theta \in \Xi^*$.

Any formula of Ξ^* is obtained from formulas of Ξ by some finite number of applications of the operators $\neg, \wedge, \exists$, or E_{a}. We have already shown that $S(n+1)$ holds for all formulas in Ξ. Let us assume inductively that ξ and ζ are formulas in Ξ^* satisfying $S(n+1)$. We must show that if $\beta \in Var$ and $\theta \in \{(\xi \wedge \zeta), \neg\xi, \exists\beta\xi, \mathsf{E}_{\mathsf{a}}\beta\xi\}$, we have: $M, v, h \models \theta$ iff $M, v, h \models \theta'$, where h is any assignment such that $h(x) = g(x)$ and $h(y) = g(y)$, v is any world such that $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m})(w, v)$ with $m + md(\theta) \leq md(\phi)$, and $\theta' \in \Sigma(\theta, x/y)$ is arbitrary. By the inductive hypothesis, it is immediate that the claim holds for $(\xi \wedge \zeta)$ and $\neg\xi$. Let us move on to consider the formula $\exists\beta\xi$. If x does not occur free in $\exists\beta\xi$, there is nothing to prove, so suppose it does. Now, because x occurs free in $\exists\beta\xi$, we have $\beta \neq x$. Further, since y is free for x in ϕ and therefore in its subformula $\exists\beta\xi$, we have $\beta \neq y$. Suppose, then, that $M, v, h \models \exists\beta\xi$ and let $\xi' \in \Sigma(\xi, x/y)$ be arbitrary. By assumption there is $\mathbf{I} \in \mathcal{P}_v$ such that $M, v, h[\beta := \mathbf{I}] \models \xi$. Because $x \neq \beta \neq y$, we may infer that $h[\beta := \mathbf{I}](x) = h(x) = g(x)$ and $h[\beta := \mathbf{I}](y) = h(y) = g(y)$, whence the inductive hypothesis applies and yields $M, v, h[\beta := \mathbf{I}] \models \xi'$. Thus, $M, v, h \models \exists\beta\xi'$. Conversely, the inductive hypothesis guarantees that if $M, v, h \models \exists\beta\xi'$, then $M, v, h \models \exists\beta\xi$. The case of $\mathsf{E}_{\mathsf{a}}\beta\theta$ can be proven similarly. □

A.3 Constant Symbols and Substitutivity of Identicals

Fact 5.3 *Let ϕ be an L-formula. Let ϕ' be the result of replacing at least one occurrence of c in ϕ by d. The following is a valid formula:*

$$\bigwedge_{\mathbf{i} \in P_\phi} (\theta_{\mathbf{i}}^c \wedge \theta_{\mathbf{i}}^d) \rightarrow ((c = d \wedge \phi) \rightarrow \phi').$$

Proof If $\chi \in L$, write $\Sigma(\chi, c/d)$ for the set of formulas obtained from χ by substituting d for at least one occurrence of c in χ. (If χ contains no occurrences of c, then $\Sigma(\chi, c/d) = \{\chi\}$.) If $md(\phi) = 0$, the formula $((c = d \wedge \phi) \rightarrow \phi')$ is valid for all $\phi' \in \Sigma(\phi, c/d)$. A fortiori, therefore, all formulas $\bigwedge_{\mathbf{i} \in P_\phi}(\theta^c_{\mathbf{i}} \wedge \theta^d_{\mathbf{i}}) \rightarrow ((c = d \wedge \phi) \rightarrow \phi')$ are valid. Suppose, then, that $md(\phi) \geq 1$.

Since physically individuated world lines do not overlap, if e is any constant symbol, the formula $\bigwedge_{\mathbf{i} \in P_\phi} \theta^e_{\mathbf{i}} = \bigwedge_{\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi} \exists x(x = e \wedge \Box_{\mathsf{i}_1} \cdots \Box_{\mathsf{i}_m} x = e)$ is equivalent to $\exists x \bigwedge_{\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi}[x = e \wedge \Box_{\mathsf{i}_1} \cdots \Box_{\mathsf{i}_m} x = e]$. Namely, if the former formula is satisfied at w, and $\mathbf{J}_1$ and $\mathbf{J}_2$ are witnesses of $\exists x$ in distinct conjuncts $\mathbf{i_1}$ and $\mathbf{i_2}$, then $\mathbf{J}_1(w) = Int(e, w) = \mathbf{J}_2(w)$. By hypothesis H1 of Sect. 3.4, this entails that $\mathbf{J}_1 = \mathbf{J}_2$. We may conclude that the formula $\bigwedge_{\mathbf{i} \in P_\phi}(\theta^c_{\mathbf{i}} \wedge \theta^d_{\mathbf{i}})$ is equivalent to the formula

$$\exists x \exists y[x = c \ \wedge \ y = d \ \wedge \ \bigwedge\nolimits_{\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi} \Box_{\mathsf{i}_1} \cdots \Box_{\mathsf{i}_m} (x = c \ \wedge \ y = d)].$$

Now, let M be an arbitrary model, let $w \in dom(M)$ be any world, and let g be any assignment in M. Suppose that $M, w, g \models \bigwedge_{\mathbf{i} \in P_\phi}(\theta^c_{\mathbf{i}} \wedge \theta^d_{\mathbf{i}})$. Thus, there are physically individuated world lines $\mathbf{I}, \mathbf{J} \in \mathcal{P}_w$ such that

(∗) $M, w, x := \mathbf{I}, y := \mathbf{J} \models x = c \ \wedge \ y = d \ \wedge \ \bigwedge_{\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi} \Box_{\mathsf{i}_1} \cdots \Box_{\mathsf{i}_m}(x = c \ \wedge \ y = \ d)$.

Assume, then, that $M, w, g \models c = d \wedge \phi$. Together with (∗) this assumption yields $\mathbf{I}(w) = Int(c, w) = Int(d, w) = \mathbf{J}(w)$, which again entails by hypothesis H1 of Sect. 3.4 that $\mathbf{I}$ and $\mathbf{J}$ are one and the same world line. Therefore (∗) allows us to infer that $M, w, x := \mathbf{I} \models \bigwedge_{\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi} \Box_{\mathsf{i}_1} \cdots \Box_{\mathsf{i}_m}(x = c \wedge x = d)$ and consequently

(†) $M, w \models c = d \wedge \bigwedge_{\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi} \Box_{i_1} \cdots \Box_{i_m} c = d$.

It remains to show that $M, w, g \models \phi'$ for all $\phi' \in \Sigma(\phi, c/d)$. Observe, first, that if θ is *atomic* and $\theta' \in \Sigma(\phi, c/d)$, then by (†) we have: $M, v, g \models \theta$ iff $M, v, g \models \theta'$ whenever $v = w$ or $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m})(w, v)$ for $\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi$. It is, then, straightforward to prove by induction on the number n the following claim: for any numbers n and m such that $n + m \leq md(\phi)$, string $\mathsf{i}_1 \ldots \mathsf{i}_m \in P_\phi$, world v with $v = w$ or $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m})(w, v)$, formula θ with $md(\theta) = n$, and formula $\theta' \in \Sigma(\theta, c/d)$, we have: $M, v, g \models \theta$ iff $M, v, g \models \theta'$. In the special case of $\theta = \phi$ and $n = md(\phi)$ and $m = 0$ and $v = w$, it follows that $M, w, g \models \phi$ iff $M, w, g \models \phi'$ for all $\phi' \in \Sigma(\phi, c/d)$. Since by the hypothesis, we have $M, w, g \models \phi$, may conclude that $M, w, g \models \phi'$. □

A.4 Variants of the Barcan Formula and Its Converse

A.4.1 Relation to Versions of Monotonicity and Anti-Monotonicity

Fact 5.5 *Let F be any frame.*

(a) $F \models$ BF-P $\Leftrightarrow$ *F is physically anti-monotonic.*
(b) $F \models$ CBF-P $\not\Rightarrow \Leftarrow$ *F is physically monotonic.*
(c) $F \models$ BF-I $\not\Rightarrow \Leftarrow$ *F is intentionally anti-monotonic relative to agent α.*
(d) $F \models$ CBF-I $\not\Rightarrow \Leftarrow$ *F is intentionally monotonic relative to agent α.*

Proof The right–left entailments of all four claims are obvious; let us consider the left–right entailments. As for claim (a), suppose for contradiction that $F = \langle W, \{R\}, \mathcal{P}, \mathcal{I}\rangle$ is a frame satisfying $F \models$ BF-P, though F is not physically anti-monotonic. Because the physical anti-monotonicity fails, there are worlds w and w' with $R(w, w')$ such that for some $\mathbf{I} \in \mathcal{P}_{w'}$, we have $\mathbf{I} \notin \mathcal{P}_w$. Since $\mathbf{I}$ is physically individuated and $\mathbf{I} \in \mathcal{P}_{w'}$, there is a local object $b \in dom(w')$ such that $b = \mathbf{I}(w')$. Define an interpretation function *Int* by setting $Int(Q, w') = \{b\}$ and $Int(Q, w'') = \varnothing$ for all $w'' \in W\backslash\{w'\}$. Letting $M = \langle F, Int\rangle$, it follows that $M, w \models \Diamond\exists x Q(x)$. Since $F \models$ BF-P, it ensues that $M, w \models \exists x\Diamond Q(x)$. That is, there is $\mathbf{J} \in \mathcal{P}_w$ such that for some w'' with $R(w, w'')$, we have $\mathbf{J}(w'') \in Int(Q, w'')$. Given the definition of *Int*, we may conclude that $w'' = w'$ and $\mathbf{J}(w'') = b$. As no two physically individuated world lines overlap, we have $\mathbf{J} = \mathbf{I}$ and $\mathbf{I} \in \mathcal{P}_w$. This is a contradiction. Regarding claim (b), we recall that CBF-P is valid. Therefore it is not valid *only* in physically monotonic frames.

For claim (c), in order to show that the validity of BF-I in a frame fails to entail its intentional anti-monotonicity, consider a frame $F_0 = \langle W, \{R\}, \mathcal{P}, \mathcal{I}\rangle$ with an arbitrarily chosen $\mathcal{P}$ such that $W = \{w_1, w_2\}$ with $w_1 \neq w_2$, $R = \{\langle w_1, w_2\rangle\}$, and $\mathcal{I} = \{\mathcal{I}^{\alpha}_{w_1}, \mathcal{I}^{\alpha}_{w_2}\}$—given that $\mathcal{I}^{\alpha}_{w_1} = \{\mathbf{I}\}$ and $\mathcal{I}^{\alpha}_{w_2} = \{\mathbf{J}\}$, where $marg(\mathbf{I}) = \{w_1, w_2\}$ and $marg(\mathbf{J}) = \{w_2\}$ and $\mathbf{I}(w_2) = \mathbf{J}(w_2)$. Now, F_0 is not intentionally anti-monotonic relative to agent α: we have $R(w_1, w_2)$ but $\{\mathbf{I}\} = \mathcal{I}^{\alpha}_{w_1} \not\supseteq \mathcal{I}^{\alpha}_{w_2} = \{\mathbf{J}\}$, because $\mathbf{I} \neq \mathbf{J}$. I proceed to show that BF-I is, however, valid in F_0. Namely, let $M = \langle F_0, Int\rangle$ be any model based on F_0, let $w \in \{w_1, w_2\}$ be arbitrary, and suppose $M, w \models \Diamond \mathsf{E_a} x Q(x)$. In fact, w must equal w_1, as w_2 has no R-successor: the formula $\Diamond \mathsf{E_a} x Q(x)$ could not be true at w_2. Because w_2 is the only world R-accessible from w_1, it follows from the truth of $\Diamond \mathsf{E_a} x Q(x)$ at w_1 that the unique world line $\mathbf{J}$ intentionally available for α in w_2 satisfies $\mathbf{J}(w_2) \in Int(Q, w_2)$. Because $\mathbf{I}(w_2) = \mathbf{J}(w_2)$, we have $M, w_1, x := \mathbf{I} \models \Diamond Q(x)$. Since $\mathbf{I}$ is available for α in w_1, we may conclude that $M, w_1 \models \mathsf{E_a} x \Diamond Q(x)$. It ensues that $F_0 \models$ BF-I. Concerning claim (d), we may note that the frame F_0 is not intentionally monotonic: we have $R(w_1, w_2)$ but $\{\mathbf{I}\} = \mathcal{I}^{\alpha}_{w_1} \not\subseteq \mathcal{I}^{\alpha}_{w_2} = \{\mathbf{J}\}$, because $\mathbf{I} \neq \mathbf{J}$. Yet, it can be checked that CBF-I is valid in F_0. □

A.4.2 Noncharacterizability of Intentional Monotonicity and Intentional Anti-Monotonicity

Theorem 5.1 *(a) There is no L-formula that is valid in a frame iff the frame is intentionally monotonic. (b) Neither is there an L-formula that is valid in a frame iff the frame is intentionally anti-monotonic.*

Proof Let us begin with statement (a). I construct two frames of which one is intentionally monotonic but the other is not, and show that any L-formula valid in the former is also valid in the latter. Let w_1, w_2, and w_3 be three worlds. For each $1 \leq i \leq 3$, let $\mathcal{P}_{w_i}$ be a singleton $\{\langle w_i, a_i\rangle\}$, where $a_i \in dom(w_i)$. Let $\mathcal{I}^{\alpha}_{w_1} = \mathcal{I}^{\alpha}_{w_2} = \mathcal{I}^{\alpha}_{w_3} = \{\mathbf{I}_0, \mathbf{J}_0\}$, where $marg(\mathbf{I}_0) = marg(\mathbf{J}_0) = \{w_1\}$ and $\mathbf{I}_0(w_1) \neq \mathbf{J}_0(w_1)$. Define a frame $F = \langle W, \{R\}, \mathcal{P}, \mathcal{I}\rangle$ by setting $W = \{w_1, w_2, w_3\}$, $R = \{\langle w_1, w_2\rangle, \langle w_2, w_3\rangle\}$, $\mathcal{P} = \{\mathcal{P}_{w_1}, \mathcal{P}_{w_2}, \mathcal{P}_{w_3}\}$, and $\mathcal{I} = \{\mathcal{I}^{\alpha}_{w_1}, \mathcal{I}^{\alpha}_{w_2}, \mathcal{I}^{\alpha}_{w_3}\}$. It is trivial that the frame F is intentionally monotonic. Next, define another frame $F' = \langle W', \{R'\}, \mathcal{P}', \mathcal{I}'\rangle$ by setting $W' = W$ and $R' = R$ and $\mathcal{P}' = \mathcal{P}$ and $\mathcal{I}' = \{\mathcal{I}'^{\alpha}_{w_1}, \mathcal{I}'^{\alpha}_{w_2}, \mathcal{I}'^{\alpha}_{w_3}\}$—given that $\mathcal{I}'^{\alpha}_{w_1} = \{\mathbf{I}_0, \mathbf{J}_0\}$ and $\mathcal{I}'^{\alpha}_{w_2} = \{\mathbf{I}_0\}$ and $\mathcal{I}'^{\alpha}_{w_3} = \{\mathbf{I}_0, \mathbf{J}_0\}$, where $\mathbf{I}_0$ and $\mathbf{J}_0$ are the same world lines as in F. The frame F' is not intentionally monotonic since $R(w_1, w_2)$ but $\mathcal{I}'^{\alpha}_{w_1} \not\subseteq \mathcal{I}'^{\alpha}_{w_2}$. The only difference between F and F' is that unlike in F, in F' the world line $\mathbf{J}_0$ is not intentionally available in w_2. Let *Int* be any interpretation over W. Let $M = \langle F, Int\rangle$ and $M' = \langle F', Int'\rangle$, where $Int' = Int$. Let us prove the following auxiliary claim:

Claim 1 For all $\phi \in L$, all $g : Free(\phi) \to \{\mathbf{I}_0, \mathbf{J}_0\}$, and all $w \in W$, we have:

$$M, w, g \models \phi \quad \text{iff} \quad M', w, g \models \phi.$$

The claim can be proven by induction on the structure of the L-formula ϕ. The base case concerns atomic predications $Q(\mathsf{t}_1, \ldots, \mathsf{t}_n)$ and identities $\mathsf{t}_1 = \mathsf{t}_2$. Let $g : Free(Q(\mathsf{t}_1, \ldots, \mathsf{t}_n)) \to \{\mathbf{I}_0, \mathbf{J}_0\}$ be an arbitrary assignment. We have $M, w, g \models Q(\mathsf{t}_1, \ldots, \mathsf{t}_n)$ iff all values $\mathsf{t}_1^{M,w,g}, \ldots, \mathsf{t}_n^{M,w,g}$ are defined and $\langle \mathsf{t}_1^{M,w,g}, \ldots, \mathsf{t}_n^{M,w,g}\rangle \in Int(Q, w) = Int'(Q, w)$ iff $M', w, g \models Q(\mathsf{t}_1, \ldots, \mathsf{t}_n)$. Further, for any assignment $g : Free(\mathsf{t}_1 = \mathsf{t}_n) \to \{\mathbf{I}_0, \mathbf{J}_0\}$, we have $M, w, g \models \mathsf{t}_1 = \mathsf{t}_2$ iff both values $\mathsf{t}_1^{M,w,g}$ and $\mathsf{t}_2^{M,w,g}$ are defined and $\mathsf{t}_1^{M,w,g}$ equals $\mathsf{t}_2^{M,w,g}$ iff $M', w, g \models \mathsf{t}_1 = \mathsf{t}_2$.

Suppose, then, inductively that for fixed formulas ψ and θ, we have: if $\chi \in \{\psi, \theta\}$, then $M, w, g \models \chi$ iff $M', w, g \models \chi$ for all $g : Free(\chi) \to \{\mathbf{I}_0, \mathbf{J}_0\}$ and all $w \in W$. Evidently, the formula $\neg\psi$ satisfies the claim. For the remaining inductive cases, let w be any world in W. Concerning $(\psi \wedge \theta)$, let $g : Free(\psi) \cup Free(\theta) \to \{\mathbf{I}_0, \mathbf{J}_0\}$ be arbitrary. Now, $M, w, g \models (\psi \wedge \theta)$ iff $M, w, g \models \psi$ and $M, w, g \models \theta$ iff $M, w, g\restriction_{Free(\psi)} \models \psi$ and $M, w, g\restriction_{Free(\theta)} \models \theta$ iff (ind. hyp.) $M', w, g\restriction_{Free(\psi)} \models \psi$ and $M', w, g\restriction_{Free(\theta)} \models \theta$ iff $M', w, g \models \psi$ and $M', w, g \models \theta$ iff $M', w, g \models (\psi \wedge \theta)$.

The case of $\Box\psi$: Let $g : Free(\psi) \to \{\mathbf{I}_0, \mathbf{J}_0\}$ be arbitrary. Now, $M, w, g \models \Box\psi$ iff for all w' with $R(w, w')$, we have $M, w', g \models \psi$ iff (ind. hyp.) for all w' with $R(w, w')$, we have $M', w', g \models \psi$ iff $M', w, g \models \Box\psi$. Observe that actually, the inductive hypothesis is not needed in the special case that $w = w_3$. This is because w_3 has no R-successors—it satisfies vacuously all formulas of the form $\Box\zeta$.

The case of $\exists x\psi$: Because there is a world line $\mathbf{I}$ with $marg(\mathbf{I}) = \{w\}$ such that $\mathcal{P}_w = \{\mathbf{I}\}$, we have $M, w, g \models \exists x\psi$ iff $M, w, g[x := \mathbf{I}] \models \psi$ iff (ind. hyp.) $M', w, g[x := \mathbf{I}] \models \psi$ iff $M', w, g \models \exists x\psi$ for any suitable assignment g.

The case of $\mathsf{E}_{\mathsf{a}}x\psi$: Let g be any map $Free(\psi)\backslash\{x\} \to \{\mathbf{I}_0, \mathbf{J}_0\}$. Suppose $M, w, g \models \mathsf{E}_{\mathsf{a}}x\psi$. Thus, there is $\mathbf{I} \in \mathcal{I}^{\alpha}_{w}$ such that $M, w, g[x := \mathbf{I}] \models \psi$. Therefore, by the inductive hypothesis, $M', w, g[x := \mathbf{I}] \models \psi$. Unless $w = w_2$ and $\mathbf{I} = \mathbf{J}_0$, we may conclude—given how the frame F' is defined—that $\mathbf{I} \in \mathcal{I}'^{\alpha}_{w}$, whence $M', w, g \models \mathsf{E}_{\mathsf{a}}x\psi$. Let us consider separately the special case that $M, w_2, g[x := \mathbf{J}_0] \models \psi$. Because the sets $\mathcal{I}^{\alpha}_{w_j}, \mathcal{I}'^{\alpha}_{w_j}$ with $j := 2, 3$ are non-empty and none of their elements is realized in w_j, the following general fact can be easily shown by induction: for all $n \geq 0$, formulas $\zeta(x_1, \ldots, x_n)$ of n free variables, assignments $f, f' : \{x_1, \ldots, x_n\} \to \{\mathbf{I}_0, \mathbf{J}_0\}$, and indices $j \in \{2, 3\}$, we have $M, w_j, f \models \zeta$ iff $M', w_j, f' \models \zeta$. Therefore, we can infer from $M, w_2, g[x := \mathbf{J}_0] \models \psi$ that $M', w_2, g[x := \mathbf{I}_0] \models \psi$, which entails $M', w_2, g \models \mathsf{E}_{\mathsf{a}}x\psi$, because $\mathbf{I}_0 \in \mathcal{I}'^{\alpha}_{w_2}$. The direction from $M', w, g \models \mathsf{E}_{\mathsf{a}}x\psi$ to $M, w, g \models \mathsf{E}_{\mathsf{a}}x\psi$ can be proven similarly, without resorting to the last-mentioned additional fact. ⧄

Having now proven Claim 1, it can be used to establish the following:

Claim 2 Any formula valid in F is valid in F', as well.

Suppose $\phi \in L$ and $F \models \phi$. Let $M' = \langle F', Int' \rangle$ be any model based on F'. Let $w \in dom(M')$ and $g : Free(\phi) \to \{\mathbf{I}_0, \mathbf{J}_0\}$ be arbitrary. Consider the model $M = \langle F, Int \rangle$ with $Int = Int'$. Since ϕ is valid in F, we have $M, w, g \models \phi$. By Claim 1, it follows $M', w, g \models \phi$. We may conclude that ϕ is valid in F'. ⧅

In order to prove statement (a), suppose for contradiction that there is an L-formula ϕ_0 such that for all frames F_0, the formula ϕ_0 is valid in F_0 iff F_0 is intentionally monotonic. Since F is intentionally monotonic, ϕ_0 is valid in F. By Claim 2, ϕ_0 is valid in F', as well. However, F' is not intentionally monotonic. This contradicts the assumption that ϕ_0 is valid only in intentionally monotonic frames.

Concerning statement (b), observe that F is not only intentionally monotonic, but also intentionally anti-monotonic. Further, not only does F' fail to be intentionally monotonic, but it also fails to be intentionally anti-monotonic: we have $R(w_2, w_3)$ but $\mathcal{I}'^{\alpha}_{w_2} \not\supseteq \mathcal{I}'^{\alpha}_{w_3}$. If there was an L-formula characterizing anti-monotonicity, it would be valid in F and—by Claim 2—in F'. So it would, after all, not be valid only in anti-monotonic frames. □

A.5 Existence-Entailment Problem Is Undecidable

Let FOL* be the fragment of FOL without identity symbol or constant symbols. Its only logical symbols are $\neg$, $\wedge$, $\exists$, and the variables in the set *Var*. It is well known that the validity problem of FOL* is undecidable; see [59]. Note that syntactically, FOL* is a fragment of L. I prove that there is no algorithm for solving the *existence-entailment problem* of L—the problem of determining whether an L-formula is existence-entailing. I do this by showing that if this problem were decidable, so would be the validity problem of FOL*.

An FOL*-formula ϕ is *FOL*-valid* if $\mathcal{M}, \Gamma \models_{\text{FOL}} \phi$ for all first-order models $\mathcal{M}$ and assignments Γ. The same formula, viewed as an L-formula, is valid if $M, w, g \models \phi$ for all models M, worlds $w \in dom(M)$, and assignments g in M. If g is an assignment and $X \subseteq dom(g) \subseteq Var$, then g is said to be *w-realized for X* provided that for all variables $x \in X$, the world line $g(x)$ is realized in w. The assignment g is *w-realized* if it is w-realized for $dom(g)$. The following lemma is useful.

Lemma *Let $\phi(x_1, \ldots, x_n) \in FOL^*$ be arbitrary. The formula $\phi(x_1, \ldots, x_n)$ is FOL*-valid iff the L-formula $\left(\bigwedge_{1 \le i \le n} x_i = x_i\right) \rightarrow \phi(x_1, \ldots, x_n)$ is valid.*

Proof Let $\phi(x_1, \ldots, x_n) \in$ FOL* and assume, first, that ϕ is FOL*-valid. Suppose for contradiction that there is a model M, a world w_0, and an assignment g_0 such that nevertheless $M, w_0, g_0 \not\models \left(\bigwedge_{1 \le i \le n} x_i = x_i\right) \rightarrow \phi(x_1, \ldots, x_n)$. This means that the assignment g_0 is w_0-realized for $Free(\phi)$ and $M, w_0, g_0 \not\models \phi(x_1, \ldots, x_n)$. If Int is the interpretation function of M, define an interpretation function **Int** by setting $\mathbf{Int}(Q) := Int(Q, w_0)$ for all predicate symbols Q occurring in ϕ. Let $\mathcal{M} := \langle dom(w_0), \mathbf{Int}\rangle$. If g is any w_0*-realized* assignment in M, let Γ_g be the function defined on the set $dom(g)$ as follows: $\Gamma_g(y) := g(y)(w_0)$ for all $y \in dom(g)$. We can prove by induction on the complexity of an FOL*-formula θ the following claim: for all w_0-realized assignments g in M with $dom(g) = Free(\theta)$, we have $M, w_0, g \models \theta$ iff $\mathcal{M}, \Gamma_g \models \theta$. The base case of atomic formulas $Q(y_1, \ldots, y_m)$ holds, because $M, w_0, g \models Q(y_1, \ldots, y_m)$ iff $\langle g(y_1)(w_0), \ldots, g(y_m)(w_0)\rangle \in Int(Q, w_0)$ iff $\langle \Gamma_g(y_1), \ldots, \Gamma_g(y_m)\rangle \in \mathbf{Int}(Q)$ iff $\mathcal{M}, \Gamma_g \models Q(y_1, \ldots, y_m)$. The inductive cases for $\neg$ and $\wedge$ are trivial. For the remaining case of physical quantifiers, we need hypothesis H4, according to which every local object is the realization of some physical object. If $\mathcal{M}, \Gamma_g \models \exists x \psi$, there is $a \in dom(w_0)$ such that $\mathcal{M}, \Gamma_g[x := a] \models \psi$. By H4, there is $\mathbf{I}_a$ in M such that $\mathbf{I}_a(w_0) = a$ and $\Gamma_g[x := a] = \Gamma_{g[x:=\mathbf{I}_a]}$, where $g[x := \mathbf{I}_a]$ is w_0-realized. By the inductive hypothesis, we have $M, w_0, g[x := \mathbf{I}_a] \models \psi$, whence it follows that $M, w_0, g \models \exists x \psi$. Further, if $M, w_0, g \models \exists x \psi$, there is $\mathbf{J}$ realized in w_0 such that $M, w_0, g[x := \mathbf{J}] \models \psi$, where $g[x := \mathbf{J}]$ is w_0-realized. The inductive hypothesis entails $\mathcal{M}, \Gamma_{g[x:=\mathbf{J}]} \models \psi$, where $\Gamma_{g[x:=\mathbf{J}]} = \Gamma_g[x := \mathbf{J}(w_0)]$. It follows that $\mathcal{M}, \Gamma_g \models \exists x \psi$. Since g_0 is w_0-realized for $Free(\phi)$, we may conclude that $M, w_0, g_0 \models \phi$ iff $\mathcal{M}, \Gamma_{g_0} \models \phi$. As ϕ is FOL*-valid, we have $\mathcal{M}, \Gamma_{g_0} \models \phi$ and therefore $M, w_0, g_0 \models \phi$, which is a contradiction.

Conversely, assume that the L-formula $\left(\bigwedge_{1 \le i \le n} x_i = x_i\right) \rightarrow \phi(x_1, \ldots, x_n)$ is valid and suppose for contradiction that $\mathcal{M}, \Gamma_0 \not\models \phi$ for some $\mathcal{M}$ and Γ_0. If $\mathcal{M} = \langle D, \mathbf{Int}\rangle$, let w_0 be a world such that $dom(w_0) = D$. Let $\mathcal{P}_{w_0} := \{\mathbf{I}_a : a \in D\}$, where each $\mathbf{I}_a = \{\langle w_0, a\rangle\}$. Note that each world line $\mathbf{I}_a$ is realized in w_0. Observe also that $\mathbf{I}_a \neq \mathbf{I}_{a'}$ whenever $a \neq a'$. Define an interpretation Int by letting $Int(Q, w_0) := \mathbf{Int}(Q)$. Finally, let $M := \langle W, \mathcal{R}, \mathcal{P}, \mathcal{I}\rangle$, where $W = \{w_0\}$ and $\mathcal{R} = \varnothing = \mathcal{I}$ and $\mathcal{P} = \{\mathcal{P}_{w_0}\}$. If Γ is any assignment with values in D, let g_Γ be the function defined on the set $dom(\Gamma)$ as follows: $g_\Gamma(y) := \mathbf{I}_{\Gamma(y)}$ for all $y \in dom(\Gamma)$. Thus, g_Γ is w_0-realized for $dom(\Gamma)$ and satisfies $g_\Gamma(y)(w_0) = \Gamma(y)$. We may, then, prove by induction on the complexity of an FOL*-formula θ that for all assignments $\Gamma : Free(\theta) \to D$, we have $\mathcal{M}, \Gamma \models \theta$ iff $M, w_0, g_\Gamma \models \theta$. The base case of atomic formulas $Q(y_1, \ldots, y_m)$ is in force, since $\mathcal{M}, \Gamma \models Q(y_1, \ldots, y_m)$ iff $\langle \Gamma(y_1), \ldots, \Gamma(y_m)\rangle \in$ **Int**(Q) iff $\langle g_\Gamma(y_1)(w_0), \ldots, g_\Gamma(y_m)(w_0)\rangle \in Int(Q, w)$ iff $M, w_0, g_\Gamma \models Q(y_1, \ldots, y_m)$. The inductive cases for $\neg$, $\wedge$, and physical quantifiers readily follow. We may conclude that $\mathcal{M}, \Gamma_0 \models \phi$ iff $M, w_0, g_{\Gamma_0} \models \phi$. Since $\left(\bigwedge_{1\le i\le n} x_i = x_i\right) \to \phi(x_1, \ldots, x_n)$ is valid, we have $M, w_0, g_{\Gamma_0} \models \left(\bigwedge_{1\le i\le n} x_i = x_i\right) \to \phi(x_1, \ldots, x_n)$. Now, the assignment g_{Γ_0} is w_0-realized for $Free(\phi)$, whence $M, w_0, g_{\Gamma_0} \models \bigwedge_{1\le i\le n} x_i = x_i$ and therefore $M, w_0, g_{\Gamma_0} \models \phi(x_1, \ldots, x_n)$. Consequently, $\mathcal{M}, \Gamma_0 \models \phi$. This is a contradiction. □

We can now prove the undecidability of the existence-entailment problem of L.

Theorem 5.2 *The problem of determining whether an L-formula is existence-entailing (safe for substitution) is undecidable.*

Proof We observe, first, that if ϕ is any L-formula such that $Free(\phi) \neq \varnothing$, we have:

$$\neg\Box_j\phi \text{ is existence-entailing iff } \neg\Box_j\phi \text{ is contradictory (not satisfiable).}$$

To begin with, if $\neg\Box_j\phi$ is satisfiable, then so is $\neg\phi$: there is a model $M = \langle W, \mathcal{R}, \mathcal{P}, \mathcal{I}\rangle$, a world $w \in W$, and an assignment g in M such that $M, w, g \models \neg\phi$. Construct a model $M' = \langle W', \mathcal{R}', \mathcal{P}', \mathcal{I}'\rangle$ from M as follows. Let $\mathcal{P}' := \mathcal{P}$, $\mathcal{I}' := \mathcal{I}$, and $W' := W \cup \{w^*\}$, where $w^* \notin W$. Further, let $\mathcal{R}'$ contain exactly the same relations as $\mathcal{R}$, except that its relation R'_j corresponding to the index j equals $R_j \cup \{\langle w^*, w\rangle\}$. Consequently, w^* cannot be accessed from w along (results of composing) relations belonging to $\mathcal{R}$, though w can be accessed from w^* along R'_j. Evidently $M', w, g \models \neg\phi$ and therefore $M', w^*, g \models \Diamond_j\neg\phi$. Note that there is no variable $x \in Free(\phi)$ such that $g(x)$ is defined on w^*. (The world w^* was brought in from outside M.) It follows that the formula $\Diamond_j\neg\phi$ and therefore its equivalent $\neg\Box_j\phi$ are not existence-entailing: these formulas contain at least one free variable x and they are satisfied in M' at w^* under g, but $w^* \notin marg(g(x))$. We have shown (by contraposition) that if $\neg\Box_j\phi$ is existence-entailing, it is contradictory. Conversely, if $Free(\phi) = \{x_1, \ldots, x_n\}$ and $\neg\Box_j\phi$ is contradictory, it is trivially existence-entailing—it satisfies trivially the condition that in *all* structures in which it is satisfied (namely, in none), also the formula $\bigwedge_{1\le i\le n} x_i = x_i$ is satisfied.

Let us proceed to observe that for every $\phi \in L$ with at least one free variable,

$$\neg\Box_j\phi \text{ is contradictory iff } \phi \text{ is valid.}$$

First, if ϕ is not valid, then $\neg\phi$ and therefore $\Diamond_j\neg\phi$ are satisfiable, whence $\neg\Box_j\phi$ is not contradictory. Second, if $\neg\Box_j\phi$ is not contradictory, then $\Diamond_j\neg\phi$ and therefore $\neg\phi$ are satisfiable, whence ϕ is not valid.

Suppose, then, for contradiction that the existence-entailment problem of L is decidable. Let $\theta(x_1, \ldots, x_n)$ be an arbitrary FOL*-formula containing at least one free variable (i.e., $n \geq 1$). Now, $\left(\bigwedge_{1\leq i\leq n} x_i = x_i\right) \rightarrow \theta$ is an L-formula with at least one free variable, so it follows in particular that

$$\neg\Box_{\mathrm{j}}[\left(\bigwedge\nolimits_{1\leq i\leq n} x_i = x_i\right) \rightarrow \theta] \text{ is existence-entailing iff } \left(\bigwedge\nolimits_{1\leq i\leq n} x_i = x_i\right) \rightarrow \theta \text{ is valid.}$$

By the Lemma above, we may infer that

$$\neg\Box_{\mathrm{j}}[\left(\bigwedge\nolimits_{1\leq i\leq n} x_i = x_i\right) \rightarrow \theta] \text{ is existence-entailing iff } \theta \text{ is FOL}^*\text{-valid.}$$

Since by assumption there is a decision algorithm for the existence-entailment problem of L, we can apply it to the formula $\neg\Box_{\mathrm{j}}[\left(\bigwedge_{1\leq i\leq n} x_i = x_i\right) \rightarrow \theta]$ and thereby determine whether θ is FOL*-valid. That is, the mentioned decision algorithm induces a decision method for the validity problem of FOL*, which is impossible. □

A.6 Translation of *L* into First-Order Logic

Theorem 5.3 (First-order translation of L) *For all* $\phi(x_1, \ldots, x_n) \in L$, *models* M, *worlds* $w \in dom(M)$, *and assignments* $g : \{x_1, \ldots, x_n\} \rightarrow \mathrm{WL}(M)$, *we have:*

$$M, w, g \models \phi \textit{ iff } \mathcal{M}, \Gamma_{t,w,g} \models T_t[\phi].$$

Proof Let M be an arbitrary model. I prove by induction on the complexity of ϕ the following slightly more general claim: for all suitable worlds w, assignments g, and variables s of sort 2, we have $M, w, g \models \phi(x_1, \ldots, x_n)$ iff $\mathcal{M}, \Gamma_{s,w,g} \models T_s[\phi]$. The base cases of induction concern atomic predications and identities.

Atomic formulas. Let us consider predications. Suppose $M, w, g \models Q(x_1, \ldots, x_n)$ with $g(x_j) := \mathbf{I}_j$. Thus, $\langle \mathbf{I}_1(w), \ldots, \mathbf{I}_n(w)\rangle \in Int(Q, w)$. Let $m := dom(w)$ and $i_j := Im(\mathbf{I}_j)$. Because $\mathbf{I}_j(w) \in m \cap i_j$ for all $1 \leq j \leq n$, we have $\langle m, i_1, \ldots, i_n\rangle \in \mathbf{Int}(\mathsf{Q})$. Since $\Gamma_{s,w,g}(s) = m$ and $\Gamma_{s,w,g}(x_j) = i_j$, we may infer that $\mathcal{M}, \Gamma_{s,w,g} \models \mathsf{Q}(s, x_1, \ldots, x_n)$. Conversely, suppose $\mathcal{M}, \Gamma_{s,w,g} \models T_s[Q(x_1, \ldots, x_n)]$. Letting $m := dom(w)$ and $i_j := Im(g(x_j))$, there are elements $b_1, \ldots, b_n \in m$ such that $b_j \in i_j$ and $\langle b_1, \ldots, b_n\rangle \in Int(Q, w)$. Since each $b_j = g(x_j)(w)$, it follows that $M, w, g \models Q(x_1, \ldots, x_n)$. We must still consider identities. Suppose $M, w, g \models x_1 = x_2$, where $g(x_j) := \mathbf{I}_j$. So, there is $b \in dom(w)$ such that $\mathbf{I}_1(w) = b = \mathbf{I}_2(w)$. Thus, the set $dom(w) \cap Im(\mathbf{I}_1) \cap Im(\mathbf{I}_2)$ is non-empty, whence $\mathcal{M}, \Gamma_{s,w,g} \models \mathsf{E}(s, x_1, x_2)$. Conversely, if $\mathcal{M}, \Gamma_{s,w,g} \models T_s[x_1 = x_2]$, there is b such that $\Gamma_{s,w,g}(s) \cap \Gamma_{s,w,g}(x_1) \cap \Gamma_{s,w,g}(x_2) = \{b\}$. This means that $g(x_1)(w) = b = g(x_2)(w)$ and $b \in dom(w)$, and therefore $M, w, g \models x_1 = x_2$.

Having now established the base cases, let us assume inductively that formulas $\psi(y_1, \ldots, y_m)$ and $\chi(z_1, \ldots, z_k)$ satisfy the claim. The inductive case of negation holds trivially. I proceed to consider the remaining cases.

Conjunction. Write $g_1 := g\restriction_{\{y_1,\ldots,y_m\}}$ and $g_2 := g\restriction_{\{z_1,\ldots,z_k\}}$. Now, $M, w, g \models \psi \wedge \chi$ iff: $M, w, g_1 \models \psi$ and $M, w, g_2 \models \psi$ iff: (ind. hyp.) $\mathcal{M}, \Gamma_{s,w,g_1} \models T_s[\psi]$ and $\mathcal{M}, \Gamma_{s,w,g_2} \models T_s[\chi]$ iff: $\mathcal{M}, \Gamma_{s,w,g_1} \cup \Gamma_{s,w,g_2} \models T_s[\psi \wedge \chi]$. Here, the union $\Gamma_{s,w,g_1} \cup \Gamma_{s,w,g_2}$ is a well-defined assignment: if there are variables $u \in \{y_1, \ldots, y_m\} \cap \{z_1, \ldots, z_k\}$, we have $\Gamma_{s,w,g_1}(u) = g_1(u) = g(u) = g_2(u) = \Gamma_{s,w,g_2}(u)$. Indeed, $\Gamma_{s,w,g_1} \cup \Gamma_{s,w,g_2} = \Gamma_{s,w,g}$.

Quantifiers. I prove the claim about physical quantifiers; the case of intentional quantifiers can be dealt with similarly. Suppose $M, w, g \models \exists x\psi$. There is, then, $\mathbf{I} \in \mathcal{P}_w$ such that $M, w, g[x := \mathbf{I}] \models \psi$. By the inductive hypothesis, $\mathcal{M}, \Gamma_{s,w,g[x:=\mathbf{I}]} \models T_s[\psi]$. Write $m := dom(w)$ and $i := Im(\mathbf{I})$. Since $\mathbf{I} \in \mathcal{P}_w$, we have $\langle m, i\rangle \in \mathcal{P}^*$, wherefore $\mathcal{M}, \Gamma_{s,w,g}[x := i] \models \mathsf{P}(s, x)$. Here, $\Gamma_{s,w,g}[x := i] = \Gamma_{s,w,g[x:=\mathbf{I}]}$ and it follows that $\mathcal{M}, \Gamma_{s,w,g} \models T_s[\exists x\psi]$. For the converse direction, suppose $\mathcal{M}, \Gamma_{s,w,g} \models \exists x(\mathsf{P}(s, x) \wedge T_s[\psi])$. Thus, there is $i \in U_2$ such that $\mathcal{M}, \Gamma_{s,w,g}[x := i] \models \mathsf{P}(s, x) \wedge T_s[\psi]$. Given how the predicate P is interpreted, there is $\mathbf{I} \in \mathcal{P}_w$ such that $i = Im(\mathbf{I})$. Note that $\Gamma_{s,w,g}[x := i] = \Gamma_{s,w,g[x:=\mathbf{I}]}$. By the inductive hypothesis, we may conclude that $M, w, g[x := \mathbf{I}] \models \psi$, which entails that $M, w, g \models \exists x\psi$.

Boxes. Suppose $M, w, g \models \Box_{\mathsf{i}}\psi$. I wish to show that $\mathcal{M}, \Gamma_{s,w,g} \models T_s[\Box_{\mathsf{i}}\psi]$. To this end, let m' be any element of U_1 such that $\mathcal{M}, \Gamma_{s,w,g}[u := m'] \models \mathsf{R}_{\mathsf{i}}(s, u)$. I must show that $\mathcal{M}, \Gamma_{s,w,g}[u := m'] \models T_u[\psi]$. Write $m := \Gamma_{s,w,g}(s)$. Because $\mathcal{M}, \Gamma_{s,w,g}[u := m'] \models \mathsf{R}_{\mathsf{i}}(s, u)$, we have $\langle m, m'\rangle = \langle dom(w), m'\rangle \in R^*_{\mathsf{i}}$, whence $m' = dom(w')$ for some w' with $R_{\mathsf{i}}(w, w')$. Since $M, w, g \models \Box_{\mathsf{i}}\psi$, we have $M, w', g \models \psi$, which yields $\mathcal{M}, \Gamma_{u,w',g} \models T_u[\psi]$ by the inductive hypothesis. Now, $\Gamma_{u,w',g}[s := m] = \Gamma_{s,w,g}[u := m']$ and s does not occur free in $T_u[\psi]$, so we may conclude that $\mathcal{M}, \Gamma_{s,w,g}[u := m'] \models T_u[\psi]$. Conversely, suppose $\mathcal{M}, \Gamma_{s,w,g} \models T_s[\Box_{\mathsf{i}}\psi]$. Let w' with $R_{\mathsf{i}}(w, w')$ be arbitrary. I wish to show that $M, w', g \models \psi$. Write $m := dom(w)$ and $m' := dom(w')$. We have, then, $\mathcal{M}, \Gamma_{s,w,g}[u := m'] \models \mathsf{R}_{\mathsf{i}}(s, u)$. Because $\mathcal{M}, \Gamma_{s,w,g} \models T_s[\Box_{\mathsf{i}}\psi]$, we have $\mathcal{M}, \Gamma_{s,w,g}[u := m'] \models T_u[\psi]$. Since $\Gamma_{s,w,g}[u := m'] = \Gamma_{u,w',g}[s := m]$ and s does not occur free in $T_u[\psi]$, it follows that $\mathcal{M}, \Gamma_{u,w',g} \models T_u[\psi]$. By the inductive hypothesis, it ensues that $M, w', g \models \psi$. □

Appendix B
Overview of Definitions

B.1 FOL and FOML

The reader is expected to be familiar with first-order logic and with first-order modal logic interpreted according to the standard Kripke semantics. However, for clarity of exposition I recall certain aspects of these languages here.

Formulas of 'first-order logic' of a relational vocabulary τ, or FOL$[\tau]$, are those formulas of $L_0[\tau]$ that do not employ the modal operator $\Box$. Its semantics uses *models* $\langle D, \mathbf{Int}\rangle$, where D is a non-empty set and **Int** is a function such that $\mathbf{Int}(Q) \subseteq D^n$ for every positive n and n-ary predicate symbol Q in τ. If clarity so demands, these models may be referred to as 'first-order models'. An *assignment* is any function of type $Var \to D$. The standard semantics of FOL$[\tau]$ specifies recursively what it means for a formula ϕ to be *satisfied* in a model $\mathcal{M}$ under an assignment Γ, denoted $\mathcal{M}, \Gamma \models_{\mathrm{FOL}} \phi$. An occurrence of x is *free* in ϕ if in ϕ it does not lie in the scope of the quantifier $\exists x$, the notion of scope being defined in the usual way. In fact, the satisfaction of ϕ under Γ depends only on the values of Γ on the free variables of ϕ: a formula ϕ can be evaluated relative to a partial assignment defined only on the free variables of ϕ. The syntax of FOL could be generalized by including constant symbols in the vocabulary and its semantics could be formulated so as to allow non-referring constant symbols. Such generalizations are not needed in this book.

The syntax of 'first-order modal logic' of a relational vocabulary τ, or FOML$[\tau]$, is that of $L_0[\tau]$. Here the notion of free variable can be defined as in FOL. The semantics of FOML$[\tau]$ can be explained as follows. If W is a non-empty set, R is a binary relation on W, and $w \mapsto D_w$ is a map assigning to each $w \in W$ a non-empty set D_w, the structure $\langle W, R, (D_w)_{w\in W}\rangle$ is a *Kripke frame*. If F is a Kripke frame, a *Kripke model* based on F is a pair $\langle F, \mathbf{Int}\rangle$, where **Int** is a function such that for every positive n and n-ary predicate symbol $Q \in \tau$, we have $\mathbf{Int}(Q, w) \subseteq (\bigcup_{w\in W} D_w)^n$. Interpretations of predicate symbols are *not* subject to the domain constraint: if $\langle a_1, \ldots, a_n\rangle \in \mathbf{Int}(Q, w)$, it is *not* required that the a_i belong to D_w. It is only required that for every a_i there be $w_i \in W$ such that $a_i \in D_{w_i}$. By contrast, the ranges of quantifiers in a world w are restricted to D_w. If M is a Kripke model, an *assignment*

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is any function of type $Var \to \bigcup_{w \in W} D_w$. The semantics of FOML[$\tau$] is defined by recursively specifying what it means for a formula ϕ to be *satisfied* in a Kripke model M at a world under an assignment g, denoted $M, w, g \models_K \phi$:

- $M, w, g \models_K Q(x_1, \dots, x_n)$ iff $\langle g(x_1), \dots, g(x_n) \rangle \in \mathbf{Int}(Q, w)$.
- $M, w, g \models_K x = y$ iff $g(x) = g(y)$.
- $M, w, g \models_K \neg\phi$ iff $M, w, g \not\models_K \phi$.
- $M, w, g \models_K (\phi \wedge \psi)$ iff $M, w, g \models_K \phi$ and $M, w, g \models_K \psi$.
- $M, w, g \models_K \Box\phi$ iff for all w' with $R(w, w')$ we have: $M, w', g \models_K \phi$.
- $M, w, g \models_K \exists x\phi$ iff there is $a \in D_w$ such that $M, w, g[x := a] \models_K \phi$.

The satisfaction of a formula ϕ in a world w under an assignment g depends only on the values of g on the free variables of ϕ.

The syntax of FOML can be generalized by allowing the use of multiple modal operators. Such a generalization is not needed for the purposes of this book. Further, the syntax can be generalized by allowing constant symbols in the same way as in the syntax of L. If c is a constant symbol, it can be stipulated that $\mathbf{Int}(c, w) \in \{*\} \cup \bigcup_{w \in W} D_w$, where $* \notin \bigcup_{w \in W} D_w$. Intuitively, $\mathbf{Int}(c, w) = *$ means that in w, c lacks a referent. A constant symbol c is a *rigid designator* if it refers to the same object in every world in which that object exists (cf. Kripke [71, pp. 48–9]). We may take this to mean that the following three conditions are met for all worlds w and v:

(i) If c refers at all in w, it refers to something that exists in w: $\mathbf{Int}(c, w) \in D_w \cup \{*\}$.
(ii) If $\mathbf{Int}(c, w) \neq * \neq \mathbf{Int}(c, v)$, then $\mathbf{Int}(c, w) = \mathbf{Int}(c, v)$.
(iii) If $\mathbf{Int}(c, w) \in D_v$, then $\mathbf{Int}(c, v) \neq *$.

By clause (ii), if c has a referent in two worlds, it refers to the same object in both worlds. By clause (i), this means that the domains of the two worlds have a non-empty intersection. By clause (iii), again, if the object to which c refers in w is present in v, then c has a referent in v, as well. By (ii), this referent $\mathbf{Int}(c, v)$ actually equals $\mathbf{Int}(c, w)$.

B.2 Modal Languages L_0 and L

The quantified modal language L_0 is introduced in Sect. 2.3, and its extension L is introduced in Sect. 3.4.

B.2.1 Syntax of L_0

Let *Var* be a set of variables and τ a set of predicate symbols, each with an associated positive arity. The language $L_0[\tau]$ of vocabulary τ is built according to the following syntax:

$$\phi ::= Q(x_1, \dots, x_n) \mid x_1 = x_2 \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box\phi \mid \exists x\phi,$$

where n is a positive integer, the symbols $x, x_1, x_2, \ldots, x_n$ all belong to *Var*, and Q is an n-ary predicate belonging to τ.

B.2.2 Syntax of L

For all $n \geq 0$, let τ_n be a set of n-ary predicate symbols. Constant symbols are elements of τ_0. The set $\tau := \bigcup_{i \in \mathbb{N}} \tau_i$ is a *vocabulary*. Variables and constant symbols are collectively referred to as *terms*. I write *Term* for the set $Var \cup \tau_0$. Further, let $\mathbb{A}$ and $\mathbb{I}$ be finite non-empty sets of *agent markers* and *indices*, respectively. The extended quantified modal language L is recursively defined like L_0, except that it is closed under applications of modal operators $\Box_\mathsf{i}$ with $\mathsf{i} \in \mathbb{I}$ and applications of quantifiers E_a with $\mathsf{a} \in \mathbb{A}$. Moreover, atomic formulas can employ arbitrary terms. That is, the language $L[\tau]$ of vocabulary τ is built according to the following syntax:

$$\phi ::= Q(\mathsf{t}_1, \ldots, \mathsf{t}_n) \mid \mathsf{t}_1 = \mathsf{t}_2 \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_\mathsf{i}\phi \mid \exists x\phi \mid \mathsf{E}_\mathsf{a} x\phi,$$

where $n \geq 1$ and $Q \in \tau_n$ and $\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_n \in Term$ and $x \in Var$ and $\mathsf{i} \in \mathbb{I}$ and $\mathsf{a} \in \mathbb{A}$. I refer to $\exists$ as a *physical quantifier* and to the E_a as *intentional quantifiers*.

B.2.3 Semantics of L_0

Models of vocabulary τ are structures $M = \langle W, R, \mathfrak{I}, Int\rangle$. Here, W is a non-empty set. Every member w of W has a specified non-empty domain $dom(w)$ whose elements are referred to as *local objects*. Further, R is a binary relation on W, and *Int* is a function assigning to every n-ary predicate Q of τ and element w of W a subset $Int(Q, w)$ of $dom(w)^n$. Finally, $\mathfrak{I}$ is a collection of sets $\mathfrak{I}_w$ with $w \in W$. Each element of $\mathfrak{I}_w$ is a non-empty partial function on W, assigning an element of $dom(w')$ to every w' on which this partial function is defined.

The elements of the sets $\mathfrak{I}_w$ are referred to as *world lines over W*, although strictly speaking these partial functions are not world lines in the sense discussed in Sect. 1.5. (Yet, there is a one-to-one correspondence between world lines and such partial functions.) A world line $\mathbf{I}$ is *available* in w iff $\mathbf{I} \in \mathfrak{I}_w$. And it is *realized* in w iff there is a local object $b \in dom(w)$ such that $\mathbf{I}(w) = b$. These features are, generally, independent. A world line may be available in w without being realized therein, and realized in w without being available in w. Quantifiers evaluated relative to w range over world lines available in w. Atomic formulas $Q(x_1, \ldots, x_n)$ are evaluated with reference to realizations of those world lines that have been assigned as values of the variables $x_1, \ldots, x_n$.

I refer to the domain of a world line $\mathbf{I}$ as its *modal margin*, denoted $marg(\mathbf{I})$. The set $\mathrm{WL}(M)$ is defined as the union $\bigcup_{w \in W} \mathfrak{I}_w$. An *assignment* in M is a function of type $Var \to \mathrm{WL}(M)$. If g is an assignment defined on x, then $g(x)$ is a world line. If this world line is realized in world w, the result $g(x)(w)$ of applying the function $g(x)$ to the world w is a local object that belongs to the domain of w. If g is an assignment and $\mathbf{I}$ is a world line, $g[x := \mathbf{I}]$ stands for the assignment that differs from g at most in that it assigns $\mathbf{I}$ to x. The satisfaction relation $M, w, g \models \phi$ is defined recursively for suitable models M, worlds w, assignments g, and L_0-formulas ϕ as follows:

- $M, w, g \models Q(x_1, \ldots, x_n)$ iff for all $1 \leq i \leq n$, the world line $g(x_i)$ is realized in the world w, and the tuple $\langle g(x_1)(w), \ldots, g(x_n)(w) \rangle$ belongs to $Int(Q, w)$.
- $M, w, g \models x_1 = x_2$ iff the world lines $g(x_1)$ and $g(x_2)$ are both realized in the world w, and the local object $g(x_1)(w)$ is the same as the local object $g(x_2)(w)$.
- $M, w, g \models \neg\phi$ iff $M, w, g \not\models \phi$.
- $M, w, g \models (\phi \wedge \psi)$ iff $M, w, g \models \phi$ and $M, w, g \models \psi$.
- $M, w, g \models \Box\phi$ iff for all worlds w' with $R(w, w')$, we have: $M, w', g \models \phi$.
- $M, w, g \models \exists x\phi$ iff there is $\mathbf{I} \in \mathcal{I}_w$ such that $M, w, g[x := \mathbf{I}] \models \phi$.

B.2.4 Semantics of L

If $\mathsf{a} \in \mathbb{A}$, then α is the agent denoted by 'a' and A is the set of agents denoted by markers in the set $\mathbb{A}$. *Frames* are structures $\langle W, \mathcal{R}, \mathcal{P}, \mathcal{I} \rangle$, where $\mathcal{R} = \{R_\mathsf{i} : \mathsf{i} \in \mathbb{I}\}$ is a family of accessibility relations, and $\mathcal{P} = \{\mathcal{P}_w : w \in W\}$ and $\mathcal{I} = \{\mathcal{I}^\alpha_w : w \in W, \alpha \in A\}$ are families of world lines over W. *Models* of vocabulary τ are structures $M = \langle F, Int \rangle$, where F is a frame and Int is an interpretation function defined otherwise as in connection with L_0, except that it associates every constant symbol c and world w with an element of the set $dom(w) \cup \{*\}$, where $* \notin \bigcup_{v \in W} dom(v)$. We define $\mathrm{WL_P}(M) := \bigcup_{w \in W} \mathcal{P}_w$ and $\mathrm{WL_I}(M) := \bigcup_{w \in W, \alpha \in A} \mathcal{I}^\alpha_w$. Further, $\mathrm{WL}(M) := \mathrm{WL_P}(M) \cup \mathrm{WL_I}(M)$. I refer to the elements of $\mathrm{WL_P}(M)$ as *physically individuated world lines* and to those of $\mathrm{WL_I}(M)$ as *intentionally individuated world lines*. The *value* $\mathsf{t}^{M,w,g}$ of term t in model M at world w under assignment $g : Var \to \mathrm{WL}(M)$ is defined as follows depending on whether t is a constant symbol or a variable:

$$\mathsf{t}^{M,w,g} = \begin{cases} Int(\mathsf{t}, w) & \text{if } \mathsf{t} \in \tau_0 \text{ and } Int(\mathsf{t}, w) \neq * \\ g(\mathsf{t})(w) & \text{if } \mathsf{t} \in Var \text{ and } g(\mathsf{t}) \text{ is realized in } w. \end{cases}$$

It is assumed that an element of the set $\mathrm{WL_P}(M)$ is available in a world iff it is realized therein, whereas for elements of $\mathrm{WL_I}(M)$, availability and realization are mutually independent properties. The following further hypotheses H1 through H4 are made concerning the sets $\mathrm{WL_P}(M)$ and $\mathrm{WL_I}(M)$:

H1. *No two physically individuated world lines overlap:* If $\mathbf{I}, \mathbf{J} \in \mathrm{WL_P}(M)$ and there is w such that $w \in marg(\mathbf{I}) \cap marg(\mathbf{J})$ and $\mathbf{I}(w) = \mathbf{J}(w)$, then for all $v \in W$ we have: [either $v \notin marg(\mathbf{I}) \cup marg(\mathbf{J})$, or $v \in marg(\mathbf{I}) \cap marg(\mathbf{J})$ and $\mathbf{I}(v) = \mathbf{J}(v)$].

H2. *Realizations of physically individuated world lines are local objects:* If $w \in W$ and $\mathbf{I} \in \mathcal{P}_w$, then $\mathbf{I}(w) \in dom(w)$.
H3. *Realizations of intentionally individuated world lines are local objects:* If $\alpha \in A$ and $w, v \in W$ and $\mathbf{I} \in \mathcal{I}_w^\alpha$ and $\mathbf{I}$ is realized in v, then $\mathbf{I}(v) \in dom(v)$.
H4. *Every local object is the realization of some physical object:* If $w \in W$ and $b \in dom(w)$, then there is $\mathbf{I} \in \mathcal{P}_w$ such that $b = \mathbf{I}(w)$.

The clauses used for defining the semantics of L are otherwise as in L_0, except that in a world w, the physical quantifier $\exists$ ranges over the set $\mathcal{P}_w$ and the intentional quantifier E_a over the set $\mathcal{I}_w^\alpha$. Further, the accessibility relation associated with the modal operator $\Box_\mathsf{i}$ depends on the index i. Finally, the clauses for atomic formulas must be modified, since atomic formulas may contain constant symbols. Here are the clauses that need modifications:

- $M, w, g \models Q(\mathsf{t}_1, \ldots, \mathsf{t}_n)$ iff for all $1 \leq i \leq n$, the value $\mathsf{t}_i^{M,w,g}$ of the term t_i in M at w under g is defined, and the tuple $\langle \mathsf{t}_1^{M,w,g}, \ldots, \mathsf{t}_n^{M,w,g} \rangle$ belongs to $Int(Q, w)$.
- $M, w, g \models \mathsf{t}_1 = \mathsf{t}_2$ iff for all $i \in \{1, 2\}$, the value $\mathsf{t}_i^{M,w,g}$ of the term t_i in M at w under g is defined, and $\mathsf{t}_1^{M,w,g}$ equals $\mathsf{t}_2^{M,w,g}$.
- $M, w, g \models \exists x\phi$ iff there is $\mathbf{I} \in \mathcal{P}_w$ such that $M, w, g[x := \mathbf{I}] \models \phi$.
- $M, w, g \models \mathsf{E}_\mathsf{a} x\phi$ iff there is $\mathbf{I} \in \mathcal{I}_w^\alpha$ such that $M, w, g[x := \mathbf{I}] \models \phi$.
- $M, w, g \models \Box_\mathsf{i}\phi$ iff for all worlds w' with $R_\mathsf{i}(w, w')$ we have: $M, w', g \models \phi$.

B.3 Semantic Values and Features of Intensional Predicates

Formulas of the quantified modal languages L_0 and L give rise to intensional predicates. Any such formula $\phi(x_1, \ldots, x_n)$ with n free variables can be considered as an intensional n-ary predicate that applies in a model M at a world w to those n-tuples of world lines that satisfy it in M at w. In Sect. 2.4, this observation leads to the following definition of the semantic value of a formula.

Definition 2.1 (*Semantic value*) Let M be a model, and let $\phi(x_1, \ldots, x_n)$ be a formula of the language L_0. The *semantic value* $|\phi(x_1, \ldots, x_n)|^M$ of ϕ in M is the set of all $(n+1)$-tuples $\langle w, \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle \in dom(M) \times \mathrm{WL}(M)^n$ such that

$$M, w, x_1 := \mathbf{I}_1, \ldots, x_n := \mathbf{I}_n \models \phi(x_1, \ldots, x_n).$$

If ϕ is a sentence, then $|\phi|^M$ is a (possibly empty) subset of $dom(M)$—namely, the set of worlds w at which ϕ is true in M. □

We say that y is *free for* x in ϕ iff x does not occur free in the scope of the quantifier $\exists y$ in ϕ. If $\phi(x_1, \ldots, x_n)$ is an L_0-formula and $y_1, \ldots, y_n$ are variables, all of which are free for every variable $x_1, \ldots, x_n$ in ϕ, then $\phi[x_1 /\!\!/ y_1, \ldots, x_n /\!\!/ y_n]$

stands for the result of uniformly replacing all free occurrences of the variable x_i in ϕ by the variable y_i for all $1 \leq i \leq n$ (see Sect. 2.4). A unary intensional predicate is *existence-entailing* if it can be satisfied in a world w only by a world line that is realized in w. It is *pro mundo* if its satisfaction in w by a world line depends only on the realization (if any) of the world line in w. More generally, these features of intensional predicates are defined as follows.

Definition 2.2 Let $\phi(x_1, \ldots, x_n)$ be a predicate in L_0. It is *existence-entailing* if the formula $\phi(x_1, \ldots, x_n) \rightarrow \bigwedge_{1 \leq i \leq n} x_i = x_i$ is valid. It is *pro mundo* if the formula $\big(\phi(x_1, \ldots, x_n) \wedge \bigwedge_{1 \leq i \leq n} x_i = y_i\big) \rightarrow \phi[x_1 /\!\!/ y_1, \ldots, x_n /\!\!/ y_n]$ is valid, given that each variable y_i is free for every variable x_j. It is *quasi-extensional* if it is both existence-entailing and *pro mundo*. □

A quasi-extensional predicate can be satisfied in a world w only by a tuple of world lines all of which are realized in w, and the satisfaction of such a predicate in w depends only on the realizations of those world lines in w. Semantic values of quasi-extensional predicates can be encoded by interpretations of extensional predicates—i.e., predicates satisfied by tuples of local objects.

B.4 Modes of Predication

Two *modes of predication* are distinguished in Sect. 4.2: the physical and the intentional. Let M be a model with the interpretation function *Int*. Suppose w_0 is a world, $\mathbf{I}_1, \ldots, \mathbf{I}_n$ are (physically or intentionally individuated) world lines, and $\phi(x_1, \ldots, x_n)$ is an intensional predicate. Further, let $R_\mathsf{i}(w_0)$ be the set of worlds compatible with the intentional state i at w_0.

- *Physical predication:* Ascribing $\phi(x_1, \ldots, x_n)$ to the tuple of world lines $\langle \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle$ in w_0 under the physical mode is to affirm that $\langle w_0, \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle \in |\phi|^M$.
- *Intentional predication:* Ascribing $\phi(x_1, \ldots, x_n)$ to the tuple of world lines $\langle \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle$ in w_0 under the intentional mode relative to state i is to affirm that $\langle w, \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle \in |\phi|^M$ for all $w \in R_\mathsf{i}(w_0) \cap \bigcap_{1 \leq j \leq n} marg(\mathbf{I}_j)$.

B.5 Contents

In terms of the following general concept of *content*, the further notions of *situated content* and *internal modal margin* of a world line are defined (Sect. 2.5). Contents may but need not be propositional.

Definition 2.3 (*Content, situated content, internal modal margin*) Let M be a model. Let $V \subseteq dom(M)$ and $\mathbf{I}_1, \ldots, \mathbf{I}_n \in \mathrm{WL}(M)$. The structure $\langle V, \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle$ is an *n-ary content* over M. The set V is its *propositional component*, and the $\mathbf{I}_j$ are its *world* *line components*. A content is *propositional* if $n = 0$, otherwise it is said to have a propositional and a non-propositional aspect. If R is a binary relation on $dom(M)$, $w^* \in dom(M)$, and $V = R(w^*)$, the structure $\langle V, \mathbf{I}_1, \ldots, \mathbf{I}_n, w^* \rangle$ is an *R-situated n-ary content*. The set $V \cap marg(\mathbf{I}_j)$ is the *internal modal margin* of $\mathbf{I}_j$. □

Let $Cont_n[M]$ be the set of all n-ary contents over M. Relative to any given model, any formula generates a set of contents.

Definition 2.4 (*Contents generated by a formula*) Let M be a model. The set $Cont(\phi, M)$ of *contents generated by* $\phi(x_1, \ldots, x_n)$ *in* M is the smallest subset of $Cont_n[M]$ satisfying the following condition: if V is non-empty and $\langle w, \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle \in |\phi|^M$ for all $w \in V$, then $\langle V, \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle \in Cont(\phi, M)$. □

We discern two ways in which a content may be related to a formula: by *locally* or *uniformly supporting* the formula.

Definition 2.5 (*Formulas supported by a content*) Let $C = \langle V, \mathbf{I}_1, \ldots, \mathbf{I}_n, w^* \rangle$ be a situated n-ary content over M, and let $\phi(x_1, \ldots, x_n)$ be an L_0-formula. C *locally supports* ϕ (in symbols $C \Vdash_{loc} \phi$) if $\langle w^*, \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle \in |\phi|^M$. It *uniformly supports* ϕ (in symbols $C \Vdash_{uni} \phi$) if $\langle V, \mathbf{I}_1, \ldots, \mathbf{I}_n \rangle \in Cont(\phi, M)$. □

Sometimes it is more convenient to speak of the converses of these relations. In Sect. 6.5 the following terminology is adopted: we say that ϕ is *true of C* iff C locally supports ϕ, and that ϕ *describes C* iff C uniformly supports ϕ.

B.6 Syntactic Properties of Modal Formulas

As defined in Sect. 5.2, the *modal depth* of an L-formula ϕ, denoted $md(\phi)$, is the maximum number of nested modal operators occurring in ϕ. The *degree* of ϕ is the number indices of modal operators occurring in ϕ. The following definition introduces the syntactic notions of modal character and modal profile.

Definition 5.1 (*Modal character, modal profile*) If $\phi \in L$, let $\mathsf{i}_1, \ldots, \mathsf{i}_k$ be the indices of modal operators occurring in ϕ. If $m \in \mathbb{N}$, let $\langle \mathsf{j}_1, \ldots, \mathsf{j}_m \rangle$ be a tuple whose members are among the elements of the set $\{\mathsf{i}_1, \ldots, \mathsf{i}_k\}$. (The tuple may contain several occurrences of one and the same index: if $k \geq 1$, we may have $m > k$.) The tuple $\langle \mathsf{j}_1, \ldots, \mathsf{j}_m \rangle$ is a *modal character in* ϕ if $m \geq 1$ and it satisfies the following: there are in ϕ modal operator tokens $\bigcirc^1, \ldots, \bigcirc^m$ with respective indices $\mathsf{j}_1, \ldots, \mathsf{j}_m$ such that for all $1 \leq r < m$, $\bigcirc^{r+1}$ is the immediate successor of $\bigcirc^r$ along the relation of syntactic subordination among modal operator tokens in ϕ, and $\bigcirc^1$ is not subordinate to any modal operator in ϕ. The *modal profile* of ϕ (denoted P_ϕ) is the set of all modal characters in ϕ. □

B.7 Relative Rigid Designators

According to the semantics of L, a constant symbol does not stand for a name of a physically individuated world line; it stands for a *realization* of such a world line. If 'rigid designators' are constant symbols that stand for the same object in all worlds in which that object exists, then by the semantics of L, there are no non-trivial cases of rigid designators: a constant symbol cannot refer to the same object in two worlds. Namely, in each world w, the interpretation of a constant symbol is a local object belonging to $dom(w)$, and no local object appears in the domains of several worlds. The notion of rigid designator can, however, be simulated as follows (Sect. 5.3).

Definition 5.2 (*Relative rigid designator*) If $\mathsf{i}_1 \dots \mathsf{i}_m$ is a finite string of indices and c is a constant symbol, let us write $\theta^c_{\mathsf{i}_1 \dots \mathsf{i}_m} := \exists x(x = c \wedge \Box_{\mathsf{i}_1} \cdots \Box_{\mathsf{i}_m}\, x = c)$. We say that c is a *relative rigid designator of type* $\mathsf{i}_1 \dots \mathsf{i}_m$ in M at w iff $M, w \models \theta^c_{\mathsf{i}_1 \dots \mathsf{i}_m}$. □

That is, c is a rigid designator relative to the string $\mathsf{i}_1 \dots \mathsf{i}_m$ of types of modalities and relative to the world w if there is a physically individuated world line $\mathbf{I} \in \mathcal{P}_w$ whose realization is named by 'c' in w and in all worlds w' such that $(R_{\mathsf{i}_1} \circ \cdots \circ R_{\mathsf{i}_m})(w, w')$ but possibly not in any further world.

B.8 Substitutions and Validity

The following definition given in Sect. 5.5 provides a notion of substitution that consists of replacing atomic formulas by specified arbitrary formulas within an L-formula. (Recall the notation $\phi[x_1 /\!/ y_1, \dots, x_n /\!/ y_n]$ for uniform substitution of y_i for x_i; see Sect. B.3 of this appendix.)

Definition 5.3 (*Substitution, base of substitution*) Let τ_1 and τ_2 be disjoint vocabularies. Let V_1 and V_2 be disjoint subsets of *Var*, with $V_2 = \{v_i : i \geq 1\}$. A *base of substitution* is a map $\upsilon : \tau_1 \rightarrow L[\tau_1 \cup \tau_2, V_2]$ that assigns to every n-ary predicate P of τ_1 an $L[\{P\} \cup \tau_2, V_2]$-formula $\upsilon(P)$ whose free variables are $v_1, \dots, v_n$. A map $\sigma : L[\tau_1, V_1] \rightarrow L[\tau_1 \cup \tau_2, V_1 \cup V_2]$ is an υ*-substitution* (or *substitution based on* υ) if it satisfies the following:

- $\sigma[P(x_1, \dots, x_n)] := \upsilon(P)[v_1 /\!/ x_1, \dots, v_n /\!/ x_n]$
- $\sigma[x = y] := x = y$
- $\sigma[\neg\psi] := \neg\sigma[\psi]$
- $\sigma[(\psi_1 \wedge \psi_2)] := (\sigma[\psi_1] \wedge \sigma[\psi_2])$
- $\sigma[\Box_{\mathsf{i}}\psi] := \Box_{\mathsf{i}}\sigma[\psi]$
- $\sigma[\mathsf{Q}x\psi] := \mathsf{Q}x\sigma[\psi]$ for $\mathsf{Q} \in \{\exists\} \cup \{\mathsf{E}_{\mathsf{a}} : \mathsf{a} \in \mathbb{A}\}$. □

The following variant of the above notion of substitution is considered, as well.

Definition 5.4 (*Strong substitution, strong base of substitution*) Let the sets τ_1, τ_2, V_1, and V_2 be as in Definition 5.3. A *strong base of substitution* is a base of substitution $\rho : \tau_1 \to L[\tau_1 \cup \tau_2, V_2]$ such that for all $P \in \tau_1$, there is $\chi_P \in L[\{P\} \cup \tau_2, V_2]$ satisfying $\rho(P) = \chi_P(v_1, \ldots, v_n) \wedge \bigwedge_{1 \leq i \leq n} v_i = v_i$. A *strong ρ-substitution* is a ρ-substitution, where ρ is a strong base of substitution. □

Two notions of validity are distinguished in Sect. 5.5. An L-formula ϕ is *model-theoretically valid* if it is satisfied in all models M in all worlds $w \in dom(M)$ under all assignments in M. The formula ϕ is *schematically valid* if for all substitutions σ, the formula $\sigma[\phi]$ is model-theoretically valid. It is shown that in L, model-theoretic validity is not preserved under uniform substitution, though it is preserved under strong uniform substitution.

References

1. Anscombe, Elizabeth (1965). 'The Intentionality of Sensation: A Grammatical Feature,' in R. J. Butler (ed.), *Analytical Philosophy: Second Series* (Oxford: Blackwell), pp. 158–80.
2. Anscombe, Elizabeth (1979). 'Under a Description,' *Noûs*, 13, pp. 219–33.
3. Areces, Carlos, and ten Cate, Balder (2007). 'Hybrid Logics,' in P. Blackburn *et al.* (eds.), *Handbook of Modal Logic* (Amsterdam: Elsevier), pp. 821–68.
4. Belnap, Nuel, and Müller, Thomas (2014). 'CIFOL: Case-Intensional First Order Logic,' *Journal of Philosophical Logic*, 43, pp. 393–437.
5. Braüner, Torben, and Ghilardi, Silvio (2007). 'First-Order Modal Logic,' in P. Blackburn *et al.* (eds.), *Handbook of Modal Logic* (Amsterdam: Elsevier), pp. 549–620.
6. Bressan, Aldo (1972). *A General Interpreted Modal Calculus* (New Haven, CT: Yale University Press).
7. Bricker, Phillip (1993). 'The Fabric of Space: Intrinsic vs. Extrinsic Distance Relations,' *Midwest Studies in Philosophy*, 18, pp. 271–94.
8. Burgess, John (1999). 'Which Modal Logic Is the Right One?,' *Notre Dame Journal of Formal Logic*, 40, pp. 81–93.
9. Burgess, John (2003). 'Which Modal Logics Are the Right Ones (for Logical Necessity)?,' *Theoria* (San Sebastián), 18, pp. 145–58.
10. Byrne, Alex (2001). 'Intentionalism Defended,' *Philosophical Review*, 110, pp. 199–240.
11. Carnap, Rudolf (1928). *Der logische Aufbau der Welt* (Berlin: Weltkreis). Translated by Rolf A. George as *The Logical Structure of the World. Pseudoproblems in Philosophy* (Berkeley: University of California Press, 1967).
12. Carnap, Rudolf (1947). *Meaning and Necessity: A Study in Semantics and Modal Logic* (Chicago: Chicago University Press).
13. Cassam, Quassim (1999). 'Self-Directed Transcendental Arguments,' in R. Stern (ed.), *Transcendental Arguments: Problems and Prospects* (Oxford: Oxford University Press), pp. 83–110.
14. Chisholm, Roderick (1967). 'Identity through Possible Worlds: Some Questions,' *Noûs*, 1, pp. 1–8.
15. Church, Alonzo (1956). *Introduction to Mathematical Logic* (Princeton, NJ: Princeton University Press).
16. Clark, Austen (1996). 'Three Varieties of Visual Field,' *Philosophical Psychology*, 9, pp. 477–95.
17. Crane, Tim (1992). 'The Nonconceptual Content of Experience,' in T. Crane (ed.), *The Contents of Experience: Essays on Perception* (Cambridge: Cambridge University Press), pp. 136–57.

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18. Crane, Tim (2009). 'Is Perception a Propositional Attitude?,' *The Philosophical Quarterly*, 59, pp. 452–69.
19. Crane, Tim (2009). 'Intentionalism,' in A. Beckermann and B. McLaughlin (eds.), *Oxford Handbook to the Philosophy of Mind* (Oxford: Oxford University Press), pp. 474–93.
20. Crane, Tim (2011). 'The Problem of Perception,' in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*: Spring 2011 edition.
21. Crane, Tim (2013). *The Objects of Thought* (Oxford: Oxford University Press).
22. Crane, Tim (2013). 'The Given,' in J. Schear (ed.) *Mind, Reason and Being-in-the-World: The McDowell-Dreyfus Debate* (London: Routledge), pp. 229–49.
23. Dummett, Michael (1978). *Truth and Other Enigmas* (London: Duckworth).
24. Evans, Gareth (1982). *The Varieties of Reference* (Oxford: Oxford University Press).
25. Fine, Kit (1999). 'Things and Their Parts,' *Midwest Studies in Philosophy*, 23, pp. 61–74.
26. Fitch, Greg, and Nelson, Michael (2013). 'Singular Propositions,' in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*: Winter 2013 edition.
27. Fitting, Melvin, and Mendelsohn, Richard (1998). *First-Order Modal Logic* (Dordrecht: Kluwer).
28. Forbes, Graeme (2013). 'Intensional Transitive Verbs,' in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*: Fall 2013 edition.
29. Frege, Gottlob (1974) [1884]. *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number*, translation by J. L. Austin (Oxford: Blackwell, second revised edition).
30. Frege, Gottlob (1980). *Philosophical and Mathematical Correspondence* (Oxford: Blackwell).
31. Gamut, L. T. F. (1991). *Logic, Language, and Meaning, vol. 2: Intensional Logic and Logical Grammar* (Chicago: The University of Chicago Press).
32. Gardner, Sebastian (1999). *Kant and the Critique of Pure Reason* (London: Routledge).
33. Garson, James (2014). 'Modal Logic,' in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*: Summer 2014 edition.
34. Geach, Peter (1967). 'Intentional Identity,' *The Journal of Philosophy*, 64, pp. 627–32.
35. Grandy, Richard (2007). 'Sortals,' in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*: Summer 2007 edition.
36. Grosu, Alexander, and Krifka, Manfred (2007). 'The Gifted Mathematician that You Claim to Be: Equational Intensional "Reconstruction" Relatives,' *Linguistics and Philosophy*, 30, pp. 445–85.
37. Hall, Ned (2010). 'David Lewis's Metaphysics,' in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*: Fall 2010 edition.
38. Hawley, Katherine (2001). *How Things Persist* (Oxford: Clarendon Press).
39. Hawley, Katherine (2015). 'David Lewis on Persistence,' in B. Loewer, and J. Schaffer (eds.), *A Companion to David Lewis* (Oxford: Wiley-Blackwell), pp. 237–49.
40. Hintikka, Jaakko (1956). 'Identity, Variables, and Impredicative Definitions,' *Journal of Symbolic Logic*, 21, pp. 225–45.
41. Hintikka, Jaakko (1962). *Knowledge and Belief* (Ithaca, NY: Cornell University Press).
42. Hintikka, Jaakko (1969). 'On the Logic of Perception,' in N. Care, and R. Grimm (eds.), *Perception and Personal Identity* (Cleveland, OH: Case Western Reserve University Press), pp. 140–73. References are to the reprint in [44, pp. 151–83]
43. Hintikka, Jaakko (1969). 'Semantics for Propositional Attitudes,' in W. Davis *et al.* (eds.), *Philosophical Logic* (Dordrecht: Reidel), pp. 21–45. References are to the reprint in [44, pp. 87–111]
44. Hintikka, Jaakko (1969). *Models for Modalities* (Dordrecht: Reidel).
45. Hintikka, Jaakko (1973). 'Carnap's Semantics in Retrospect,' *Synthese*, 25, pp. 372–97.
46. Hintikka, Jaakko (1975). *The Intentions of Intentionality and Other New Models for Modalities* (Dordrecht: Reidel).
47. Hintikka, Jaakko (1998). 'Perspectival Identification, Demonstratives and "Small Worlds",' *Synthese*, 114, pp. 203–32.

48. Hintikka, Jaakko (1998). *Paradigms for Language Theory and Other Essays*, Selected Papers, vol. 4 (Dordrecht: Kluwer).
49. Hintikka, Jaakko (2006). 'Intellectual Autobiography' and Hintikka's replies to the contributors, in R. E. Auxier, and L. E. Hahn (eds.), *The Philosophy of Jaakko Hintikka, The Library of Living Philosophers*, vol. 30 (Chicago: Open Court).
50. Hintikka, Jaakko, and Hintikka, Merrill B. (1982). 'Towards a General Theory of Individuation and Identification,' in W. Leinfellner *et al.* (eds.), *Language and Ontology: Proceedings of the Sixth International Wittgenstein Symposium* (Vienna: Hölder-Pichler-Tempsky), pp. 137–50.
51. Hintikka, Jaakko, and Sandu, Gabriel (1995). 'The Fallacies of the New Theory of Reference,' *Synthese*, 104, pp. 245–83.
52. Hintikka, Jaakko, and Symons, John (2003). 'Systems of Visual Identification in Neuroscience: Lessons from Epistemic Logic,' *Philosophy of Science*, 70, pp. 89–104.
53. Hornstein, Norbert (1990). *As Time Goes By: Tense and Universal Grammar* (Cambridge, MA: MIT Press).
54. Humberstone, Lloyd (1992). 'Direction of Fit,' *Mind*, 101, pp. 59–83.
55. Jespersen, Bjørn (2011). 'An Intensional Solution to the Bike Puzzle of Intentional Identity,' *Philosophia*, 39, pp. 297–307.
56. Jespersen, Bjørn (2015). 'Qualifying Quantifying-in,' in A. Torza (ed.), *Quantifiers, Quantifiers, and Quantifiers: Themes in Logic, Metaphysics, and Language* (Heidelberg: Springer), pp. 241–69.
57. Jónsson, Ólafur Páll (2005). 'The Bike Puzzle,' *Mind*, 114, pp. 929–32.
58. Jubien, Michael (1993). *Ontology, Modality, and the Fallacy of Reference* (Cambridge: Cambridge University Press).
59. Kalmár, László, and Surányi, János (1950). 'On the Reduction of the Decision Problem, Third Paper: Pepis Prefix, a Single Binary Predicate,' *Journal of Symbolic Logic*, 15, pp. 161–73.
60. Kant, Immanuel (1933) [first edition 1781 (A), second edition 1787 (B)]. *Critique of Pure Reason*, translation by N. Kemp Smith (London: Macmillan).
61. Kaplan, David (1989). 'Demonstratives,' in J. Almog, H. Wettstein, and J. Perry (eds.), *Themes From Kaplan* (Oxford: Oxford University Press), pp. 481–563.
62. Kirjavainen, Heikki (2008). 'How is God-Talk Logically Possible? A Sketch for an Argument on the Logic of "God",' *International Journal for Philosophy of Religion*, 64, pp. 75–88.
63. Koslicki, Kathrin (2008). *The Structure of Objects* (Oxford: Oxford University Press).
64. Kraut, Robert (1979). 'Attitudes and Their Objects,' *Journal of Philosophical Logic*, 8, pp. 197–217.
65. Kraut, Robert (1983). 'There Are No *De Dicto* Attitudes,' *Synthese*, 54, pp. 275–94.
66. Kracht, Marcus, and Kutz, Oliver (2007). 'Logically Possible Worlds and Counterpart Semantics for Modal Logic,' in D. Jacquette (ed.), *Philosophy of Logic* (Amsterdam: Elsevier), pp. 943–95.
67. Kripke, Saul (1963). 'Semantical Considerations in Modal Logic,' *Acta Philosophica Fennica*, 16, pp. 83–94.
68. Kripke, Saul (1971). 'Identity and Necessity,' in M. Munitz (ed.), *Identity and Individuation* (New York: New York University Press), pp. 135–64.
69. Kripke, Saul (1972). 'Naming and Necessity,' in D. Davidson, and G. Harman (eds.), *Semantics of Natural Language* (Dordrecht: Reidel), pp. 253–355.
70. Kripke, Saul (1976). 'Is There a Problem About Substitutional Quantification?,' in G. Evans, and J. McDowell (eds.), *Truth and Meaning* (Oxford: Oxford University Press), pp. 324–419.
71. Kripke, Saul (1980). *Naming and Necessity*. Oxford: Blackwell.
72. Kripke, Saul (2013). *Reference and Existence: The John Locke Lectures* (Oxford: Oxford University Press).
73. Lewin, Kurt (1922). *Der Begriff der Genese in Physik, Biologie und Entwicklungsgeschichte* (Berlin: Springer).
74. Lewin, Kurt (1923). 'Die zeitliche Geneseordnung,' *Zeitschrift für Physik*, 13, pp. 62–81.
75. Lewis, David (1968). 'Counterpart Theory and Quantified Modal Logic,' *The Journal of Philosophy*, 65, pp. 113–26.

76. Lewis, David (1986). *On the Plurality of Worlds* (Oxford: Blackwell).
77. Lewis, David (1986). *Philosophical Papers*, vol. 2 (Oxford: Oxford University Press).
78. Lewis, David (1994). 'Humean Supervenience Debugged,' *Mind*, 103, pp. 473–90.
79. Lewis, David (1999). *Papers in Metaphysics and Epistemology* (Cambridge: Cambridge University Press).
80. Lowe, E. Jonathan (1989). 'What Is a Criterion of Identity?,' *The Philosophical Quarterly*, 39, pp. 1–21.
81. Lowe, E. Jonathan (1989). *Kinds of Being: A Study of Individuation, Identity, and the Logic of Sortal Terms* (Oxford: Basil Blackwell).
82. Marcus, Ruth Barcan (1961). 'Modalities and Intensional Languages,' *Synthese*, 13, pp. 303–22.
83. Markosian, Ned (2000). 'What are Physical Objects?,' *Philosophy and Phenomenological Research*, 61, pp. 375–95.
84. McDowell, John (1977). '*De Re* Senses,' *The Philosophical Quarterly*, 34, pp. 283–94.
85. McDowell, John (1998). *Meaning, Knowledge and Reality* (Cambridge, MA: Harvard University Press).
86. Moltmann, Friederike (1997). 'Intensional Verbs and Quantifiers,' *Natural Language Semantics*, 5, pp. 1–52.
87. Moltmann, Friederike (2011). 'Tropes, Intensional Relative Clauses, and the Notion of a Variable Object'. Draft retrieved February 11, 2015, from http://semanticsarchive.net/Archive/zg0ZmYyN. Appeared with modifications in M. Aloni *et al.* (eds.), *Logic, Language and Meaning. Proceedings of the 18th Amsterdam Colloquium (2011)*, (Berlin: Springer, 2012), pp. 431–40.
88. Moltmann, Friederike (2015). 'Quantification with Intentional and with Intensional Verbs,' in A. Torza (ed.), *Quantifiers, Quantifiers, and Quantifiers: Themes in Logic, Metaphysics, and Language* (Heidelberg: Springer), pp. 141–68.
89. Moltmann, Friederike 3(2015). 'Variable Objects and Truth-Making'. To appear in M. Dumitru (ed.), *Metaphysics, Meaning, and Modality: Themes from Kit Fine* (Oxford: Oxford University Press). References are to the preprint retrieved February 11, 2015, from http://semanticsarchive.net/Archive/zQ2YjQzM.
90. Montague, Michelle (2007). 'Against Propositionalism,' *Noûs*, 41, pp. 503–18.
91. Montague, Richard (1973). 'The Proper Treatment of Quantification in Ordinary English,' in J. Hintikka *et al.* (eds.), *Approaches to Natural Language* (Reidel: Dordrecht), pp. 221–42.
92. Niiniluoto, Ilkka (1982). 'Remarks on the Logic of Perception,' *Acta Philosophica Fennica*, 35, pp. 116–29.
93. Pereboom, Derek (2013). 'Kant's Transcendental Arguments,' in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*: Winter 2013 edition.
94. Plantinga, Alvin (1974). *The Nature of Necessity* (Oxford: Oxford University Press).
95. Priest, Graham (2005). *Towards Non-Being: The Logic and Metaphysics of Intentionality* (Oxford: Clarendon Press).
96. Prior, Arthur (1957). *Time and Modality* (Oxford: Clarendon Press).
97. Prior, Arthur (1967). *Past, Present and Future* (Oxford: Clarendon Press).
98. Prior, Arthur (1968). *Papers on Time and Tense* (Oxford: Clarendon Press).
99. Prior, Arthur (1971). *Objects of Thought*, edited by P. Geach, and A. Kenny (Oxford: Clarendon Press).
100. Prior, Arthur (1976). *Papers in Logic and Ethics*, edited by P. Geach, and A. Kenny (London: Duckworth).
101. Quine, Willard Van Orman (1970). *Philosophy of Logic* (Englewood Cliffs, NJ: Prentice-Hall).
102. Rebuschi, Manuel, and Tulenheimo, Tero (2011). 'Between *De Dicto* and *De Re*: *De Objecto* Attitudes,' *The Philosophical Quarterly*, 61, pp. 828–38.
103. Recanati, François (2012). *Mental Files* (Oxford: Oxford University Press).
104. Russell, Bertrand (1910). 'Knowledge by Acquaintance and Knowledge by Description,' *Proceedings of the Aristotelian Society*, 11, pp. 108–28.
105. Russell, Bertrand (1912). *The Problems of Philosophy* (Oxford: Oxford University Press).

106. Russell, Bertrand (1914). *Our Knowledge of the External World* (London: George Allen & Unwin).
107. Saarinen, Esa (1982). 'Propositional Attitudes Are Not Attitudes Towards Propositions,' *Acta Philosophica Fennica*, 35, pp. 130–62.
108. Salmon, Nathan (1986). *Frege's Puzzle* (Cambridge, MA: MIT Press).
109. Schurz, Gerhard (1997). *The Is-Ought Problem* (Dordrecht: Kluwer).
110. Searle, John (1983). *Intentionality* (Cambridge: Cambridge University Press).
111. Sider, Theodore (1996). 'All the World's a Stage,' *Australasian Journal of Philosophy*, 74, pp. 433–53.
112. Sider, Theodore (2001). *Four-Dimensionalism: An Ontology of Persistence and Time* (Oxford: Oxford University Press).
113. Smith, David, and McIntyre, Ronald (1982). *Husserl and Intentionality. A Study of Mind, Meaning, and Language* (Dordrecht: Reidel).
114. Soames, Scott (1985). 'Lost Innocence,' *Linguistics and Philosophy*, 8, pp. 59–71.
115. Soames, Scott (1987). 'Direct Reference, Propositional Attitudes, and Semantic Content,' *Philosophical Topics*, 15, pp. 47–87.
116. Stroud, Barry (1968). 'Transcendental Arguments,' *Journal of Philosophy*, 65, pp. 241–56.
117. Stroud, Barry (1999). 'The Goal of Transcendental Arguments,' in R. Stern (ed.), *Transcendental Arguments: Problems and Prospects* (Oxford: Oxford University Press), pp. 155–72.
118. Tulenheimo, Tero (2009). 'Remarks on Individuals in Modal Contexts,' *Revue Internationale de Philosophie*, 63, pp. 383–94.
119. Tulenheimo, Tero (2015). 'Cross-World Identity, Temporal Quantifiers and the Question of Tensed Contents,' in A. Torza (ed.), *Quantifiers, Quantifiers, and Quantifiers: Themes in Logic, Metaphysics, and Language* (Heidelberg: Springer), pp. 409–61.
120. Tulenheimo, Tero (2016). 'Worlds, Times and Selves Revisited,' *Synthese*, 193, pp. 3713–25.
121. Tye, Michael (2000). *Consciousness, Color, and Content* (Cambridge, MA: MIT Press).
122. van Inwagen, Peter (1990). *Material Beings* (Ithaca: Cornell University Press).
123. Wasserman, Ryan (2013). 'Material Constitution,' in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*: Summer 2013 edition.
124. Weatherson, Brian (2015). 'Humean Supervenience,' in B. Loewer, and J. Schaffer (eds.), *A Companion to David Lewis* (Oxford: Wiley-Blackwell), pp. 101–15.
125. Wehmeier, Kai (2012). 'How to Live Without Identity—And Why,' *Australasian Journal of Philosophy*, 90, pp. 761–77.
126. Wiggins, David (2001). *Sameness and Substance Renewed* (Cambridge: Cambridge University Press).
127. Williamson, Timothy (2013). *Modal Logic as Metaphysics* (Oxford: Oxford University Press).
128. Zimmerman, Thomas Ede (1993). 'On the Proper Treatment of Opacity in Certain Verbs,' *Natural Language Semantics*, 1, pp. 149–79.

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