# Appendix A
# Selected Problems

Appendix A provides a set of selected problems. The solutions to these problems are summarized in Appendix B. Appendix A starts with some introductory problems, followed by problems which need additional considerations for obtaining a solution. Furthermore some problems are devoted to the choice of the base system and to model theory and also on similarity solutions. The problems are thought to provide a good basis for exercises in order to very well understand the beauty of dimensional analysis.

## A.1 Problems Which Lead to Optimal Results

### *A1.1 Speed of Sound in Gases*

The state of a gas is characterized by the pressure $p$ and density $\rho$. Determine the equation for the speed of sound $a$, using dimensional analysis.

### *A1.2 Speed of Sound in Elastic Bodies*

An elastic body shall have the elasticity modulus $E$ and density $\rho$. Determine the speed of sound $a$ in dimensionless form.

### *A1.3 Natural Frequency of a Taut String*

The natural frequency $n$ of a string (e.g. the string of a guitar or a violin) depends on the force $F$ with which the string is strained, its density $q$ (in this case: mass per unit length) and the length of the string $L$. Determine the natural frequency $n$ of the string using dimensional analysis.



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### *A1.4 Viscosity of Gases*

The macroscopically observed property of viscosity of gases originates in the thermal motion of the molecules. The molecular motion is affected by the molecular weight $m$, the average molecular velocity $v$ and the intermolecular force $F$ written in the form

$$F = K\,r^{-n}, \quad K, n = \text{const.},$$

whereas $r$ denotes the distance between the molecules. According to kinetic theory of gases, the temperature is proportional to the average kinetic energy of the molecule, i.e.

$$\left\langle \frac{mv^2}{2} \right\rangle = k_B \vartheta,$$

with the Boltzmann constant $k_B$. Determine the dependency between the gas viscosity and its temperature.

## A.2 Problems That Require Additional Considerations

### *A2.1 Oscillations of a Pendulum*

At the end of a pendulum of length $L$ a mass $m$ is attached (see Fig. A.1). Determine with the help of dimensional analysis, the formula for the period of the oscillations $\tau$ of the pendulum, for very small initial displacements $\varphi_0$.

### *A2.2 Velocity of a Bubble Rising Upward*

An upright standing tube of diameter $d$ is filled with a liquid of density $\rho_w$. In the liquid a bubble with the density $\rho_L$ rises upwards with the speed $U$. It can be

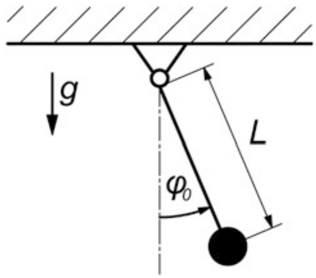



**Fig. A.1** Mathematical pendulum

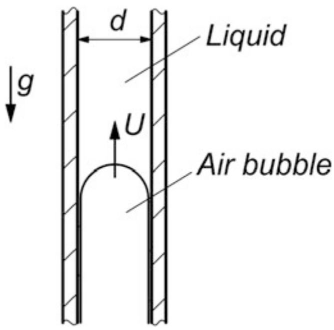



**Fig. A.2** Bubble in a tube filled with liquid

assumed that the air bubble almost entirely fills the circular cross section of the pipe and is very long (Fig. A.2). Determine the rising velocity of the air bubble $U$ by dimensional analysis. At first specify the dimension matrix, and determine its rank and the number of dimensionless products. Deduce the dimensionless products, and discuss the special case $\rho_L/\rho_w \ll 1$.

## *A2.3 Capillary Height in a Tube*

At the interface between two liquids (such as water and air) always surface tension occurs. In still liquid, the pressure difference between two fluids at the separating interface is proportional to the surface tension and the mean curvature. In general, the separating surface between two fluids inflects at the edges, when two liquids—for example water and air—touch a solid wall. The angle $\alpha$ enclosed between interface and the wall is a function of the considered liquids and the wall material (see Fig. A.3). The interface bends and a pressure difference develops between the liquids. In capillary tubes with small diameters, the pressure difference causes the liquid in the tube to rise upwards against gravity or to be pressed downwards. The height difference $h$ in the tube of diameter $d$ thus is a function of surface tension $\sigma$ and the liquid density $\rho$, the gravitational acceleration $g$ and the angle $\alpha$. Determine, the height $h$ using dimensional analysis.

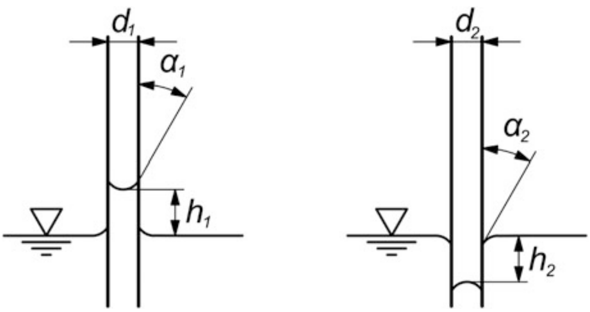



**Fig. A.3** Capillary tube for water (*left*) and mercury (*right*)

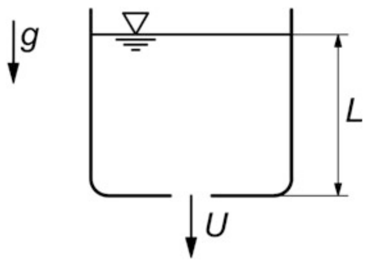



**Fig. A.4** Liquid outflow from a large container

## A2.4 Liquid Outflow from a Large Container

A large container is filled with liquid of density $\rho$ and viscosity $\eta$. This is schematically shown in Fig. A.4. The liquid is released at the bottom of the container through an opening of diameter $d$ at speed $u$. Determine the outflow velocity of the liquid.

## A2.5 Volume Flow Measurement at Open Water Courses

To measure the volume flux at open water courses a vertical wall is used that has an opening, through which the water passes (see Fig. A.5). For a rectangular opening of width $b$ the flow rate depends not only on the width but also on the water level $h$ to the bottom edge of the opening and the forcing of gravity. Determine the volume flow rate.

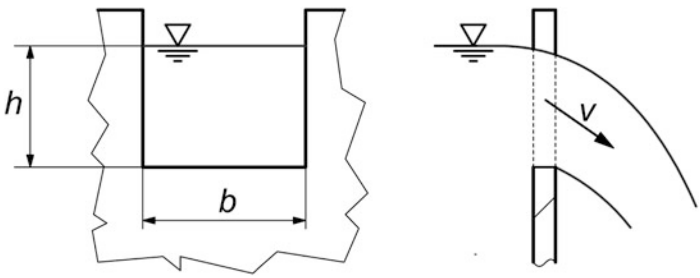



**Fig. A.5** Volume flow measurement

# A.3 Problems Related with the Choice of the Base System

## A3.1 Pressure Drop in a Pipe

An incompressible fluid with density $\rho$ and viscosity $\eta$flows through a straight circular pipe of diameter $d$. The pressure gradient is given by $K = -\Delta p/L$. Determine the mass flow $\dot{m}$ using the $[LMFT]$ system and discuss the result.

## A3.2 Rotating Cylinder

An infinitely long cylinder of radius $R$ is located in a liquid with viscosity $\eta$ and density $\rho$ and rotates with the angular velocity $\Omega$ around its axis. Determine the moment $M$ that acts on the cylinder by dimensional analysis in the $[LMFT]$system. Determine the dimension matrix, and determine its rank and the number of dimensionless products. Specify the dimensionless products, and discuss both the general case as well as the special case occurring when the dimensional constant $C$ is not considered in the $[LMFT]$ system.

# A.4 Problems to Exercise the Formal Determination of Dimensionless Products

## A4.1 Resistance of a Body Under Water

A sphere of diameter $d$ moves with the velocity $U$ through an incompressible fluid with density $\rho$ and viscosity $\eta$. This is shown in Fig. A.6. Determine the resistance of the sphere when it moves at a certain distance $h$ below the free surface of the liquid including the surface tension $\sigma$. The influence of the atmosphere can be neglected.

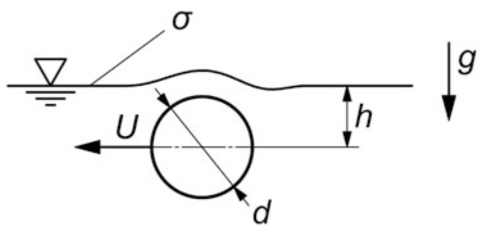



**Fig. A.6** Resistance of a body under water

# A.5 Problems Related to Model Theory

## *A5.1 Buckling Load of a Steel Column*

In order to determine the buckling load of a steel column ($E = 21.1 \times 10^{10}\ \mathrm{N/mm^2}$), a load test is conducted on a ten times smaller, geometrically similar model from aluminum ($E = 0.7 \times 10^{10}\ \mathrm{N/mm^2}$). Figure A.7 shows a schematic of the experiment. The experiments reveal a buckling load $P_{kr} = 2.5$ kN of the model. What is the critical load of the original element?

## *A5.2 Deformation and Stress of a Cantilever Beam*

A cantilever that is anchored at one end and carries a load $P$ at the other end is considered (see Fig. A.8). A small scale model is investigated, to draw conclusions on the maximum deflection and the stresses in the beam of the full size version. Specify the required scale factors, if the scale factor is $M_L = 0.1$ for the beam length.

**Fig. A.7** Buckling load of a steel column

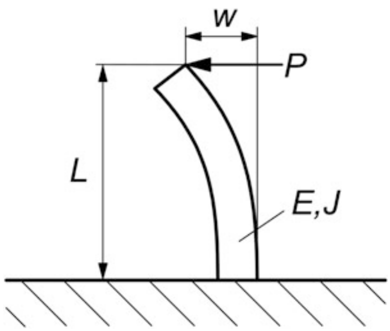
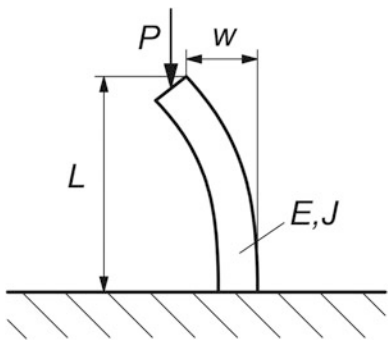



**Fig. A.8** Deformation of a cantilever beam



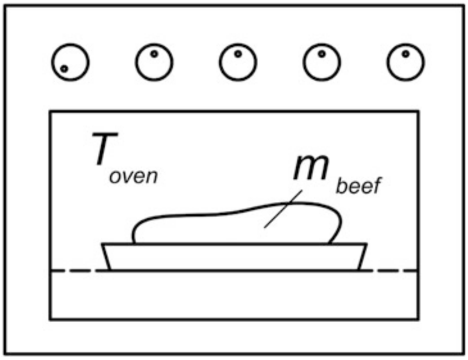



**Fig. A.9** Piece of roast beef in an oven

### *A5.3 Cooking of Roast Beef*

Following a cookbook recipe (for four people), a 1 kg heavy piece of roast beef must fry in a preheated 175 °C oven for 1 h (see Fig. A.9). Because eight guests are expected, the chef doubles the piece of meat to 2 kg. How long must the piece roast with geometric similarity at the same oven temperature?

## A.6 Problems which lead to Similarity Solutions of Partial Differential Equations

### *A6.1 Vortex Decay*

In an incompressible fluid of density $\rho$ and viscosity $\eta$ an infinitely long straight vortex filament with constant circulation $\Gamma$. The velocity field induced by the vortex has only one component $v$ in the circumferential direction, while the radial and axial velocity components are zero. At time $t = 0$, the velocity field is given by

$$v(r, t = 0) = \frac{\Gamma}{2\pi r}. \tag{A.1}$$

Without further supplying energy to the vortex, as a results of the fluid viscosity, it decays. Besides the initial condition (A.1) the velocity field must satisfy the boundary conditions

$$v(r = 0, t) = 0 \tag{A.2}$$

and

$$v(r \to \infty, t) = \frac{\Gamma}{2\,\pi\,r}. \tag{A.3}$$

This flow identically satisfies the continuity equation and the axial and radial components of the equation of motion. The circumferential component of the momentum equation simplifies to

$$\frac{\partial v}{\partial t} = \frac{\eta}{\rho}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right). \tag{A.4}$$

(a) Determine the dimension matrix for the velocity distribution $v = v(r, t, \Gamma, \eta, \rho)$ and determine the dimensionless products.
(b) Show that the partial differential equation (A.4) can be converted into an ordinary differential equation using the dimensionless products obtained in (a), and determine the solution of this differential equation that satisfies initial and boundary conditions according to Eqs. (A.1), (A.2) and (A.3).

# Appendix B
# Solutions to the Problems

Appendix B provides the solution of all problems given in Appendix A. The solution is presented in a way to show all the major steps in the solution process of the problem.

**Problem A1.1** The speed of sound $a$ in gas depends on the local pressure $p$, and the local density $\rho$. It has therefore a relationship of the form

$$a = f(p, \rho). \tag{B.1}$$

The variables describing the problem are summarized in the following table.

| Variable | Symbol | Dimension formula |
|---|---|---|
| Speed of sound | $a$ | $\mathrm{L\,T^{-1}}$ |
| Pressure | $p$ | $\mathrm{L^{-1}\,M\,T^{-2}}$ |
| Density | $\rho$ | $\mathrm{L^{-3}\,M}$ |

The product $p/\rho$ has the dimension

$$\left[\frac{p}{\rho}\right] = \mathrm{L^2\,T^{-2}} \quad \text{and the unit} \quad \left\{\frac{p}{\rho}\right\} = \mathrm{m^2\,s^{-2}}, \tag{B.2}$$

that is, with $\left[\sqrt{p/\rho}\right] = LT^{-1}$ the speed of sound can be made dimensionless, and we obtain

$$\prod = a\sqrt{\frac{\rho}{p}} \quad \text{respectively} \quad a\sqrt{\frac{\rho}{p}} = \text{const.} \tag{B.3}$$



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The speed of sound is then

$$a = \text{const.}\sqrt{\frac{p}{\rho}}. \tag{B.4}$$

This relation holds strictly only for calorically perfect gases, for which const. $= \sqrt{\gamma}$, where $\gamma$ is the isentropic exponent of the gas.

**Problem A1.2** The speed $a$ at which sound propagates in an elastic body, is dependent on the elasticity modulus $E$ and the density $\rho$ of the body. It is therefore

$$a = \mathrm{f}(E, \rho). \tag{B.5}$$

Thus, one can formulate the following table.

| Variable | Symbol | Dimension formula |
|---|---|---|
| Speed of sound | $a$ | $\mathrm{L\,T^{-1}}$ |
| Modulus of elasticity | $E$ | $\mathrm{L^{-1}\,M\,T^{-2}}$ |
| Density | $\rho$ | $\mathrm{L^{-3}\,M}$ |

The product $E/\rho$ has the dimension of a velocity again

$$\left[\frac{E}{\rho}\right] = \mathrm{L^2T^{-2}} \quad \text{with the unit} \quad \left\{\frac{E}{\rho}\right\} = \mathrm{m^2\,s^{-2}}, \tag{B.6}$$

so that the speed of sound can be non-dimensionalized with $\sqrt{E/\rho}$ yielding

$$a = \text{const.}\sqrt{\frac{E}{\rho}}. \tag{B.7}$$

The theory provides const. = 1. If in addition to the sound propagation also a displacement velocity occurs as a result of an external force, the result is the so called Cauchy number

$$\mathrm{Ca} = \left(\frac{v}{a}\right)^2. \tag{B.8}$$

In the theory of elasticity the Cauchy number plays an analogous role as the Mach number in fluid mechanics.

**Problem A1.3** The natural frequency $n$ of the string is dependent on the force $F$ with which the string is biased, the density $q$ (in this case, mass per unit length) and the length $L$ of the string. The unknown function thus has the form

$$n = f(F, q, L), \tag{B.9}$$

and the physical quantities are summarized in the following table of dimension formulas.

| Variable | Symbol | Dimension formula |
|---|---|---|
| Natural frequency | $n$ | $\mathrm{T}^{-1}$ |
| Force | $F$ | $\mathrm{L\,M\,T}^{-2}$ |
| Density | $q$ | $\mathrm{L}^{-1}\,\mathrm{M}$ |
| Length | $L$ | L |

The only dimensionless product is thus

$$nL\sqrt{\frac{q}{F}} = \text{const.}, \tag{B.10}$$

and we obtain the equation for the desired natural frequency as

$$n = \text{const.}\,\frac{1}{L}\sqrt{\frac{F}{q}}, \tag{B.11}$$

from which one deduces that the frequency increases as the force increases and decreases with increasing density. The solution of the corresponding physical equation yields the value of the constant to const. $= 1/2$.

**Problem A1.4** The macroscopically observed property of the viscosity of gases is originated in the thermal motion of the molecules that causes the smearing of velocity gradients. As a result, a momentum exchange takes place so that from a continuum theoretical point of view the viscosity $\eta$ becomes noticeable. Except for the molecular weight $m$ and the mean molecular velocity $v$, the molecular motion is also influenced by the intermolecular forces. Molecules of a gas repel each other with a force $F$ that decreases with increasing distance $r$ between the molecules that can be written in the form

$$F = K\,r^{-n}, \quad K, n = \text{const.}, \tag{B.12}$$

where the constant $K$ has the dimension $L^{1+n}\,M\,T^{-2}$ and alike the exponent $n$ is a characteristic property of the considered molecules. We now ask for the viscosity of gases as a function of the molecular mass $m$, the speed $v$ and the force $F$ and expect a correlation of the form

$$\eta = f(m, v, F). \tag{B.13}$$

Within dimensional analysis, the force $F$ is described by the dimensionless constant $K$, whose dimension contains the exponent $n$ and thus all necessary information on the forcing between the molecules. Equation (B.13) thus takes the form

$$\eta = f(m, v, K). \tag{B.14}$$

Furthermore, the dependency of viscosity on temperature, which is proportional to the average kinetic energy of a molecule according to the kinetic theory of gases, is given by

$$\left\langle \frac{m\,v^2}{2} \right\rangle = k_B\,\vartheta. \tag{B.15}$$

Here, $k_B$ denotes the Boltzmann constant. Since the mass of the molecule already appears in the argument of the function, we will replace the speed by the temperature and the Boltzmann constant, and write Eq. (B.14) in the form

$$\eta = f(m, \vartheta, K, k_B). \tag{B.16}$$

The dimension matrix is then

| | $\eta$ | $m$ | $\vartheta$ | $K$ | $k_B$ |
|---|---|---|---|---|---|
| L | −1 | 0 | 0 | $n+1$ | 2 |
| M | 1 | 1 | 0 | 1 | 1 |
| T | −1 | 0 | 0 | −2 | −2 |
| $\theta$ | 0 | 0 | 1 | 0 | −1 |

and may be transformed into the form

| | $\eta$ | $m$ | $k_B\,\vartheta$ | $K$ | $k_B$ |
|---|---|---|---|---|---|
| L | −1 | 0 | 2 | $n+1$ | 2 |
| M | 1 | 1 | 1 | 1 | 1 |
| T | −1 | 0 | −2 | −2 | −2 |
| $\theta$ | 0 | 0 | 0 | 0 | −1 |

From which one deduces that $k_B$ may no longer appear in the problem, since no further variable contains the dimension of temperature. Further rearrangement results to

| | $\eta/m$ | $m$ | $k_B\,\vartheta/K$ | $K$ |
|---|---|---|---|---|
| L | −1 | 0 | $1-n$ | $n+1$ |
| M | 0 | 1 | 0 | 1 |
| T | −1 | 0 | 0 | −2 |

and thus

| | $\eta/m$ | $m$ | $k_B\,\vartheta/K$ | $K/m$ |
|---|---|---|---|---|
| L | −1 | 0 | $1-n$ | $n+1$ |
| M | 0 | 1 | 0 | 0 |
| T | −1 | 0 | 0 | −2 |

so that the mass cannot enter the problem and thus yielding

| | $\eta/m\,(m/K)^{1/2}$ | $k_B\,\vartheta/K$ | $K/m$ |
|---|---|---|---|
| L | $-(n+3)/2$ | $1-n$ | $n+1$ |
| M | 0 | 0 | 0 |
| T | 0 | 0 | −2 |
| $\theta$ | 0 | 0 | 0 |

The only dimensionless product is therefore

$$\frac{\eta}{m}\left(\frac{m}{K}\right)^{1/2}\left(\frac{k_B\,\vartheta}{K}\right)^{(n+3)/(2-2n)} = \text{const.} \tag{B.17}$$

that solved for the viscosity becomes

$$\eta = \text{const.}(K\,m)^{1/2}(k_B\,\vartheta/K)^{(n+3)/(2n-2)}. \tag{B.18}$$

Thus, for $n = 5$ the viscosity is directly proportional to the temperature, while in the limiting case $n \to \infty$ the viscosity increases as the square root of the temperature.

**Problem A2.1** The initial deflection of the pendulum is $\varphi(t=0)=\varphi_0$. The requested period $\tau$ is a function of

$$\tau = f(L, m, g, \varphi_0). \tag{B.19}$$

The dimension matrix then takes the form

| | $\tau$ | $L$ | $m$ | $g$ | $\varphi_0$ |
|---|---|---|---|---|---|
| L | 0 | 1 | 0 | 1 | 0 |
| M | 0 | 0 | 1 | 0 | 0 |
| T | 1 | 0 | 0 | −2 | 0 |

Since only $m$ contains the dimension of mass, no combination of parameters can be found to render the quantity dimensionless. Therefore, the mass $m$ does not enter the problem, and the corresponding row can be deleted from the dimension matrix. Besides the already non-dimensional initial displacement $\varphi_0$, the remaining variables the dimensionless product

$$\prod = \tau\sqrt{\frac{g}{L}} \tag{B.20}$$

yielding the relation

$$f\left(\tau\sqrt{\frac{g}{L}}, \varphi_0\right) = 0 \quad \text{respectively} \quad \tau\sqrt{\frac{g}{L}} = f(\varphi_0). \tag{B.21}$$

So far only dimension analytical approaches are applied. Further progress requires additional considerations. The function $f(\varphi_0)$ must be an even function, i.e.

$$f(\varphi_0) = f(-\varphi_0). \tag{B.22}$$

A Taylor expansion around $\varphi_0 = 0$ therefore must have the form

$$f(\varphi_0) = a_0 + a_2\,\varphi_0^2 + a_4\,\varphi_0^4 + \cdots \tag{B.23}$$

with constant coefficients $a_i$. Neglecting terms which are quadratic in $\varphi_0$, we obtain

$$f(\varphi_0) \doteq a_0 \tag{B.24}$$

and thus

$$\tau\sqrt{\frac{g}{L}} = a_0 \quad \text{for} \quad \varphi_0 \to 0, \tag{B.25}$$

respectively

$$\tau = a_0\sqrt{\frac{L}{g}} \quad \text{for} \quad \varphi_0 \to 0. \tag{B.26}$$

The equation of motion for small amplitudes around the idle state

$$\ddot{\varphi} + \frac{g}{L}\varphi = 0 \tag{B.27}$$

yields the natural frequency

$$\omega = \sqrt{\frac{g}{L}} \tag{B.28}$$

and the period

$$\tau = \frac{2\,\pi}{\omega} = 2\,\pi\sqrt{\frac{L}{g}} \tag{B.29}$$

so that the constant $a_0$ is determined to $2\,\pi$.

**Problem A2.2** We consider an upright standing cylinder of diameter $d$ filled with a liquid of density $\rho_w$. Under certain circumstances an air bubble with the lower density $\rho_L$ is formed and rises upwards at the speed $U$. We assume that the air bubble entirely fills the circular cross section of the pipe and is very long. The speed of the rising bubble will depend on the pipe diameter $d$, the densities $\rho_w$ and $\rho_L$ as well as the acceleration of gravity $g$ so that

$$U = f(d, g, \rho_w, \rho_L). \tag{B.30}$$

The dimension matrix then reads

| | $U$ | $d$ | $g$ | $\rho_w$ | $\rho_L$ |
|---|---|---|---|---|---|
| L | 1 | 1 | 1 | −3 | −3 |
| M | 0 | 0 | 0 | 1 | 1 |
| T | −1 | 0 | −2 | 0 | 0 |

yielding the nondimensional products

$$\prod_1 = \frac{U}{\sqrt{gd}} \quad \text{and} \quad \prod_2 = \frac{\rho_L}{\rho_w}. \tag{B.31}$$

so that the relationship may be written as

$$f\left(\prod_1, \prod_2\right) = 0 \quad \text{respectively} \quad \prod_1 = f\left(\prod_2\right) \tag{B.32}$$

or

$$\frac{U}{\sqrt{gd}} = f\left(\frac{\rho_L}{\rho_w}\right). \tag{B.33}$$

In the limiting case that $\rho_L/\rho_w < < 1$ one may simplify the relation to

$$f\left(\frac{\rho_L}{\rho_w}\right) = f(0) = \text{const.}, \tag{B.34}$$

where the constant value follows from measurements to const. = 0.35.

**Problem A2.3** The height $h$ to which the liquid rises in a capillary tube of diameter $d$, is a function of the surface tension $\sigma$ and the density, the gravitational acceleration $g$ and the angle $\alpha$, according to

$$h = f(d, \sigma, g, \rho, \alpha). \tag{B.35}$$

The dimension matrix thus reads

| | $h$ | $d$ | $\sigma$ | $g$ | $\rho$ | $\alpha$ |
|---|---|---|---|---|---|---|
| L | 1 | 1 | 0 | 1 | −3 | 0 |
| M | 0 | 0 | 1 | 0 | 1 | 0 |
| T | 0 | 0 | −2 | −2 | 0 | 0 |

and can be transformed to

| | $h/d$ | $d$ | $\sigma/\rho$ | $g$ | $\rho$ | $\alpha$ |
|---|---|---|---|---|---|---|
| L | 0 | 1 | 3 | 1 | −3 | 0 |
| M | 0 | 0 | 0 | 0 | 1 | 0 |
| T | 0 | 0 | −2 | −2 | 0 | 0 |

and subsequently to

| | $h/d$ | $d$ | $\sigma/(\rho g)$ | $g$ | $\alpha$ |
|---|---|---|---|---|---|
| L | 0 | 1 | 2 | 1 | 0 |
| M | 0 | 0 | 0 | 0 | 0 |
| T | 0 | 0 | 0 | −2 | 0 |

where in the first step the dimension of mass is eliminated by combining the gravitational acceleration and the density. The product

$$a = \sqrt{\frac{\sigma}{\rho g}} \tag{B.36}$$

possesses the dimensions of a length and is often referred to as Laplace length. Besides gravitational acceleration it only depends on material properties. For water at 20 °C it takes the value 0.39 m. The dimensionless products are

$$\prod\nolimits_1 = \frac{h}{d}, \quad \prod\nolimits_2 = \frac{\sigma}{\rho g d^2}, \quad \prod\nolimits_3 = \alpha, \tag{B.37}$$

yielding the relation

$$\frac{h}{d} = f\left(\frac{\sigma}{\rho g d^2}, \alpha\right) \tag{B.38}$$

or

$$\frac{h}{d} = f\left(\frac{a}{d}, \alpha\right). \tag{B.39}$$

Further improvement requires additional physical insight. From observation, we know that the height increases as the diameter of the tube becomes smaller. Assuming that $h \propto 1/d$ or that the height occurs only in the combination $hd$, besides $\alpha$ the dimensionless product

$$\prod\nolimits_1^* = \prod\nolimits_1 \prod\nolimits_2^{-1} = \frac{\rho g h d}{\sigma} \tag{B.40}$$

is obtained and the functional relation can be transformed to

$$\frac{\rho g h d}{\sigma} = f(\alpha). \tag{B.41}$$

By analytical means one finds $f(\alpha) = 4\cos(\alpha)$.

**Problem A2.4** A large container contains a liquid of density $\rho$ and viscosity $\eta$. The liquid is released at the bottom of the container through an opening of diameter $d$ at speed $u$. It is sought for the liquid outflow velocity in the form

$$u = f(h, d, g, \eta, \rho) \tag{B.42}$$

The dimension matrix reads

| | $u$ | $h$ | $d$ | $g$ | $\eta$ | $\rho$ |
|---|---|---|---|---|---|---|
| L | 1 | 1 | 1 | 1 | −1 | −3 |
| M | 0 | 0 | 0 | 0 | 1 | 1 |
| T | −1 | 0 | 0 | −2 | −1 | 0 |

and is reshaped to

| | $u$ | $h$ | $d/h$ | $gh$ | $\eta$ | $\rho/\eta$ |
|---|---|---|---|---|---|---|
| L | 1 | 1 | 0 | 2 | −1 | −2 |
| M | 0 | 0 | 0 | 0 | 1 | 0 |
| T | −1 | 0 | 0 | −2 | −1 | 1 |

At this point the viscosity must vanish, since no other variable containing the dimension of mass is left. After further transformation, one obtains

| | $u/\sqrt{gh}$ | $h$ | $d/h$ | $gh$ | $h\rho/\eta$ |
|---|---|---|---|---|---|
| L | 0 | 1 | 0 | 2 | −1 |
| M | 0 | 0 | 0 | 0 | 0 |
| T | 0 | 0 | 0 | −2 | 1 |

and finally the dimensionless products result to

$$\prod_1 = \frac{u}{\sqrt{gd}}, \quad \prod_2 = \frac{\rho h\sqrt{gh}}{\eta} \quad \text{and} \quad \prod_3 = \frac{d}{h}. \tag{B.43}$$

The relation between the products is then

$$f\left(\prod_1, \prod_2, \prod_3\right) = 0 \quad \text{respectively} \quad \prod_1 = f\left(\prod_2, \prod_3\right) \tag{B.44}$$

or

$$\frac{u}{\sqrt{gh}} = f\left(\frac{\sqrt{gh}\rho h}{\eta}, \frac{d}{h}\right). \tag{B.45}$$

Additional physical considerations allow further simplification. The product $\prod_2$ can be identified as the Reynolds number. Since it is expected that for large Reynolds numbers, i.e. in the limit of negligible friction, a non-trivial result is obtained, the product $\prod_2$ is retracted, yielding

$$\frac{u}{\sqrt{gh}} = f\left(\frac{d}{h}\right). \tag{B.46}$$

However, even this relationship can be further simplified if one considers that the volume flow $\dot{V}$ is proportional to the outflow area and, therefore, the speed cannot depend on the diameter of the outlet opening. We finally obtain

$$u = \text{const.}\sqrt{gh}, \tag{B.47}$$

whereas theoretical considerations provide the value const. $= \sqrt{2}$. A related problem is given by the sand descending in an hourglass or the outflow of grain from a silo. As long as the grain diameter is comparatively small thus $k \ll d$ and $k \ll h$, the dry sand or the cereal can be well approximated as a continuum. However, other than for liquids it is found, that the pressure at the bottom of the vessel is independent of the filling level $h$. In fact the sand flows only very close to the opening hole. Close to the opening the sand moves downwards in a funnel-shaped tube, or in a cylindrical tube at larger distances. Outside that tube the sand does not move. As a result of this observation and the independence of the ground pressure from the container height yields a relation of the form

$$u = f(g, d) \tag{B.48}$$

thus

$$u = \text{const.}\sqrt{gd} \tag{B.49}$$

resulting in the relation $\dot{V} \propto d^{5/2}$ for the volume flow, also known as "5/2-law" (see e.g. Wieghardt 1952).

**Problem A2.5** The water course is characterized by its width $b$ and the height of the water level $h$. From the previous example it is apparent, that for calculating the outflow velocity of liquids from large containers, the influence of viscosity can be neglected. But then the density can no longer be part of the solution, as it becomes the only variable of the problem that contains the dimension of mass. Accordingly the sought relation for the volume flow is of the form

$$\dot{V} = f(g, b, h), \tag{B.50}$$

with the corresponding dimension matrix

| | $\dot{V}$ | $g$ | $b$ | $h$ |
|---|---|---|---|---|
| L | 3 | 1 | 1 | 1 |
| M | 0 | 0 | 0 | 0 |
| T | −1 | −2 | 0 | 0 |

with

| | $\dot{V}/h^2$ | $gh$ | $b/h$ | $h$ |
|---|---|---|---|---|
| L | 1 | 2 | 0 | 1 |
| M | 0 | 0 | 0 | 0 |
| T | −1 | −2 | 0 | 0 |

it follows that besides the length ratio

$$\prod_1 = \frac{b}{h} \tag{B.51}$$

the combination

$$\prod_2 = \frac{\dot{V}}{h^2\sqrt{gh}} \tag{B.52}$$

occurs, revealing the functional relation

$$\frac{\dot{V}}{h^2\sqrt{gh}} = f\left(\frac{b}{h}\right). \tag{B.53}$$

An improved statement is obtained by the assumption that the flow is proportional to the width of the opening, so that

$$\frac{\dot{V}}{b} = f(g, h). \tag{B.54}$$

In that case the dimension matrix reads

| | $\dot{V}/b$ | $g$ | $h$ |
|---|---|---|---|
| L | 2 | 1 | 1 |
| M | 0 | 0 | 0 |
| T | −1 | −2 | 0 |

yielding the only dimensionless product

$$\frac{\dot{V}}{bh\sqrt{gh}} = \text{const.} \tag{B.55}$$

**Problem A3.1** An incompressible fluid with the density $\rho$ and viscosity $\eta$ flows through a straight circular pipe of diameter $d$. The pressure gradient is given by $K = -\Delta p/L$. The mass flow is then a function of the following form

$$\dot{m} = f(K, \eta, \rho, d). \tag{B.56}$$

The dimension matrix in the $[LMFT]$ system reads

| | $\dot{m}$ | $K$ | $\eta$ | $\rho$ | $d$ |
|---|---|---|---|---|---|
| L | 0 | −3 | −2 | −3 | 1 |
| M | 1 | 0 | 0 | 1 | 0 |
| F | 0 | 1 | 1 | 0 | 0 |
| T | −1 | 0 | 1 | 0 | 0 |

and after transformation it becomes

| | $\dot{m}$ | $K/\eta$ | $\eta$ | $\rho$ | $d$ |
|---|---|---|---|---|---|
| L | 0 | −1 | −2 | −3 | 1 |
| M | 1 | 0 | 0 | 1 | 0 |
| F | 0 | 0 | 1 | 0 | 0 |
| T | −1 | −1 | 1 | 0 | 0 |

This matrix can be rewritten, because $\eta$ can not be part of the solution. This results in

| | $\dot{m}/\rho$ | $K\,d/\eta$ | $\rho$ | $d$ |
|---|---|---|---|---|
| L | 3 | 0 | −3 | 1 |
| M | 0 | 0 | 1 | 0 |
| F | 0 | 0 | 0 | 0 |
| T | −1 | −1 | 0 | 0 |

Thus, only one dimensionless product is obtained

$$\frac{\dot{m}\eta}{\rho\,K\,d^4} = \text{const.} \tag{B.57}$$

If we solve this equation for the mass flow, we obtain

$$\dot{m} = \text{const.}\frac{\rho K d^4}{\eta}. \tag{B.58}$$

By means of the $[LMFT]$ system corresponding to Newton's law the introduction of a dimensional constant $C$ is required, thus

$$F = Cma. \tag{B.59}$$

Otherwise, Newton's law would not be dimensionally homogeneous. This constant is however not considered in the derivation of Eqs. (B.57) and (B.58), therefore implying that Newton's law is not relevant for the considered problem, i.e. the flow is not accelerated. The solution given by Eq. (B.57) or (B.58) thus corresponds to the laminar Hagen-Poiseuille flow and the constant may be determined from theoretical considerations to be const. $= \pi/128$ (see Schlichting 1979).

To account for the validity of Newton's law in the $[LMFT]$ system, the dependence of the constant $C$ must be included and the function given by Eq. (B.56) is extended to

$$\dot{m} = f(K, \eta, \rho, d, C). \tag{B.60}$$

The dimension matrix becomes

| | $\dot{m}$ | $K$ | $\eta$ | $\rho$ | $d$ | $C$ |
|---|---|---|---|---|---|---|
| L | 0 | −3 | −2 | −3 | 1 | −1 |
| M | 1 | 0 | 0 | 1 | 0 | −1 |
| F | 0 | 1 | 1 | 0 | 0 | 1 |
| T | −1 | 0 | 1 | 0 | 0 | 2 |

Transformation yields

| | $\dot{m}$ | $K/\eta$ | $\eta$ | $\rho$ | $d$ | $C/\eta$ |
|---|---|---|---|---|---|---|
| L | 0 | −1 | −2 | −3 | 1 | 1 |
| M | 1 | 0 | 0 | 1 | 0 | −1 |
| F | 0 | 0 | 1 | 0 | 0 | 0 |
| T | −1 | −1 | 1 | 0 | 0 | 1 |

This shows that $\eta$ cannot be part of the solution and one obtains

| | $\dot{m}/\rho$ | $K/\eta$ | $\rho$ | $d$ | $C\dot{m}/\eta$ |
|---|---|---|---|---|---|
| L | 3 | −1 | −3 | 1 | 1 |
| M | 0 | 0 | 1 | 0 | 0 |
| F | 0 | 0 | 0 | 0 | 0 |
| T | −1 | −1 | 0 | 0 | 0 |

From the dimension matrix we see that also $\rho$ cannot be part of the solution and thus one obtains

| | $\dot{m}\,\eta/(\rho K d^4)$ | $K/\eta$ | $d$ | $C\,\dot{m}/(\eta d)$ |
|---|---|---|---|---|
| L | 0 | −1 | 1 | 0 |
| M | 0 | 0 | 0 | 0 |
| F | 0 | 0 | 0 | 0 |
| T | 0 | −1 | 0 | 0 |

The two dimensionless products result to

$$\prod\nolimits_1 = \frac{\dot{m}\eta}{\rho K d^4} \quad \text{and} \quad \prod\nolimits_2 = \frac{C\dot{m}}{\eta d}. \tag{B.61}$$

Equation (B.60) then becomes

$$\frac{\dot{m}\eta}{\rho K d^4} = f\left(\frac{C\dot{m}}{\eta d}\right). \tag{B.62}$$

Both dimensionless products contain the viscosity. It is however beneficial to choose the products such that the viscosity appears only in a single product, and therefore $\prod_1$ is replaced by

$$\prod_1^* = \prod_1 \prod_2 = \frac{C\dot{m}^2}{\rho K d^5}. \tag{B.63}$$

Further introducing a mean velocity defined by

$$\dot{m} = \rho\,\dot{V} = \rho\,\overline{U}\frac{\pi}{4}d^2 \tag{B.64}$$

results to

$$\prod_2 = C\frac{\rho\overline{U}d}{\eta} = C\,\mathrm{Re} \quad \text{with} \quad \mathrm{Re} = \frac{\rho\overline{U}d}{\eta}. \tag{B.65}$$

To return to the $[LMT]$-system, the constant and its dimension are set to unity, thus $[C] = 1$. From Eq. (B.62) the functional dependence then reads

$$\frac{\dot{m}^2}{\rho\,K\,d^5} = f(\mathrm{Re}) \tag{B.66}$$

and solving for the mass flow reveals

$$\dot{m} = \sqrt{\rho K d^5} f(\mathrm{Re}). \tag{B.67}$$

In contrast to Eq. (B.58), the mass flow is no longer directly proportional to the pressure gradient, but increases proportionally to its square root.

**Problem A3.2** The moment $M$ acting on a cylinder of radius $R$ resulting from a rotation with the angular velocity $\Omega$ in a liquid with the viscosity $\eta$ and density $\rho$ follows to

$$M = f(\eta, \rho, \Omega, R). \tag{B.68}$$

In the $[LMFT]$-System the corresponding dimension matrix reads

| | $M$ | $\eta$ | $\rho$ | $\Omega$ | $R$ | $C$ |
|---|---|---|---|---|---|---|
| L | 1 | −2 | −3 | 0 | 1 | −1 |
| M | 0 | 0 | 1 | 0 | 0 | −1 |
| F | 1 | 1 | 0 | 0 | 0 | 1 |
| T | 0 | 1 | 0 | −1 | 0 | 2 |

The dimension matrix is of the rank $r = 4$, so that the number of dimensionless products results to $d = n - r = 2$. After transformation

| | $M/\eta$ | $\eta$ | $\rho$ | $\Omega$ | $R$ | $C\rho/\eta$ |
|---|---|---|---|---|---|---|
| L | 3 | −2 | −3 | 0 | 1 | −2 |
| M | 0 | 0 | 1 | 0 | 0 | 0 |
| F | 0 | 1 | 0 | 0 | 0 | 0 |
| T | −1 | 1 | 0 | −1 | 0 | 1 |

One notices that $\rho$ and $\eta$ cannot be part of the solution. Thus, we obtain

| | $M/(\eta\Omega R^3)$ | $\Omega$ | $R$ | $C\rho\,\Omega R^2/\eta$ |
|---|---|---|---|---|
| L | 0 | 0 | 1 | 0 |
| M | 0 | 0 | 0 | 0 |
| F | 0 | 0 | 0 | 0 |
| T | 0 | −1 | 0 | 0 |

The two dimensionless products are identified to be

$$\prod\nolimits_1 = \frac{M}{\eta\Omega R^3} \quad \text{and} \quad \prod\nolimits_2 = C\frac{\rho\,\Omega R^2}{\eta}. \tag{B.69}$$

Equation (B.68) can thus be written in the form

$$\frac{M}{\eta\Omega R^3} = f\left(C\frac{\rho\,\Omega R^2}{\eta}\right). \tag{B.70}$$

The product $\prod_2$ corresponds to the Reynolds number, so that for Re $\rightarrow 0$ one obtains

$$\frac{M}{\eta\Omega R^3} = \text{const.} \tag{B.71}$$

The same result is found, when not accounting for the constant $C$, implying that Newton's law is disregarded and therefore momentum is neglected. For finite, non-zero Reynolds number, it is helpful to choose the products such that the viscosity occurs only in one product, hence

$$\prod\nolimits_1^* = \prod\nolimits_1 \prod\nolimits_2^{-1} = \frac{1}{C}\frac{M}{\rho\,(\Omega R)^2 R^3} \tag{B.72}$$

or in a form equivalent to Eq. (B.70)

$$\frac{M}{\rho(\Omega R)^2 R^3} = f\left(\frac{\rho\,\Omega R^2}{\eta}\right). \tag{B.73}$$

**Problem A4.1** It is sought for the resistance $W$ of a sphere in the form

$$W = f(g, v, \sigma, h, d, U, \rho). \tag{B.74}$$

In the $[LMT]$-system the dimension matrix results in

| | $W$ | $g$ | $v$ | $\sigma$ | $h$ | $d$ | $U$ | $\rho$ |
|---|---|---|---|---|---|---|---|---|
| | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ | $k_7$ | $k_8$ |
| L | 1 | 1 | 2 | 0 | 1 | 1 | 1 | −3 |
| M | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| T | −2 | −2 | −1 | −2 | 0 | 0 | −1 | 0 |

The quantities that are to occur linearly in the dimensionless products are put to the front columns. The rank $r$ of the dimension matrix is $r = 3$ so that $d = n - r = 5$ dimensionless products have to be determined. For the exponent of the basic variables to vanish the relation

$$\sum_{j=1}^{n} a_{ij} k_j = 0 \tag{B.75}$$

must hold. In the present case this yields the system of equations

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 1 & 1 & -3 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -2 & -2 & -1 & -2 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \end{bmatrix} = 0 \tag{B.76}$$

and accordingly

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} k_6 \\ k_7 \\ k_8 \end{bmatrix} = 0 \tag{B.77}$$

respectively after transformation

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} k_6 \\ k_7 \\ k_8 \end{bmatrix} = -\begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix} \tag{B.78}$$

and

$$\begin{bmatrix} k_6 \\ k_7 \\ k_8 \end{bmatrix} = -\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix}. \tag{B.79}$$

Thus, we obtain

$$\begin{bmatrix} k_6 \\ k_7 \\ k_8 \end{bmatrix} = -\begin{bmatrix} 2 & -1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix}. \tag{B.80}$$

This results in a set of $d = n - r = 5$ linearly independent solution vectors $k_{(i),j}, (i = 1\ldots d, j = 1\ldots n)$ by assigning the $k_{(i),j}, (i,j = 1\ldots d)$ on the right side of Eq. (B.80) to unit vectors according to

$$\begin{bmatrix} k_{(1),1} & k_{(1),2} & k_{(1),3} & k_{(1),4} & k_{(1),5} \\ k_{(2),1} & k_{(2),2} & k_{(2),3} & k_{(2),4} & k_{(2),5} \\ k_{(3),1} & k_{(3),2} & k_{(3),3} & k_{(3),4} & k_{(3),5} \\ k_{(4),1} & k_{(4),2} & k_{(4),3} & k_{(4),4} & k_{(4),5} \\ k_{(5),1} & k_{(5),2} & k_{(5),3} & k_{(5),4} & k_{(5),5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{B.81}$$

Substituting Eq. (B.81) into Eq. (B.80), allows to determine the $k_{(i),1}, k_{(i),2}, k_{(i),3}, (i = 1\ldots d)$ according to the following table.

| | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ | $k_7$ | $k_8$ |
|---|---|---|---|---|---|---|---|---|
| $\prod_1$ | 1 | 0 | 0 | 0 | 0 | −2 | −2 | −1 |
| $\prod_2$ | 0 | 1 | 0 | 0 | 0 | 1 | −2 | 0 |
| $\prod_3$ | 0 | 0 | 1 | 0 | 0 | −1 | −1 | 0 |
| $\prod_4$ | 0 | 0 | 0 | 1 | 0 | −1 | −2 | −1 |
| $\prod_5$ | 0 | 0 | 0 | 0 | 1 | −1 | 0 | 0 |

The dimensionless products then follow from

$$\prod_i = W^{k_{(i),1}}\, g^{k_{(i),2}}\, \nu^{k_{(i),3}}\, \sigma^{k_{(i),4}}\, h^{k_{(i),5}}\, d^{k_{(i),6}}\, U^{k_{(i),7}}\, \rho^{k_{(i),8}}, \quad (i = 1\ldots d) \tag{B.82}$$

to

$$\prod_1 = \frac{W}{d^2\, U^2\, \rho} = c_w, \tag{B.83}$$

$$\prod_2 = \frac{g\, d}{U^2} = \frac{1}{Fr}, \tag{B.84}$$

$$\prod_3 = \frac{\nu}{d\, U} = \frac{1}{\mathrm{Re}}, \tag{B.85}$$

$$\prod_4 = \frac{\sigma}{d\, U^2 \rho} = We, \tag{B.86}$$

and

$$\prod_5 = \frac{h}{d}. \tag{B.87}$$

Thus, Eq. (B.74) can be written in the equivalent form

$$c_w = f(\mathrm{Re}, \mathrm{Fr}, \mathrm{We}, h/d). \tag{B.88}$$

In most practical applications, the Laplace length $a \equiv \sqrt{\sigma/\rho\, g}$ is much smaller than the sphere diameter, thus

$$\frac{a}{d} \ll 1. \tag{B.89}$$

Then, the surface tension becomes insignificant for the problem and the Weber number may be neglected. Formally, the products are combined to

$$\prod_4^* = \sqrt{\mathrm{Fr}}\sqrt{\mathrm{We}} = \frac{U}{\sqrt{g\, d}}\sqrt{\frac{\sigma}{\rho\, d}}\frac{1}{U} = \frac{a}{d} \tag{B.90}$$

with $\prod_4^* \ll 1$ resulting in

$$c_w = f(\mathrm{Re}, \mathrm{Fr}, h/d). \tag{B.91}$$

**Problem A5.1** In consideration of geometric similarity the buckling load is only a function of the elasticity modulus $E$ and a characteristic length $L$, thus

$$P = f(E, L). \tag{B.92}$$

The corresponding dimension matrix reads

| | $P$ | $E$ | $L$ |
|---|---|---|---|
| L | 1 | −1 | 1 |
| M | 1 | 1 | 0 |
| T | −2 | −2 | 0 |

and the only dimensionless product is identified to be

$$\frac{P}{E\,L^2} = \text{const.} \tag{B.93}$$

To achieve complete similarity, all dimensionless products for model and full-scale must be equal. The ratios of the physical quantities in the model $p'_j$ to those of the full-scale $p_j$ determine the scale factors $M_j$, i.e.

$$p'_j = M_j p_j. \tag{B.94}$$

In the present case, the dimensionless quantity given by Eq. (B.93) for model and full-scale must be equal, i.e.

$$\frac{P}{E\,L^2} = \frac{P'}{E'\,L'^2} \tag{B.95}$$

or

$$\frac{P}{P'} = \frac{E}{E'}\left(\frac{L}{L'}\right)^2 \tag{B.96}$$

resulting in the solution

$$P = 7.5 \times 10^6\ \text{N}. \tag{B.97}$$

**Problem A5.2** A cantilever that is anchored at one end and carries a load $P$ at the other end is considered and it is sought for the deflection of the beam. The deflection of the beam $w$ depends on the force $P$, the beam length $L$, the density $\rho$ and the gravitational acceleration $g$ so that the relation is of the form

$$w = f(L, P, E, \rho, g). \tag{B.98}$$

The corresponding dimension matrix reads

| | $w$ | $L$ | $P$ | $E$ | $\rho$ | $g$ |
|---|---|---|---|---|---|---|
| L | 1 | 1 | 1 | −1 | −3 | 1 |
| M | 0 | 0 | 1 | 1 | 1 | 0 |
| T | 0 | 0 | −2 | −2 | 0 | −2 |

and after transformation

| | $w/L$ | $L$ | $P/E$ | $E$ | $\rho/E$ | $g$ |
|---|---|---|---|---|---|---|
| L | 0 | 1 | 2 | −1 | −2 | 1 |
| M | 0 | 0 | 0 | 1 | 0 | 0 |
| T | 0 | 0 | 0 | −2 | 2 | −2 |

and

| | $w/L$ | $L$ | $P/(E\,L^2)$ | $E$ | $\rho\,g\,L/E$ | $g$ |
|---|---|---|---|---|---|---|
| L | 0 | 1 | 0 | −1 | 0 | 1 |
| M | 0 | 0 | 0 | 1 | 0 | 0 |
| T | 0 | 0 | 0 | −2 | 0 | −2 |

three dimensionless product are obtained and Eq. (B.98) is rewritten to

$$\frac{w}{L} = f\left(\frac{P}{E\,L^2}, \frac{\rho\,g\,L}{E}\right). \tag{B.99}$$

In order for the model and full-scale are loaded in a physically similar way, the dimensionless products for both versions must be equal, i.e.

$$\frac{P}{E\,L^2} = \frac{P'}{E'\,L'^2} \quad \text{and} \quad \frac{\rho\,g\,L}{E} = \frac{\rho'\,g\,L'}{E'}, \tag{B.100}$$

or

$$\frac{P'}{P} = \frac{E'}{E}\left(\frac{L'}{L}\right)^2 \quad \text{and} \quad \frac{\rho'}{\rho} = \frac{E'}{E}\,\frac{L}{L'}. \tag{B.101}$$

Accordingly the scale factors are related by

$$M_P = M_E M_L^2 \quad \text{and} \quad M_\rho = \frac{M_E}{M_L} \tag{B.102}$$

or

$$M_P = M_E M_L^2 = M_\rho M_L^3. \tag{B.103}$$

From Eq. (B.102) it follows that for the same material ($M_E = M_\rho = 1$) model and full-scale must be the same, if the deflection as a result of its own weight is considered. If the deformation caused by its own weight can be neglected over the single force deformation, the force in the model is smaller than that for the full scale by a factor of $M_L^2$.

To determine the stresses $\sigma$, a relation of the form

$$\sigma = f(L, P, E, \rho, g), \tag{B.104}$$

is examined, corresponding to the non-dimensional formulation

$$\frac{\sigma L^2}{P} = f\left(\frac{P}{E L^2}, \frac{\rho g L}{E}\right). \tag{B.105}$$

The demand for the equality of the dimensionless products for model and full scale yields the relations for the scale factors to

$$M_\sigma = \frac{M_P}{M_L^2}, \quad M_P = M_E M_L^2 \quad \text{and} \quad M_\rho = \frac{M_E}{M_L}, \tag{B.106}$$

respectively

$$M_\sigma = M_E \quad \text{and} \quad M_P = M_E M_L^2 = M_\rho M_L^3. \tag{B.107}$$

It is again found that when using the same material, model and full-scale must be the same if stresses caused by the own weight cannot be neglected. In case that the own weight may be neglected over the added loading it is found that the stresses in the model and in the full scale are equal.

**Problem A5.3** The temperature of the roast beef piece $\vartheta$ depends on the initial temperature $\vartheta_0$ at which it has been pushed into the oven, the oven temperature $\vartheta_\infty$, the time $t$, the thermal conductivity $k$, the heat capacity $c$, the density $\rho$, and a characteristic length $L$. We are therefore looking for a functional relation of the form

$$\vartheta = f(\vartheta_0, \vartheta_\infty, t, k, \rho, c, L). \tag{B.108}$$

Since heat conduction problems are linear with regard to the temperature and therefore only temperature differences are significant, we write Eq. (B.108) in the form

$$\vartheta - \vartheta_0 = (\vartheta - \vartheta_\infty) \quad f(t, k, \rho, c, L). \tag{B.109}$$

So that the dimension matrix yields

| | $t$ | $k$ | $\rho$ | $c$ | $L$ |
|---|---|---|---|---|---|
| L | 0 | 1 | −3 | 2 | 1 |
| M | 0 | 1 | 1 | 0 | 0 |
| T | 1 | −3 | 0 | −2 | 0 |
| $\theta$ | 0 | −1 | 0 | −1 | 0 |

and after transformation

| | $t$ | $k/(\rho\, c)$ | $\rho$ | $c$ | $L$ |
|---|---|---|---|---|---|
| L | 0 | 2 | −3 | 2 | 1 |
| M | 0 | 0 | 1 | 0 | 0 |
| T | 1 | −1 | 0 | −2 | 0 |
| $\theta$ | 0 | 0 | 0 | −1 | 0 |

The only dimensionless product then becomes

$$\frac{t}{L^2}\frac{k}{\rho\, c} = \text{const.} \tag{B.110}$$

or

$$t = \text{const.} L^2 \frac{\rho\, c}{k}. \tag{B.111}$$

The mass of the roast beef piece is

$$m \propto \rho\, L^3 \quad \text{thus} \quad L \propto \left(\frac{m}{\rho}\right)^{1/3}, \tag{B.112}$$

so that accordingly Eq. (B.111) becomes

$$t = \text{const.} \; \frac{\rho\, c}{k} \left(\frac{m}{\rho}\right)^{2/3} = \text{const.} \; \frac{c}{k}\, \rho^{1/3}\, m^{2/3}. \tag{B.113}$$

The time $t'$ that the roast beef piece with $m' = 2\,kg$ must fry, can then be determined from the roasting time $t$ of the lighter piece with $m = 1\ kg$

$$\frac{t'}{t} = \left(\frac{m'}{m}\right)^{2/3} \quad \text{or} \quad t' = t\left(\frac{m'}{m}\right)^{2/3} = 95\ \text{min}. \tag{B.114}$$

**Problem A6.1** The sought circumferential speed is a function of the form

$$v = v(r, t, \Gamma, \eta, \rho). \tag{B.115}$$

(a) The dimension matrix reads

| | $v$ | $r$ | $t$ | $\Gamma$ | $\eta$ | $\rho$ |
|---|---|---|---|---|---|---|
| L | 1 | 1 | 0 | 2 | −1 | −3 |
| M | 0 | 0 | 0 | 0 | 1 | 1 |
| T | −1 | 0 | 1 | −1 | −1 | 0 |

and may be transformed to

| | $v\,r$ | $r$ | $t$ | $\Gamma$ | $\eta/\rho = \nu$ | $\rho$ |
|---|---|---|---|---|---|---|
| L | 2 | 1 | 0 | 2 | 2 | −3 |
| M | 0 | 0 | 0 | 0 | 0 | 1 |
| T | −1 | 0 | 1 | −1 | −1 | 0 |

and further to

| | $v\,r/\Gamma$ | $r/\sqrt{\nu\,t}$ | $t$ | $\Gamma/\nu$ | $\nu$ |
|---|---|---|---|---|---|
| L | 0 | 0 | 0 | 0 | 2 |
| M | 0 | 0 | 0 | 0 | 0 |
| T | 0 | 0 | 1 | 0 | −1 |

So that the nondimensional products result to

$$\prod\nolimits_1 = \frac{v\,r}{\Gamma}\ ,\quad \prod\nolimits_2 = \frac{r}{\sqrt{\nu\,t}} \quad \text{and} \quad \prod\nolimits_3 = \frac{\Gamma}{\nu}. \tag{B.116}$$

With Eq. (B.116) the relation

$$v = f(r, t, \Gamma, \eta, \rho) \tag{B.117}$$

takes the form

$$\frac{v\,r}{\Gamma} = f\left(\frac{r}{\sqrt{\nu\,t}}, \frac{\Gamma}{\nu}\right). \tag{B.118}$$

Accordingly, a similarity variable is introduced by

$$\eta = \frac{r}{\sqrt{\nu\,t}}. \tag{B.119}$$

and we set

$$\frac{v\,r}{\Gamma} = F(\eta). \tag{B.120}$$

The initial and boundary conditions Eqs. (A.1), (A.2) and (A.3) then result in

$$F(0) = 0 \quad \text{and} \quad F(\infty) = \frac{1}{2\,\pi}. \tag{B.121}$$

Introducing the similarity variable

$$\frac{\partial\,\eta}{\partial\,t} = -\frac{\eta}{2\,t} \quad \text{and} \quad \frac{\partial\,\eta}{\partial\,r} = \frac{1}{\sqrt{\nu\,t}} \tag{B.122}$$

into the differential equation

$$\frac{\partial\,v}{\partial\,t} = \nu\left(\frac{\partial^2\,v}{\partial\,r^2} + \frac{1}{r}\frac{\partial\,v}{\partial\,r} - \frac{v}{r^2}\right) \tag{B.123}$$

results in the following ordinary differential equation

$$-\frac{\eta^3}{2}F' = -\eta\,F' + \eta^2\,F'' \tag{B.124}$$

that may be simplified to

$$\frac{F''}{F'} = -\frac{\eta}{2} + \frac{1}{\eta}. \tag{B.125}$$

This equation can be solved by using the method of separation of variables. Integration yields

$$\ln(F') = -\frac{\eta^2}{4} + \ln(\eta) + \ln(c_1) \tag{B.126}$$

with the integration constant $c_1$. Equation (B.126) may be simplified to

$$\ln\left(\frac{F'}{c_1\eta}\right) = -\frac{\eta^2}{4} \tag{B.127}$$

or

$$F' = c_1\eta\ \exp\left(-\frac{\eta^2}{4}\right). \tag{B.128}$$

Integration of Eq. (B.128) results to

$$F = 2\,c_1\ \exp\left(-\frac{\eta^2}{4}\right) + c_2. \tag{B.129}$$

Using the initial and boundary conditions of Eq. (B.121) the constants are determined to be

$$c_2 = \frac{1}{2\,\pi} \quad \text{and} \quad 2\,c_1 = -c_2 \tag{B.130}$$

so that the solution becomes

$$F = \frac{1}{2\,\pi}\left(1 - \exp\left(-\frac{\eta^2}{4}\right)\right) \tag{B.131}$$

respectively

$$v = \frac{\Gamma}{2\,\pi\,r}\left(1 - \exp\left(-\frac{\eta^2}{4}\right)\right). \tag{B.132}$$

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